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1901. If $x,y,z,t>0$, $x+y+z+t=4$ then:

$$\prod\left(1+\frac{1}{x}\right)^{x^2}\geq 16$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\log\left(1+\frac{1}{x}\right)^{x^2}=x^2\log\left(1+\frac{1}{x}\right)=x^2\int_0^1\frac{du}{x+u}$$

$$\log\prod_{cyc}\left(1+\frac{1}{x}\right)^{x^2}=\sum\log\left(1+\frac{1}{x^2}\right)^{x^2}=\sum\frac{x^2}{x+u}du\overset{CBS}{\geq}$$

$$\geq\int_0^1\frac{(x+y+z+t)^2}{(x+y+z+t)+4u}du\overset{x+y+z+t=4}{=}\int_0^1\frac{4^2}{4+4u}du=$$

$$=4\int_0^1\frac{du}{u+1}=4(\log|u+1|)_0^1=4\log 2=\log 16$$

$$\prod\left(1+\frac{1}{x}\right)^{x^2}\geq 16$$

Equality holds for x=y=z=t=1.

1902. If $x,y,z>0$ then:

$$\sqrt{2x^2+xy+2y^2}+\sqrt{2y^2+zy+2z^2}+\sqrt{2z^2+xz+2x^2}\geq\sqrt{5}(x+y+z)$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

Lemma:

$$\forall a,b>0\,,\sqrt{2a^2+ab+2b^2}\geq\frac{\sqrt{5}}{2}(a+b)$$

Proof:

$$We\ need\ to\ show:$$

$$\sqrt{2a^2+ab+2b^2}\geq\frac{\sqrt{5}}{2}(a+b)$$

$$4(2a^2+ab+2b^2)\overset{squaring}{\geq}5(a+b)^2$$

$$8a^2+4ab+8b^2\geq 5(a^2+2ab+b^2)$$

$$3(a^2-2ab+b^2)\geq 0\ or\ 3(a-b)^2\geq 0\ \ true$$

$$\sqrt{2x^2+xy+2y^2}+\sqrt{2y^2+zy+2z^2}+\sqrt{2z^2+xz+2x^2}=$$

$$=\sum\sqrt{2x^2+xy+2y^2}\overset{Lemma}{\geq}\sum\frac{\sqrt{5}}{2}(x+y)=\sqrt{5}(x+y+z)$$

$$Equality\ holds\ for\ x=y=z.$$

1903. If $x,y,z>0$ then:

$$\sqrt{5x^2+14xy+5y^2}+\sqrt{5y^2+14zy+5z^2}+\sqrt{5z^2+14xz+5x^2}\leq 2\sqrt{6}(x+y+z)$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$Lemma$:

$$\forall a,b>0\ ,\sqrt{5a^2+14ab+5b^2}\leq\sqrt{6}(a+b)$$

$Proof$:

$$We\ need\ to\ show:$$

$$\sqrt{5a^2+14ab+5b^2}\leq\sqrt{6}(a+b)$$

$$\overset{squaring}{5a^2+14ab+5b^2\leq 6(a+b)^2}$$

$$5a^2+14ab+5b^2\leq 6(a^2+2ab+b^2)$$

$$a^2-2ab+b^2\geq 0\ or\ (a-b)^2\geq 0\ true$$

$$\sqrt{5x^2+14xy+5y^2}+\sqrt{5y^2+14zy+5z^2}+\sqrt{5z^2+14xz+5x^2}=$$

$$=\sum\sqrt{5x^2+14xy+5y^2}\overset{lemma}{\leq}\sum\sqrt{6}(x+y)=2\sqrt{6}(x+y+z)$$

Equality holds for x=y=z.

1904. If $x,y,z>0,\ x+y+z\leq 1$ then:

$$2(x+y+z)+3\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\geq 29$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$We\ need\ to\ show:$$

$$2(x+y+z)+3\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\geq 29$$

$$2(x+y+z)+3\frac{(1+1+1)^2}{x+y+z}\overset{CBS}{\geq}29$$

$$2t+\frac{27}{t}\overset{x+y+z=t}{\geq}29$$

$$2t^2-29t+27\geq 0\ or\ (2t-27)(t-1)\geq 0\ true$$
$$since\ t=x+y+z\leq 1(given)so,(t-1)\leq 0\ and$$
$$(2t-27)<0\ consequently:\ (2t-27)(t-1)\geq 0$$
$$Equality\ holds\ for\ x=y=z=\frac{1}{3}$$

1905. If $a,b>0$, $a+b=2$ then:

$$\frac{a}{\sqrt{2-a}}+\frac{b}{\sqrt{2-b}}\geq 2$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\frac{a}{\sqrt{2-a}}+\frac{b}{\sqrt{2-b}}=\frac{a^{\frac{3}{2}}}{\sqrt{2a-a^2}}+\frac{b^{\frac{3}{2}}}{\sqrt{2b-b^2}}\overset{Radon}{\geq}$$
$$\geq\frac{(a+b)^{\frac{3}{2}}}{\sqrt{2(a+b)-(a^2+b^2)}}\overset{CBS}{\geq}\frac{(a+b)^{\frac{3}{2}}}{\sqrt{2(a+b)-\frac{(a+b)^2}{2}}}\overset{a+b=2}{=}\frac{2^{\frac{3}{2}}}{\sqrt{4-2}}=2$$

$$Equaliy\ holds\ for\ a=b=1$$

1906. If $a,b,c>0$, $\sqrt{a+1}+\sqrt{b+1}+\sqrt{c+1}=6$ then:

$$\sqrt{a^2+ab+b^2}+\sqrt{b^2+bc+b^2}+\sqrt{c^2+ca+a^2}\geq 9\sqrt{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\sqrt{a+1}+\sqrt{b+1}+\sqrt{c+1}=6\ or$$
$$\sqrt{3(a+1+b+1+c+1)}\overset{CBS}{\geq}6\ or\ 3(a+b+c+3)\overset{squaring}{\geq}36$$
$$or\ a+b+c\geq 9\ (1)$$

Lemma:

$$\forall x,y>0,\sqrt{x^2+xy+y^2}\geq\frac{\sqrt{3}}{2}(x+y)$$

Proof:

$$We\ need\ to\ show:$$
$$\sqrt{x^2+xy+y^2}\geq\frac{\sqrt{3}}{2}(x+y)\ or\ x^2+xy+y^2\overset{squring}{\geq}\frac{3}{4}(x+y)^2$$
$$4(x^2+y^2+xy)\geq 3(x^2+y^2+2xy)\ or$$
$$x^2-2xy+y^2\geq 0\ or(x-y)^2\geq 0\ true$$

$$\sqrt{a^2+ab+b^2}+\sqrt{b^2+bc+b^2}+\sqrt{c^2+ca+a^2}=$$
$$=\sum\sqrt{a^2+ab+b^2}\overset{lemma}{\geq}\sum\frac{\sqrt{3}}{2}(a+b)=\sqrt{3}(a+b+c)\overset{(1)}{\geq}9\sqrt{3}.$$
$$Equality\ holds\ for\ a=b=c=3$$

1907. If $a,b,c>0,\ a+b+c=3$ then:

$$\frac{x^2}{\sqrt{15x^2+26xy+8y^2}}+\frac{y^2}{\sqrt{15y^2+26yz+8z^2}}+\frac{z^2}{\sqrt{15z^2+26zx+8x^2}}\geq\frac{3}{7}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

Lemma:
$$\forall\ a,b>0\ ,\sqrt{15a^2+26ab+8b^2}\leq 4a+3b$$

Proof:
$$We\ need\ to\ show:$$
$$\sqrt{15a^2+26ab+8b^2}\leq 4a+3b$$

$$or,15a^2+26ab+8b^2\overset{squaring}{\leq}16a^2+24ab+9b^2$$

$$or,a^2-2ab+b^2\geq 0\ or\ (a-b)^2\geq 0\ true.$$

$$\frac{x^2}{\sqrt{15x^2+26xy+8y^2}}+\frac{y^2}{\sqrt{15y^2+26yz+8z^2}}+\frac{z^2}{\sqrt{15z^2+26zx+8x^2}}=$$

$$=\sum\frac{x^2}{\sqrt{15x^2+26xy+8y^2}}\overset{Lemma}{\geq}\sum\frac{x^2}{4x+3y}\overset{Bergstrom}{\geq}$$

$$\geq\frac{(x+y+z)^2}{4(x+y+z)+3(x+y+z)}=\frac{x+y+z}{7}\overset{x+y+z=3}{=}\frac{3}{7}$$
$$Equality\ holds\ for\ x=y=z=1.$$

1908. If $a,b,c>0,\ \sqrt{a}+\sqrt{b}+\sqrt{c}=3$ then:

$$\sqrt{3a^2+2ab+3b^2}+\sqrt{3b^2+2bc+3c^2}+\sqrt{3c^2+2ca+3a^2}\geq 6\sqrt{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\sqrt{a}+\sqrt{b}+\sqrt{c}=3 \ \ or \ \sqrt{3(a+b+c)} \overset{CBS}{\geq} \ 3$$

$$3(a+b+c)\geq 9 \ or \ \ a+b+c\geq 3 \ \ (1)$$

Lemma :

$$\forall x,y>0\ , \sqrt{3x^2+2xy+3y^2}\geq \sqrt{2}(x+y)$$

Proof:

$$We\ need\ to\ show:$$
$$\sqrt{3x^2+2xy+3y^2}\geq \sqrt{2}(x+y)$$
$$3x^2+2xy+3y^2 \overset{squaring}{\geq} 2(x^2+2xy+y^2)$$

$$x^2-2xy+y^2\geq 0 \ or \ (x-y)^2\geq 0 \ true$$

$$\sqrt{3a^2+2ab+3b^2}+\sqrt{3b^2+2bc+3c^2}+\sqrt{3c^2+2ca+3a^2}=$$
$$=\sum\sqrt{3a^2+2ab+3b^2} \overset{lemma}{\geq} \sum\sqrt{2}(a+b)=2\sqrt{2}(a+b+c) \overset{(1)}{\geq} 6\sqrt{2}$$

Equality holds for a=b=c=1.

1909. If $x,y,z>0$ then:

$$\sqrt{22x^2+36xy+6y^2}+\sqrt{22y^2+36xy+6x^2}\leq x^2+y^2+32$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

Lemma:

$$\forall\ a,b>0\ , \sqrt{22a^2+36ab+6b^2}\leq (5a+3b)$$

Proof:

$$We\ need\ to\ show:$$
$$\sqrt{22a^2+36ab+6b^2}\leq (5a+3b)$$
$$22a^2+36ab+6b^2 \overset{squaring}{\leq} 25a^2+30ab+9b^2$$
$$3(a^2-2ab+b^2)\geq 0 \ or \ 3(a-b)^2\geq 0 \ true.$$

$$\sqrt{22x^2+36xy+6y^2}+\sqrt{22y^2+36xy+6x^2} \overset{Lemma}{\leq}$$
$$\leq (5x+3y)+(5y+3x)=8(x+y)\ (1)$$

$$We\ need\ to\ show:$$
$$\sqrt{22x^2+36xy+6y^2}+\sqrt{22y^2+36xy+6x^2}\leq x^2+y^2+32$$

$$8(x+y) \overset{(1)}{\leq} x^2+y^2+32 \ or \ \frac{(x+y)^2}{2}+32 \overset{CBS}{\geq} 8(x+y)$$

$$(x+y)^2-16(x+y)+64 \geq 0 \ or \ (x+y-8)^2 \geq 0$$

$$Equality \ holds \ for \ x=y=4$$

1910. If $a,b,c>0, \frac{1}{a}+\frac{1}{b}+\frac{1}{c}=3$ then:

$$\frac{1}{\sqrt{a^2-ab+3b^2+1}}+\frac{1}{\sqrt{b^2-bc+3c^2+1}}+\frac{1}{\sqrt{c^2-ca+3a^2+1}} \leq \frac{3}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$a^2-ab+3b^2+1=(a^2+b^2)+(b^2+1)+b^2-ab \overset{AM-GM}{\geq}$$

$$\geq 2ab+2b+b^2-ab=b(a+b+2) \ (1)$$

$$\frac{1}{\sqrt{a^2-ab+3b^2+1}}+\frac{1}{\sqrt{b^2-bc+3c^2+1}}+\frac{1}{\sqrt{c^2-ca+3a^2+1}}=$$

$$=\sum \frac{1}{\sqrt{a^2-ab+3b^2+1}} \overset{(1)}{\leq} \sum \frac{1}{\sqrt{b(a+b+2)}}=$$

$$=2\sum \sqrt{\frac{1}{4b}\cdot\frac{1}{a+b+2}} \overset{AM-GM}{\leq} \sum\left(\frac{1}{4b}+\frac{1}{a+b+2}\right)=$$

$$=\sum \frac{1}{4b}+\sum \frac{1}{a+b+1+1} \overset{AM-HM}{\leq} \frac{1}{4}\sum \frac{1}{b}+\frac{1}{16}\sum\left(\frac{1}{a}+\frac{1}{b}+1+1\right)=$$

$$=\frac{1}{4}\sum \frac{1}{b}+\frac{1}{16}\left(\sum \frac{1}{a}+\sum \frac{1}{b}+\sum 1+\sum 1\right) \overset{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=3}{=}$$

$$=\frac{3}{4}+\frac{1}{16}(3+3+3+3)=\frac{3}{2}$$

Equality holds for a=b=c=1.

1911. If $a,b,c>0, abc=1$ and $\lambda \geq 2$ then :

$$\sum_{cyc} \frac{1}{\sqrt{a^3+\lambda^2-1}} \leq \frac{3}{\lambda}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$a^3=x,b^3=y,c^3=z\,(xyz=1)\Rightarrow\sum_{cyc}\frac{1}{\sqrt{a^3+\lambda^2-1}}\le\frac{3}{\lambda}$$

$$\Leftrightarrow\sum_{cyc}\frac{1}{\sqrt{x+\lambda^2-1}}\le\frac{3}{\lambda}\Leftrightarrow\sum_{cyc}\frac{1}{\sqrt{x+3+\lambda^2-4}}\le\frac{3}{\sqrt{4+\lambda^2-4}}$$

$$\Leftrightarrow\sum_{cyc}\frac{1}{\sqrt{x+3+m}}\overset{①}{\le}\frac{3}{\sqrt{4+m}}\ (\because m=\lambda^2-4)$$

$$\text{Now},\sum_{cyc}\frac{1}{\sqrt{x+3+m}}\overset{CBS}{\le}\sqrt{3\sum_{cyc}\frac{1}{x+3+m}}\overset{?}{\le}\frac{3}{\sqrt{4+m}}\Leftrightarrow\sum_{cyc}\frac{1}{x+3+m}\overset{?}{\le}\frac{3}{4+m}$$

$$\Leftrightarrow 3\prod_{cyc}(x+3+m)\overset{?}{\ge}(m+4)\sum_{cyc}\big((y+3+m)(z+3+m)\big)$$

$$\Leftrightarrow m^2\left(\sum_{cyc}x-3\right)+m\left(2\sum_{cyc}xy+4\sum_{cyc}x-18\right)+$$

$$3xyz+5\sum_{cyc}xy+3\sum_{cyc}x-27\overset{?}{\ge}0\to\text{true}\because m=\lambda^2-4\overset{\lambda\ge2}{\ge}0\ and$$

$$\because\sum_{cyc}x\overset{AM-GM}{\ge}3.\sqrt[3]{xyz}\overset{xyz=1}{\ge}3\ and\ also,\sum_{cyc}xy\overset{AM-GM}{\ge}3.\sqrt[3]{x^2y^2z^2}\overset{xyz=1}{\ge}3$$

$$\Rightarrow\left(\sum_{cyc}x-3\right),\left(2\sum_{cyc}xy+4\sum_{cyc}x-18\right),\left(3xyz+5\sum_{cyc}xy+3\sum_{cyc}x-27\right)\ge0$$

$$\Rightarrow ①\ \text{is true}\therefore\sum_{cyc}\frac{1}{\sqrt{a^3+\lambda^2-1}}\le\frac{3}{\lambda}\ \forall\ abc=1\ and\ \lambda\ge2,$$

$$''='' \text{ iff } a=b=c=1\ (QED)$$

1912. If $a,b,c>0,abc=1\ and\ \lambda\ge2$ then :

$$\sum_{cyc}\frac{1}{\sqrt{a+\lambda^2-1}}\le\frac{3}{\lambda}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{1}{\sqrt{a+\lambda^2-1}}\le\frac{3}{\lambda}\Leftrightarrow\sum_{cyc}\frac{1}{\sqrt{a+3+\lambda^2-4}}\le\frac{3}{\sqrt{4+\lambda^2-4}}$$

$$\Leftrightarrow \sum_{cyc}\frac{1}{\sqrt{a+3+m}}\overset{①}{\leq}\frac{3}{\sqrt{4+m}}\ (\because m=\lambda^2-4)$$

$$\text{Now, }\sum_{cyc}\frac{1}{\sqrt{a+3+m}}\overset{CBS}{\leq}\sqrt{3\sum_{cyc}\frac{1}{a+3+m}}\overset{?}{\leq}\frac{3}{\sqrt{4+m}}\Leftrightarrow\sum_{cyc}\frac{1}{a+3+m}\overset{?}{\leq}\frac{3}{4+m}$$

$$\Leftrightarrow 3\prod_{cyc}(a+3+m)\overset{?}{\geq}(m+4)\sum_{cyc}\big((b+3+m)(c+3+m)\big)$$

$$\Leftrightarrow m^2\left(\sum_{cyc}a-3\right)+m\left(2\sum_{cyc}ab+4\sum_{cyc}a-18\right)+$$

$$3abc+5\sum_{cyc}ab+3\sum_{cyc}a-27\overset{?}{\geq}0\to\text{true}\because m=\lambda^2-4\overset{\lambda\geq 2}{\geq}0\ \text{and}$$

$$\because\sum_{cyc}a\overset{AM-GM}{\geq}3.\sqrt[3]{abc}\overset{abc=1}{\geq}3\ \text{and also},\sum_{cyc}ab\overset{AM-GM}{\geq}3.\sqrt[3]{a^2b^2c^2}\overset{abc=1}{\geq}3$$

$$\Rightarrow\left(\sum_{cyc}a-3\right),\left(2\sum_{cyc}ab+4\sum_{cyc}a-18\right),\left(3abc+5\sum_{cyc}ab+3\sum_{cyc}a-27\right)\geq 0$$

$$\Rightarrow ①\ \text{is true}\therefore\sum_{cyc}\frac{1}{\sqrt{a+\lambda^2-1}}\leq\frac{3}{\lambda}\ \forall\ abc=1\ \text{and}\ \lambda\geq 2,$$

$$''='' \text{ iff } a=b=c=1\ (\text{QED})$$

1913. If $a,b>0$ *and* $a+b=2,\lambda\geq\frac{1}{2}$ **then:**

$$ab+\frac{1}{a^2+b^2+\lambda}-\frac{1}{a^3+b^3+\lambda}+\frac{1}{a^4+b^4+\lambda}\leq\frac{\lambda+3}{\lambda+2}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$ab\overset{AM-GM}{\leq}\left(\frac{a+b}{2}\right)^2\overset{a+b=2}{=}1\ (1)$$

$$a^2+b^2=\frac{a^2}{1}+\frac{b^2}{1}\overset{Radon}{\geq}\frac{(a+b)^2}{2}\overset{a+b=2}{=}2\ \ (2)$$

$$\frac{a^4}{1^3}+\frac{b^4}{1^3}\overset{Radon}{\geq}\frac{(a+b)^4}{8}\overset{a+b=2}{=}2\ (3)$$

$$a^3+b^3=(a+b)^3-3ab(a+b)\overset{a+b=2}{=}8-6ab\ (4)$$

We need to show:

$$ab+\frac{1}{a^2+b^2+\lambda}-\frac{1}{a^3+b^3+\lambda}+\frac{1}{a^4+b^4+\lambda}$$

$$\overset{(2),(3)\&(4)}{\leq} ab+\frac{1}{2+\lambda}-\frac{1}{8+\lambda-6ab}+\frac{1}{\lambda+2}=\frac{8ab+\lambda ab-6ab-1}{8+\lambda-6ab}+\frac{2}{\lambda+2}=$$

$$=\frac{2ab+\lambda ab-1}{8+\lambda-6ab}+\frac{2}{\lambda+2}\overset{(1)}{\leq}\frac{2+\lambda-1}{8+\lambda-6}+\frac{2}{\lambda+2}=$$

$$=\frac{1+\lambda}{2+\lambda}+\frac{2}{\lambda+2}=\frac{\lambda+3}{\lambda+2}$$

Equality holds for a=b=1.

1914. If $x,y,z>0, xyz=1$ and $\lambda\geq 0$ then :

$$\sum_{cyc}\frac{x^2}{x+\lambda y+y^3z}\geq\frac{3}{\lambda+2}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{x^2}{x+\lambda y+y^3z}=\sum_{cyc}\frac{x\left(x+\lambda y+y^3z-(\lambda y+y^3z)\right)}{x+\lambda y+y^3z}=$$

$$\sum_{cyc}x-\sum_{cyc}\frac{xy(\lambda+y^2z)}{\lambda y+x+y^3z}\overset{AM-GM}{\geq}\sum_{cyc}x-\sum_{cyc}\frac{xy(\lambda+y^2z)}{\lambda y+2.\sqrt{xy^3z}}\overset{xyz=1}{=}$$

$$\sum_{cyc}x-\frac{xy(\lambda+y^2z)}{\lambda y+2y}=\sum_{cyc}x-\frac{1}{\lambda+2}.\left(\lambda\sum_{cyc}x+xyz\sum_{cyc}x\right)$$

$$\overset{xyz=1}{=}\sum_{cyc}x-\frac{1}{\lambda+2}.\left(\lambda\sum_{cyc}x+\sum_{cyc}x\right)=\frac{\lambda\sum_{cyc}x+2\sum_{cyc}x-\lambda\sum_{cyc}x-\sum_{cyc}x}{\lambda+2}$$

$$=\frac{\sum_{cyc}x}{\lambda+2}\overset{AM-GM}{\geq}\frac{3.\sqrt[3]{xyz}}{\lambda+2}\overset{xyz=1}{=}\frac{3}{\lambda+2}\therefore\sum_{cyc}\frac{x^2}{x+\lambda y+y^3z}\geq\frac{3}{\lambda+2}$$

$$\forall x,y,z>0\,\big|\,xyz=1 \text{ and } \lambda\geq 0,'' =''\text{ iff } x=y=z=1\ (\text{QED})$$

1915. If $a,b,c>0, a+b+c=\frac{3}{2}$ and $\lambda\geq 0$ then :

$$\sum_{cyc}\frac{1+\lambda b}{1+4a^2}\geq\frac{3}{4}(\lambda+2)$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{1+4a^2}\overset{?}{\geq}1-a\Leftrightarrow a(2a-1)^2\overset{?}{\geq}0\rightarrow \text{true}$$

$$\therefore\frac{1+\lambda b}{1+4a^2}\geq(1+\lambda b)(1-a)\ \text{and analogs}$$

$$\therefore\sum_{cyc}\frac{1+\lambda b}{1+4a^2}\geq\sum_{cyc}(1-a+\lambda b-\lambda ab)\overset{a+b+c=\frac{3}{2}}{=}3-\frac{3}{2}+\lambda.\frac{3}{2}-\lambda\sum_{cyc}ab\geq$$

$$\frac{3}{2}+\lambda.\frac{3}{2}-\frac{\lambda}{3}\left(\sum_{cyc}a\right)^2\ (\because\lambda\geq 0)=\frac{3}{2}+\lambda.\frac{3}{2}-\frac{\lambda}{3}.\frac{9}{4}=\frac{3}{4}(\lambda+2)$$

$$\therefore\sum_{cyc}\frac{1+\lambda b}{1+4a^2}\geq\frac{3}{4}(\lambda+2)\ \forall\ a,b,c>0\ \Big|\ a+b+c=\frac{3}{2}\ \text{and}\ \lambda\geq 0,$$

$$''='' \text{ iff } a=b=c=\frac{1}{2}\ (\text{QED})$$

1916. If $x,y,z>0$ *and* $xy+yz+zx+2xyz=1$, $\lambda\geq\frac{1}{2}$ then:

$$\sum\sqrt{\lambda-x^2}\leq\frac{3\sqrt{4\lambda-1}}{2}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$xy+yz+zx+2xyz=1$$

$$xy+yz+zx+2\left(\sqrt[3]{xyz}\right)^3=1$$

$$\frac{(x+y+z)^2}{3}+2\left(\frac{(x+y+z)}{3}\right)^3\overset{AM-GM}{\geq}1$$

$$\frac{p^2}{3}+\frac{2p^3}{27}\overset{p=x+y+z}{\geq}1\ or\ 2p^3+9p^2-27\geq 0$$

$$(2p-3)(p+3)^2\geq 0\ or\ 2p-3\geq 0\ (since\ (p+3)^2>0)$$

$$or\ p\geq\frac{3}{2}\ or\ x+y+z\geq\frac{3}{2}\ (1)$$

$$\sum\sqrt{\lambda-x^2}\overset{CBS}{\leq}\sqrt{3\left(3\lambda-(x^2+y^2+z^2)\right)}\overset{CBS}{\leq}\sqrt{3\left(3\lambda-\frac{(x+y+z)^2}{3}\right)}\overset{(1)}{\leq}$$

$$\leq\sqrt{3\left(3\lambda-\frac{\left(\frac{3}{2}\right)^2}{3}\right)}=\sqrt{3\left(3\lambda-\frac{1}{3}.\frac{9}{4}\right)}=\sqrt{\frac{9}{4}(4\lambda-1)}=\frac{3\sqrt{4\lambda-1}}{2}$$

$$Equality\ holds\ for\ x=y=z=\frac{1}{2}.$$

1917. If $a,b,c>0, a+b+c+2=abc$ and $\lambda\geq 1$, then :

$$\sum_{cyc}\frac{1}{\sqrt{a+\lambda}}\leq\frac{3}{\sqrt{\lambda+2}}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{1}{\sqrt{a+\lambda}}\overset{CBS}{\leq}\sqrt{3\sum_{cyc}\frac{1}{a+\lambda}}=\sqrt{3.\frac{\sum_{cyc}ab+2\lambda\sum_{cyc}a+3\lambda^2}{\lambda^3+abc+\lambda^2\sum_{cyc}a+\lambda\sum_{cyc}ab}}$$

$$\overset{?}{\leq}\frac{3}{\sqrt{\lambda+2}}\overset{a+b+c+2=abc}{\Leftrightarrow}3\left(\sum_{cyc}a+2\right)+3\lambda^3+3\lambda^2.\sum_{cyc}a+3\lambda.\sum_{cyc}ab\overset{?}{\geq}$$

$$(\lambda+2).\sum_{cyc}ab+(2\lambda^2+4\lambda).\sum_{cyc}a+6\lambda^2$$

$$\Leftrightarrow(\lambda^2-4\lambda+3)\left(\sum_{cyc}a\right)+2(\lambda-1)\left(\sum_{cyc}ab\right)\overset{?}{\underset{①}{\geq}}6(\lambda^2-1)$$

$$\text{Let}\sum_{cyc}a=m\ and\ \because\lambda-1\geq 0\ \therefore LHS\ of\ ①\geq$$

$$(\lambda^2-4\lambda+3)m+2(\lambda-1).\sqrt{3abc\sum_{cyc}a}$$

$$\overset{a+b+c+2=abc}{=}(\lambda^2-4\lambda+3)m+2(\lambda-1).\sqrt{3m(m+2)}\overset{?}{\geq}6(\lambda^2-1)$$

$$\Leftrightarrow(m-6)\lambda^2-\left(4m-2.\sqrt{3m(m+2)}\right)\lambda+3m+6-2.\sqrt{3m(m+2)}\overset{?}{\underset{②}{\geq}}0$$

$$Now,\ abc=\sum_{cyc}a+2\overset{AM-GM}{\geq}3.\sqrt[3]{abc}+2\Rightarrow t^3-3t-2\geq 0\ \left(t=\sqrt[3]{abc}\right)\Rightarrow$$

$(t-2)(t+1)^2 \geq 0 \Rightarrow \sqrt[3]{abc} \geq 2 \Rightarrow \sum_{cyc} a + 2 \geq 8 \Rightarrow m \geq 6, "=" \text{ iff } a = b = c = 2$

and so, when $m = 6$, LHS of $② = 2.\sqrt{3m(m+2)}.(\lambda - 1) \overset{\lambda \geq 1}{\geq} 0 \Rightarrow ②$ is true for $m = 6$ and we now consider $m > 6$ and we have discrimin$a$nt δ of LHS of $② =$

$$\left(4m - 2.\sqrt{3m(m+2)}\right)^2 - 4(m-6)\left(3m + 6 - 2.\sqrt{3m(m+2)}\right)$$

$$= 4\left(4m^2 + 18m + 36 - 2(m+6).\sqrt{3m(m+2)}\right)$$

$$= \frac{8((2m^2+9m+18)^2 - 3m(m+2)(m+6)^2)}{2m^2+9m+18+(m+6).\sqrt{3m(m+2)}}$$

$$= \frac{8(m-6)^2(m+3)^2}{2m^2+9m+18+(m+6).\sqrt{3m(m+2)}} \overset{m>6}{>} 0$$

$\therefore \sqrt{\delta} \overset{(*)}{=} 2.\sqrt{4m^2+18m+36-2(m+6).\sqrt{3m(m+2)}}$ and we notice th$a$t, in order to prove $②$, it suffices to prove : $\lambda \overset{?}{\geq} \frac{4m - 2.\sqrt{3m(m+2)} + \sqrt{\delta}}{2(m-6)} \overset{via\ (*)}{\Leftrightarrow}$

$$\lambda(m-6) - 2m + \sqrt{3m(m+2)} \overset{?}{\geq} \sqrt{4m^2+18m+36-2(m+6).\sqrt{3m(m+2)}} \quad ③$$

Now, $\because m - 6 > 0$ and $\lambda \geq 1 \therefore$ LHS of $③ \geq m - 6 - 2m + \sqrt{3m(m+2)}$

$$= \sqrt{3m(m+2)} - (m+6) = \sqrt{\left(\sqrt{3m(m+2)} - (m+6)\right)^2}$$

$$\left(\because \sqrt{3m(m+2)} - (m+6) = \frac{2(m-6)(m+3)}{\sqrt{3m(m+2)} + m + 6} \overset{m>6}{>} 0\right)$$

$$= \sqrt{3m(m+2) + (m+6)^2 - 2(m+6).\sqrt{3m(m+2)}}$$

$$= \sqrt{4m^2+18m+36-2(m+6).\sqrt{3m(m+2)}} \Rightarrow ③ \text{ is true}$$

$\Rightarrow ②$ is true $\forall m \geq 6 \Rightarrow ①$ is true $\therefore \sum_{cyc} \frac{1}{\sqrt{a+\lambda}} \leq \frac{3}{\sqrt{\lambda+2}}$

$\forall a, b, c > 0 \mid a + b + c + 2 = abc \wedge \lambda \geq 1, "=" \text{ iff } a = b = c = 2$ (QED)

1918. If $a, b, c > 0, \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = 3$ and $\lambda \geq 0$ then :

$$\sum_{cyc} \frac{1}{a + \lambda\sqrt{bc} + 1} \leq \frac{3}{\lambda + 2}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } \sqrt{a}=x,\sqrt{b}=y,\sqrt{c}=z \text{ and then : } \sum_{cyc}\frac{1}{a+\lambda.\sqrt{bc}+1}\le\frac{3}{\lambda+2}\Leftrightarrow$$

$$\lambda^3\left(3x^2y^2z^2-xyz\sum_{cyc}x\right)+\lambda^2\left(\begin{matrix}3xyz\sum_{cyc}x^3-\sum_{cyc}x^3y-\sum_{cyc}xy^3+\\ xyz\sum_{cyc}x-2\sum_{cyc}xy\end{matrix}\right)+$$

$$\lambda\left(3\sum_{cyc}x^3y^3+\sum_{cyc}x^3y+\sum_{cyc}xy^3-\sum_{cyc}x^2y^2-2\sum_{cyc}x^2-\sum_{cyc}xy-3\right)+3x^2y^2z^2$$

$$-3+\sum_{cyc}x^2y^2-\sum_{cyc}x^2\overset{(*)}{\boxed{\ge}}0 \text{ and, } 3=\sum_{cyc}\frac{1}{x^4}\overset{\text{Radon}}{\ge}\frac{3^5}{\left(\sum_{cyc}x\right)^4}\Rightarrow\sum_{cyc}x\ge 3\to(1)$$

$$\text{and also, } 3=\sum_{cyc}\frac{1}{x^4}\overset{AM-GM}{\ge}\frac{3}{\sqrt[3]{x^4y^4z^4}}\Rightarrow xyz\ge 1\to(2)\ \&\ \text{now, } 3xyz\overset{?}{\ge}\sum_{cyc}x\Leftrightarrow$$

$$9x^2y^2z^2\overset{?}{\ge}\left(\sum_{cyc}x\right)^2.\frac{3x^4y^4z^4}{\sum_{cyc}x^4y^4}\left(\because 3=\sum_{cyc}\frac{1}{x^4}\right)\Leftrightarrow 3\sum_{cyc}x^4y^4\overset{?}{\ge}x^2y^2z^2\left(\sum_{cyc}x\right)^2$$

$$\to\text{true }\because 3\sum_{cyc}x^4y^4\ge 3x^2y^2z^2\sum_{cyc}x^2\ge x^2y^2z^2\left(\sum_{cyc}x\right)^2\ \therefore 3xyz\overset{(\cdot)}{\boxed{\ge}}\sum_{cyc}x$$

$$\Rightarrow xyz\sum_{cyc}x\ge\frac{1}{3}\left(\sum_{cyc}x\right)^2\ge\sum_{cyc}xy\therefore xyz\sum_{cyc}x\overset{(\cdot\cdot)}{\boxed{\ge}}\sum_{cyc}xy\ \text{and}\left(\sum_{cyc}x^2y^2\right)^2\overset{?}{\ge}$$

$$\left(\sum_{cyc}x^2\right)^2.\frac{3x^4y^4z^4}{\sum_{cyc}x^4y^4}\Leftrightarrow\left(\sum_{cyc}x^4y^4\right)\left(\sum_{cyc}x^2y^2\right)^2\overset{?}{\ge}3x^4y^4z^4\left(\sum_{cyc}x^2\right)^2\to\text{true}$$

$$\because\left(\sum_{cyc}x^4y^4\right)\left(\sum_{cyc}x^2y^2\right)^2\ge x^2y^2z^2\left(\sum_{cyc}x^2\right)\left(3x^2y^2z^2\left(\sum_{cyc}x^2\right)\right)$$

$$\therefore\sum_{cyc}x^2y^2\overset{(\cdot\cdot\cdot)}{\boxed{\ge}}\sum_{cyc}x^2\ \text{and}\ 3xyz\sum_{cyc}x^3-\sum_{cyc}x^3y-\sum_{cyc}xy^3+xyz\sum_{cyc}x-2\sum_{cyc}xy$$

$$\overset{\text{via (2) and (••)}}{\ge}2xyz\sum_{cyc}x^3+\sum_{cyc}x^3-\sum_{cyc}xy-\sum_{cyc}x^3y-\sum_{cyc}xy^3\overset{\text{via Chebyshev and (1)}}{\ge}$$

$$2xyz\sum_{cyc}x^3+\sum_{cyc}x^2-\sum_{cyc}xy-\sum_{cyc}x^3y-\sum_{cyc}xy^3\ge 2xyz\sum_{cyc}x^3-\sum_{cyc}x^3y$$

$$-\sum_{cyc} xy^3 \overset{?}{\geq} 0 \Leftrightarrow 4x^2y^2z^2\left(\sum_{cyc} x^3\right)^2 \overset{?}{\geq} \left(\sum_{cyc} x^3y + \sum_{cyc} xy^3\right)^2 \cdot \frac{3x^4y^4z^4}{\sum_{cyc} x^4y^4}$$

$$\Leftrightarrow 4\left(\sum_{cyc} x^4y^4\right)\left(\sum_{cyc} x^3\right)^2 \overset{?}{\geq} 3x^2y^2z^2\left(\sum_{cyc} x^3y + \sum_{cyc} xy^3\right)^2 \rightarrow \text{true} \because \text{LHS} \geq$$

$$\frac{4}{3}x^2y^2z^2\left(\sum_{cyc} x\right)^2\left(\sum_{cyc} x^3\right)^2 \overset{?}{\geq} \text{RHS} \Leftrightarrow 2\sum_{cyc} x^4 \overset{?}{\geq} \sum_{cyc} x^3y + \sum_{cyc} xy^3 \rightarrow \text{true } via$$

$$Schur + \text{A} - \text{G} \therefore 3xyz\sum_{cyc} x^3 - \sum_{cyc} x^3y - \sum_{cyc} xy^3 + xyz\sum_{cyc} x - 2\sum_{cyc} xy \geq 0 \rightarrow \text{(i)}$$

$$and\ again, 3\sum_{cyc} x^3y^3 + \left(\sum_{cyc} x^3y + \sum_{cyc} xy^3 - 2\sum_{cyc} x^2\right) - \sum_{cyc} x^2y^2 - \sum_{cyc} xy - 3$$

$$\overset{\text{AM-GM \& via (•••)}}{\geq} 3\sum_{cyc} x^3y^3 - \sum_{cyc} x^2y^2 - \sum_{cyc} xy - 3 \overset{\text{Chebyshev and AM-GM}}{\geq}$$

$$\left(\frac{1}{3}.3\sqrt[3]{x^2y^2z^2}.\sum_{cyc} x^2y^2 - \sum_{cyc} x^2y^2\right) + \left(\frac{1}{3}.3\sqrt[3]{x^4y^4z^4}.\sum_{cyc} xy - \sum_{cyc} xy\right) +$$

$$3x^2y^2z^2 - 3 \overset{via\ (2)}{\geq} 0 \therefore 3\sum_{cyc} x^3y^3 + \sum_{cyc} x^3y + \sum_{cyc} xy^3 - \sum_{cyc} x^2y^2 - 2\sum_{cyc} x^2$$

$$-\sum_{cyc} xy - 3 \rightarrow \text{(ii)}\ and\ 3x^2y^2z^2 - 3 + \sum_{cyc} x^2y^2 - \sum_{cyc} x^2 \overset{via\ (2)\ and\ (•••)}{\geq} 0 \rightarrow \text{(iii)}$$

$$\therefore (\bullet)\ and\ (\text{i}) + (\text{ii}) + (\text{iii}) \Rightarrow (*)\ \text{is true}\ (\because \lambda \geq 0) \therefore \sum_{cyc} \frac{1}{a + \lambda.\sqrt{bc} + 1} \leq \frac{3}{\lambda + 2}$$

$$\forall\ a, b, c > 0 \mid \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = 3\ and\ \lambda \geq 0, '' = ''\ \text{iff}\ a = b = c = 1\ (\text{QED})$$

1919. If $x, y > 0, x + y = 2$ then:

$$\sqrt{10x^2 + 4xy + 2y^2} + \sqrt{10y^2 + 4xy + 2x^2} \geq 8$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

Lemma:

$$\forall\ a, b > 0, \sqrt{10a^2 + 4ab + 2b^2} \geq 3a + b$$

$\boldsymbol{Proof}$:

$$\boldsymbol{We\ need\ to\ show}:$$
$$\sqrt{10a^2+4ab+2b^2}\geq 3a+b$$
$$10a^2+4ab+2b^2 \overset{squaring}{\geq} (3a+b)^2$$
$$10a^2+4ab+2b^2\geq 9a^2+6ab+b^2$$
$$a^2-2ab+b^2\geq 0\ or, (a-b)^2\geq 0\ true$$

$$\sqrt{10x^2+4xy+2y^2}+\sqrt{10y^2+4xy+2x^2}\overset{Lemma}{\geq}$$
$$\geq (3x+y)+(3y+x)=4(x+y)\overset{x+y=2}{=}8$$
Equality holds for x=y=2.

1920. If $x,y\in(0,1)$ then prove that :

$$\sqrt{x}+\sqrt{y}+\sqrt[4]{12}\cdot\sqrt{y\sqrt{1-x^2}+x\sqrt{1-y^2}}\leq 4\sqrt[4]{\frac{2}{3}}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sqrt{x}+\sqrt{y}+\sqrt[4]{12}.\sqrt{y.\sqrt{1-x^2}+x.\sqrt{1-y^2}}\overset{CBS}{\leq}$$
$$\sqrt{2}.\sqrt{\sqrt{2(x^2+y^2)}}+\sqrt[4]{12}.\sqrt{\sqrt{x^2+y^2}.\sqrt{2-(x^2+y^2)}}$$
$$=\sqrt[4]{8(x^2+y^2)}+\sqrt[4]{24(x^2+y^2)-12(x^2+y^2)^2}$$
$$=t+\sqrt[4]{24.\frac{t^4}{8}-12.\left(\frac{t^4}{8}\right)^2}\ \left(t=\sqrt[4]{8(x^2+y^2)}\right)=t+\frac{t}{2}.\sqrt[4]{48-3t^4}$$
$$\therefore LHS\overset{①}{\leq} t+\frac{t}{2}.\sqrt[4]{48-3t^4}\ and\ since\ x,y\in(0,1)\ \therefore t=\sqrt[4]{8(x^2+y^2)}\in(0,2)$$
$$Let\ f(t)=t+\frac{t}{2}.\sqrt[4]{48-3t^4}\ \forall\ t\in(0,2);\ then: f'(t)=\frac{(48-3t^4)^{\frac{3}{4}}-3(t^4-8)}{(48-3t^4)^{\frac{3}{4}}}>0$$
$$(trivially)\ if\ t^4\leq 8 nd\ \forall\ t^4\in(8,16), f'(t)=$$
$$\frac{(48-3t^4)^3-81(t^4-8)^4}{(48-3t^4)^{\frac{3}{4}}.\left((48-3t^4)^{\frac{3}{4}}+3(t^4-8)\right).\left((48-3t^4)^{\frac{3}{2}}+9(t^4-8)^2\right)}$$

$$=\frac{27(32-3m)\left((m-5)(m-8)^2+64\right)}{(48-3t^4)^{\frac{3}{4}}.\left((48-3t^4)^{\frac{3}{4}}+3(t^4-8)\right).\left((48-3t^4)^{\frac{3}{2}}+9(t^4-8)^2\right)}\overset{(*)}{=}f'(t)$$

$$(m=t^4)\ and\ \because m=t^4>8\therefore f'(t)=0\Rightarrow 32=3m=3t^4\Rightarrow t=\sqrt[4]{\frac{32}{3}}\ and\ also,$$

$$(*)\Rightarrow\forall\, m=t^4\in\left(0,\frac{32}{3}\right],f'(t)\geq 0\ and\ \forall\, m=t^4\in\left(\frac{32}{3},16\right),f'(t)<0$$

$$\therefore f(t)\ is\uparrow on\left(0,\sqrt[4]{\frac{32}{3}}\right]\ and\ f(t)\ is\downarrow on\left(\sqrt[4]{\frac{32}{3}},2\right)\ with\ f'\left(\sqrt[4]{\frac{32}{3}}\right)=0$$

$$\therefore f(t)_{max}=f\left(\sqrt[4]{\frac{32}{3}}\right)=4.\sqrt[4]{\frac{2}{3}}\overset{via\ (1)}{\Rightarrow}\sqrt{x}+\sqrt{y}+\sqrt[4]{12}.\sqrt{y.\sqrt{1-x^2}+x.\sqrt{1-y^2}}$$

$$\leq 4.\sqrt[4]{\frac{2}{3}},''='' \text{ iff } x=y=\sqrt{\frac{2}{3}}\ (QED)$$

1921. If $x,y,z>0$ *and* $x^2+y^2+z^2\leq 12$ then:

$$\sqrt{3x^2+12xy+y^2}+\sqrt{3y^2+12yz+z^2}+\sqrt{3z^2+12zx+x^2}\leq 24$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\sqrt{3x^2+12xy+y^2}+\sqrt{3y^2+12yz+z^2}+\sqrt{3z^2+12zx+x^2}\leq$$

$$\overset{CBS}{\leq}\sqrt{3.\sum_{cyc}(3x^2+12xy+y^2)}=\sqrt{3\left(3\sum x^2+12\sum xy+\sum y^2\right)}\leq$$

$$\leq\sqrt{3\left(3\sum x^2+12\sum x^2+\sum y^2\right)}\overset{x^2+y^2+z^2\leq 12}{\leq}$$

$$\leq\sqrt{3(3\times 12+12\times 12+12)}=\sqrt{576}=24$$

Equality holds for x=y=z=2.

1922. *If* $a,b,c>0$ *then prove:*

$$\sum_{cyc}\sqrt{\frac{a^4+b^4}{a}}\left(\sqrt{\frac{b+c}{b}}+\sqrt{\frac{c+a}{c}}\right)\geq 12\sqrt{abc}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Mirsadix Muzefferov-Azerbaijan

$$\frac{a^4+b^4}{a}\geq\frac{\frac{(a^2+b^2)^2}{2}}{a}=\frac{(a^2+b^2)^2}{2a}\geq\frac{4a^2b^2}{2a}\rightarrow\sqrt{\frac{a^4+b^4}{a}}\geq\frac{2ab}{\sqrt{2a}}$$

$$Analogously:\ \sqrt{\frac{c^4+b^4}{b}}\geq\frac{2bc}{\sqrt{2b}},\quad\sqrt{\frac{a^4+c^4}{c}}\geq\frac{2ca}{\sqrt{2c}}$$

$$LHS\geq\frac{2ab}{\sqrt{2a}}\left(\sqrt{\frac{b+c}{b}}+\sqrt{\frac{c+a}{c}}\right)+\frac{2bc}{\sqrt{2b}}\left(\sqrt{\frac{a+c}{c}}+\sqrt{\frac{b+a}{b}}\right)+$$

$$+\frac{2ca}{\sqrt{2c}}\left(\sqrt{\frac{b+a}{a}}+\sqrt{\frac{b+a}{b}}\right)\overset{A-G}{\geq}\frac{2ab}{\sqrt{2a}}.2\left(\sqrt{\frac{b+c}{b}}.\frac{c+a}{c}\right)^{\frac{1}{2}}+$$

$$+\frac{2bc}{\sqrt{2b}}.2\left(\sqrt{\frac{a+c}{c}}.\frac{b+a}{a}\right)^{\frac{1}{2}}+\frac{2ca}{\sqrt{2c}}.2\left(\sqrt{\frac{b+a}{a}}.\frac{c+b}{b}\right)^{\frac{1}{2}}\overset{A-G}{\geq}$$

$$\geq\frac{4ab}{\sqrt{2a}}.\left(\sqrt{\frac{4\sqrt{bc}}{b}}.\frac{\sqrt{ca}}{c}\right)^{\frac{1}{2}}+\frac{4ac}{\sqrt{2c}}.\left(\sqrt{\frac{4\sqrt{ba}}{a}}.\frac{\sqrt{cb}}{b}\right)^{\frac{1}{2}}+\frac{4bc}{\sqrt{2b}}.\left(\sqrt{\frac{4\sqrt{ca}}{c}}.\frac{\sqrt{ab}}{a}\right)^{\frac{1}{2}}\overset{A-G}{\geq}$$

$$\geq 3\left(\frac{4ab}{\sqrt{2a}}.\frac{4ac}{\sqrt{2c}}.\frac{4bc}{\sqrt{2b}}.\left(\sqrt{\frac{64a^2b^2c^2}{a^2b^2c^2}}\right)^{\frac{1}{2}}\right)^{\frac{1}{3}}=12\sqrt{abc}$$

$$Equality\ holds\ for:\ a=b=c$$

1923. If $a,b,c>0$ then:

$$\sum\frac{1}{a^2+b^2}\left(b^4\sqrt{b^2+c^2}+c^4\sqrt{c^2+a^2}\right)\geq\frac{9\sqrt{2}(abc)^{\frac{5}{3}}}{a^2+b^2+c^2}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

$$(a^2+b^2)(b^2+c^2)(c^2+a^2)\overset{Cesaro}{\geq}8a^2b^2c^2\ (1)$$

$$\sqrt[3]{(a^2+b^2)(b^2+c^2)(c^2+a^2)}\overset{AM-GM}{\leq}\frac{2(a^2+b^2+c^2)}{3}\ (2)$$

$$b^4\sqrt{b^2+c^2}+c^4\sqrt{c^2+a^2}\overset{AM-GM}{\geq}2\sqrt{b^4c^4}.\sqrt[4]{(b^2+c^2)(c^2+a^2)}=$$

$$=2b^2c^2\sqrt[4]{(b^2+c^2)(c^2+a^2)}(3)$$

$$\sum\frac{1}{a^2+b^2}\left(b^4\sqrt{b^2+c^2}+c^4\sqrt{c^2+a^2}\right)\overset{(2),(3)}{\geq}\sum\frac{2b^2c^2.\sqrt[4]{(b^2+c^2)(c^2+a^2)}}{a^2+b^2}\overset{AM-GM}{\geq}$$

$$\geq 6\sqrt[3]{\frac{a^4b^4c^4\sqrt{(a^2+b^2)(b^2+c^2)(c^2+a^2)}}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}}\overset{(1)}{\geq}$$

$$\geq 6\sqrt[3]{\frac{a^4b^4c^4 2^{\frac{3}{2}}abc}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}}\overset{(2)}{\geq}\frac{6\sqrt{2}(abc)^{\frac{5}{3}}}{\frac{2(a^2+b^2+c^2)}{3}}=\frac{9\sqrt{2}(abc)^{\frac{5}{3}}}{a^2+b^2+c^2}$$

Equality holds for $a=b=c$.

1924. For $x,y,z>0, k\geq\sqrt{2}$ prove that :

$$\left(k+1-k.\frac{x}{y}\right)^2+\left(k+1-k.\frac{y}{z}\right)^2+\left(k+1-k.\frac{z}{x}\right)^2\geq 3$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\left(k+1-k.\frac{x}{y}\right)^2+\left(k+1-k.\frac{y}{z}\right)^2+\left(k+1-k.\frac{z}{x}\right)^2\geq 3$$

$$\Leftrightarrow 3k^2+6k+k^2\sum_{cyc}\frac{x^2}{y^2}-2k(k+1)\sum_{cyc}\frac{x}{y}\geq 0$$

$$\Leftrightarrow 3k+6+k\sum_{cyc}a^6-2(k+1)\sum_{cyc}a^3\geq 0\left(\sqrt[3]{\frac{x}{y}}=a,\sqrt[3]{\frac{y}{z}}=b,\sqrt[3]{\frac{z}{x}}=c\right)$$

$$\Leftrightarrow (3k+6)a^2b^2c^2+k\sum_{cyc}a^6-2(k+1)abc\sum_{cyc}a^3\geq 0\ (\because abc=1)$$

$$\Leftrightarrow k\left(\sum_{cyc}a^6+3a^2b^2c^2-2abc\sum_{cyc}a^3\right)-2abc\left(\sum_{cyc}a^3-3abc\right)\geq 0$$

$$\Leftrightarrow k\left(\begin{array}{l}6a^2b^2c^2+\left(\sum_{cyc}a^2\right)\left(\left(\sum_{cyc}a^2\right)^2-3\sum_{cyc}a^2b^2\right)-\\ 2abc\left(3abc+\left(\sum_{cyc}a\right)\left(\left(\sum_{cyc}a\right)^2-3\sum_{cyc}ab\right)\right)\end{array}\right)$$

$$-2abc\left(\left(\sum_{cyc}a\right)\left(\left(\sum_{cyc}a\right)^2-3\sum_{cyc}ab\right)\right)\geq 0\Leftrightarrow$$

$$k\left((p^2-2q)^3-3(p^2-2q)(q^2-2rp)-2r(p^3-3pq)\right)-2r(p^3-3pq)\geq 0$$

$$\left(p=\sum_{cyc}a,q=\sum_{cyc}ab,r=abc\right)\Leftrightarrow$$

$$k\left((p^2-2q)^3-3q^2(p^2-2q)\right)+r(k(4p^3-6pq)-2p^3+6pq)\geq 0$$

$$\Leftrightarrow Ar^2+r.G(p,q)+H(p,q)\overset{(*)}{\geq}0$$

$$\left(\begin{array}{c}A=0,G(p,q)=k(4p^3-6pq)+2p^3-6pq,\\ H(p,q)=k\left((p^2-2q)^3-3q^2(p^2-2q)\right)\end{array}\right)$$

and $\because F(a,b,c)=Ar^2+r.G(p,q)+H(p,q)\geq 0$ for $A\leq 0\leq a,b,c$ holds iff $F(a,1,1)\geq 0$ and $F(0,b,c)\geq 0\therefore (*)$ is true iff $F(a,1,1)\geq 0$ and $F(0,b,c)\geq 0$;

where $F(a,b,c)=k\left(\sum_{cyc}a^6+3a^2b^2c^2-2abc\sum_{cyc}a^3\right)-2abc\left(\sum_{cyc}a^3-3abc\right)$

Now, $F(0,b,c)=k(b^6+c^6)>0$ and $F(a,1,1)=$

$$k\left(a^6+2+3a^2-2a(a^3+2)\right)-2a(a^3+2-3a)$$

$$=k(a-1)^2.(a^4+2a^3+a^2+2)-2a(a-1)^2.(a+2)\overset{?}{\geq}0$$

$$\Leftrightarrow k^2(a^4+2a^3+a^2+2)^2\overset{?}{\geq}4a^2(a+2)^2\ (\because (a-1)^2\geq 0)\text{ and }\because k^2\geq 2$$

$$\therefore \text{it suffices to prove}: 2(a^4+2a^3+a^2+2)^2\overset{?}{\geq}4a^2(a+2)^2$$

$$\Leftrightarrow 2\left(a^8+4a^7+6a^6+4a^5+2a^4+(a^2-2)^2\right)\overset{?}{\geq}0\to\text{true (strict)}\because a>0$$

and so, $F(a,1,1)\geq 0$ and $F(0,b,c)>0\Rightarrow F(a,b,c)\geq 0\Rightarrow (*)$ is true

$$\therefore\left(k+1-k.\frac{x}{y}\right)^2+\left(k+1-k.\frac{y}{z}\right)^2+\left(k+1-k.\frac{z}{x}\right)^2\geq 3\ \forall x,y,z>0\text{ and }k\geq\sqrt{2},$$

$''=''$ iff $x=y=z$ (QED)

1925. If $a,b,c>0$ *then prove that* :

$$\sum_{cyc} a^2 \geq \sum_{cyc} ab + \frac{1}{8}\sum_{cyc}(|a-c|+|b-c|)^2$$

Proposed by Neculai Stanciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$2\left(\sum_{cyc} a^2 - \sum_{cyc} ab\right) = \sum_{cyc}(a-b)^2 = \sum_{cyc}\left(\frac{|b-c|^2+|c-a|^2}{2}\right) \geq$$
$$\sum_{cyc}\frac{(|b-c|+|c-a|)^2}{4} \Rightarrow \sum_{cyc} a^2 - \sum_{cyc} ab \geq \frac{1}{8}\sum_{cyc}(|a-c|+|b-c|)^2$$
$$\Rightarrow \sum_{cyc} a^2 \geq \sum_{cyc} ab + \frac{1}{8}\sum_{cyc}(|a-c|+|b-c|)^2, '' = '' \text{ iff } a=b=c \text{ (QED)}$$

1926. If $a,b,c>0, abc=1, m\geq 0$ **then:**

$$\sum \frac{a^3}{\sqrt{a^2+2ma+bc}} \geq \frac{3}{\sqrt{2(m+1)}}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$a+b+c \overset{AM-GM}{\geq} 3\sqrt[3]{abc} \overset{abc=1}{=} 3 \ (1)$$

$$ab+bc+ca \overset{AM-GM}{\geq} 3\sqrt[3]{a^2b^2c^2} \overset{abc=1}{=} 3 \ (2)$$

$$\sum \frac{a^3}{\sqrt{a^2+2ma+bc}} \overset{Holder}{\geq} \frac{1}{3}\frac{(a+b+c)^3}{\sum\sqrt{a^2+2ma+bc}} \overset{CBS}{\geq}$$
$$\geq \frac{(a+b+c)^3}{3\sqrt{3\left((a^2+b^2+c^2)+2m(a+b+c)+(ab+bc+ca)\right)}} \overset{a+b+c=x \ \& \ (2)}{\geq}$$
$$\geq \frac{x^3}{3\sqrt{3x^2+6mx-9}}$$

We need to show:

$$\frac{x^3}{3\sqrt{3x^2+6mx-9}} \geq \frac{3}{\sqrt{2(m+1)}} \; or \; \frac{x^6}{9(3x^2+6mx-9)} \overset{squaring}{\geq} \frac{9}{2(m+1)}$$

$$x^6(2m+2) \geq 81(3x^2+6mx-9)$$

$$x^6(2m+2) - 243x^2 - 486mx + 729 \geq 0$$

$$(x-3)^2[2(m+1)x^2(x^2+6x+27) + 216(m+1)x + 810x + 567] +$$

$$+486(5m+3)(x-3) \geq 0 \; true \; by \; (1) as \; x = a+b+c \geq 3$$

Equality holds for a=b=c=1.

1927. If $a, b, c > 0$ *with* $a+b+c \geq 3$ then prove that :

$$\sum_{cyc} \frac{3a^2}{b+c+1} \geq \left(\sum_{cyc} a\right)\left(1 + 16\sum_{cyc} \frac{(a-b)^2}{(a+4b+4c)(4a+b+4c)}\right)$$

Proposed by Neculai Stanciu, Titu Zvonaru-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} \frac{3a^2}{b+c+1} - \sum_{cyc} a \overset{a+b+c\geq 3}{\geq} \sum_{cyc} \frac{3a^2}{b+c+\frac{a+b+c}{3}} - \sum_{cyc} a$$

$$= 4\sum_{cyc} \frac{2a^2-ab-ac}{a+4b+4c} \overset{?}{\geq} 16\left(\sum_{cyc} a\right)\sum_{cyc} \frac{(a-b)^2}{(a+4b+4c)(4a+b+4c)}$$

$$\Leftrightarrow \sum_{cyc} \frac{(2a^2-ab-ac)(b+4c+4a)(c+4a+4b)}{(a+4b+4c)(b+4c+4a)(c+4a+4b)} \overset{?}{\geq}$$

$$4\left(\sum_{cyc} a\right)\sum_{cyc} \frac{(a-b)^2(c+4a+4b)}{(a+4b+4c)(b+4c+4a)(c+4a+4b)} \Leftrightarrow$$

$$\sum_{cyc}\left((2a^2-ab-ac)\left(4\left(\sum_{cyc} a\right)^2 + 12a\left(\sum_{cyc} a\right) + 9bc\right)\right) \overset{?}{\geq}$$

$$4\left(\sum_{cyc} a\right)\left(4\left(\sum_{cyc} a\right)\sum_{cyc}(a-b)^2-3\sum_{cyc} c(a-b)^2\right)$$

$$\Leftrightarrow 4\left(\sum_{cyc} a\right)^2\cdot\sum_{cyc}(a-b)^2+12\left(\sum_{cyc} a\right)\left(2\sum_{cyc} a^3-\sum_{cyc} a^2b-\sum_{cyc} ab^2\right)+$$

$$9\left(2abc\sum_{cyc} a-abc\sum_{cyc} a-abc\sum_{cyc} a\right)\overset{?}{\geq}$$

$$4\left(\sum_{cyc} a\right)^2\cdot\sum_{cyc}(a-b)^2-12\left(\sum_{cyc} a\right)\left(\sum_{cyc} a^2b+\sum_{cyc} ab^2-6abc\right)$$

$$\Leftrightarrow 2\sum_{cyc} a^3-\sum_{cyc} a^2b-\sum_{cyc} ab^2\overset{?}{\geq}-\sum_{cyc} a^2b-\sum_{cyc} ab^2+6abc\Leftrightarrow\sum_{cyc} a^3\overset{?}{\geq}3abc$$

$$\rightarrow \text{true via } AM-GM\therefore\sum_{cyc}\frac{3a^2}{b+c+1}\geq$$

$$\left(\sum_{cyc} a\right)\left(1+16\sum_{cyc}\frac{(a-b)^2}{(a+4b+4c)(4a+b+4c)}\right)\ \forall a,b,c>0\,|\,a+b+c\geq 3,$$

$$''=''\ \text{iff } a=b=c=1\ (QED)$$

1928. If $a,b,c>0, a+b+c\geq 3$ then:

$$\frac{1}{a^3+b^2+c^2}+\frac{1}{b^3+c^2+a^2}+\frac{1}{c^3+a^2+b^2}\leq 1$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$(a^3+b^2+c^2)(a+b^2+c^2)\overset{Cauchy-Schwarz}{\geq}(a^2+b^2+c^2)^2$$

$$\frac{1}{a^3+b^2+c^2}\leq\frac{(a+b^2+c^2)}{(a^2+b^2+c^2)^2}\ (1)$$

$$a^2+b^2+c^2\geq\frac{1}{3}(a+b+c)^2\overset{a+b+c\geq 3}{\geq}=3\ (2)$$

$$\frac{1}{a^3+b^2+c^2}+\frac{1}{b^3+c^2+a^2}+\frac{1}{c^3+a^2+b^2}=\sum\frac{1}{a^3+b^2+c^2}\overset{(1)}{\leq}$$

$$\leq\sum\frac{(a+b^2+c^2)}{(a^2+b^2+c^2)^2}=\frac{a+b+c+2(a^2+b^2+c^2)}{(a^2+b^2+c^2)^2}=$$

$$=\frac{\sqrt{(a+b+c)^2}+2(a^2+b^2+c^2)}{(a^2+b^2+c^2)^2}\leq$$

$$\leq\frac{\sqrt{3(a^2+b^2+c^2)}+2(a^2+b^2+c^2)}{(a^2+b^2+c^2)^2}\overset{a^2+b^2+c^2=x^2}{=}\frac{\sqrt{3}x+2x^2}{x^4}$$

$$\boldsymbol{We\ need\ to\ show}:$$

$$\frac{\sqrt{3}x+2x^2}{x^4}\leq 1\ or\ x^3-2x-\sqrt{3}\geq 0\ or\ \left(x-\sqrt{3}\right)\left(x^2+\sqrt{3}x+1\right)\geq 0$$

$$true\ as\ x^2=a^2+b^2+c^2\geq 3\ by\ (2)$$

$$Equality\ holds\ for\ a=b=c=1.$$

1929. If $x,y,z>0,\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\leq 12$ then:

$$\frac{1}{\sqrt{8x^2+4xy+13y^2}}+\frac{1}{\sqrt{8y^2+4yz+13z^2}}+\frac{1}{\sqrt{8z^2+4xz+13x^2}}\leq\frac{6}{5}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$12\geq\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\overset{CBS}{\geq}\frac{1}{3}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^2\ or\ 36\geq\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^2\ or,\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\leq 6\ (1)$$

$\boldsymbol{Lemma}$:

$$\forall a,b>0\ ,\sqrt{8a^2+4ab+13b^2}\geq(2a+3b)$$

$\boldsymbol{Proof}$:

$$\boldsymbol{We\ need\ to\ show}:$$

$$\sqrt{8a^2+4ab+13b^2}\geq(2a+3b)$$

$$8a^2+4ab+13b^2\overset{squaring}{\geq}4a^2+12ab+9b^2$$

$$4(a^2-2ab+b^2)\geq 0\ or\ 4(a-b)^2\geq 0\ true$$

$$\frac{1}{\sqrt{8x^2+4xy+13y^2}}+\frac{1}{\sqrt{8y^2+4yz+13z^2}}+\frac{1}{\sqrt{8z^2+4xz+13x^2}}=$$

$$=\sum\frac{1}{\sqrt{8x^2+4xy+13y^2}}\overset{Lemma}{\leq}\sum\frac{1}{2x+3y}=\sum\frac{1}{x+x+y+y+y}\leq$$

$$\overset{AM-HM}{\leq}\frac{1}{25}\left(\sum\frac{1}{x}+\sum\frac{1}{x}+\sum\frac{1}{y}+\sum\frac{1}{y}+\sum\frac{1}{y}\right)\overset{(1)}{\leq}\frac{1}{25}(5\times 6)=\frac{6}{5}$$

$$Equality\ holds\ for\ x=y=z=\frac{1}{2}.$$

1930. If $x,y,z>0,\ xy+yz+zx\geq 3$ then:

$$\frac{x^2}{\sqrt{7x^2+28xy+14y^2}}+\frac{y^2}{\sqrt{7y^2+28yz+14z^2}}+\frac{z^2}{\sqrt{7z^2+28zx+14x^2}}\geq\frac{3}{7}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$Lemma$:

$$\forall a,b>0\ ,\sqrt{7a^2+28ab+14b^2}\leq 3a+4b$$

$Proof$:

$$We\ need\ to\ show:$$

$$\sqrt{7a^2+28ab+14b^2}\leq 3a+4b$$

$$7a^2+28ab+14b^2\overset{squaring}{\leq}9a^2+24ab+16b^2$$

$$2(a^2-2ab+b^2)\geq 0\ or\ 2(a-b)^2\geq 0\ true$$

$$\frac{x^2}{\sqrt{7x^2+28xy+14y^2}}+\frac{y^2}{\sqrt{7y^2+28yz+14z^2}}+\frac{z^2}{\sqrt{7z^2+28zx+14x^2}}=$$

$$=\sum\frac{x^2}{\sqrt{7x^2+28xy+14y^2}}\overset{lemma}{\geq}\sum\frac{x^2}{3x+4y}\overset{Bergstrom}{\geq}$$

$$\geq\frac{(x+y+z)^2}{3(x+y+z)+4(x+y+z)}=\frac{x+y+z}{7}=\frac{\sqrt{(x+y+z)^2}}{7}\geq$$

$$\geq\frac{\sqrt{3(xy+yz+zx)}}{7}\overset{xy+yz+zx\geq 3}{\geq}\frac{3}{7}$$

Equality holds for $x = y = z = 1$.

1931. If $a, b, c > 0$, *then prove that* :

$$\sum_{cyc} \frac{a}{\sqrt{4b^2 + bc + 4c^2}} \geq 1$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} \frac{a}{\sqrt{4b^2 + bc + 4c^2}} = \sum_{cyc} \frac{a^{\frac{3}{2}}}{\sqrt{4ab^2 + abc + 4c^2 a}} \overset{\text{Radon}}{\geq}$$

$$\frac{\left(\sum_{cyc} a\right)^{\frac{3}{2}}}{\sqrt{3abc + 4\sum_{cyc} a^2 b + 4\sum_{cyc} ab^2}} \overset{?}{\geq} 1 \Leftrightarrow \left(\sum_{cyc} a\right)^3 \overset{?}{\geq} 3abc + 4\sum_{cyc} a^2 b + 4\sum_{cyc} ab^2$$

$$\Leftrightarrow \sum_{cyc} a^3 + 3\left(2abc + \sum_{cyc} a^2 b + \sum_{cyc} ab^2\right) \overset{?}{\geq} 3abc + 4\sum_{cyc} a^2 b + 4\sum_{cyc} ab^2$$

$$\Leftrightarrow \sum_{cyc} a^3 + 3abc \overset{?}{\geq} \sum_{cyc} a^2 b + \sum_{cyc} ab^2 \rightarrow \text{true via Schur}$$

$$\therefore \sum_{cyc} \frac{a}{\sqrt{4b^2 + bc + 4c^2}} \geq 1 \; \forall \, a, b, c > 0, '' = '' \text{ iff } a = b = c \; (\text{QED})$$

1932. For $x, y, z, t \geq 0$ prove that :

$$(x^2 + t)(y^2 + t)(z^2 + t) \geq \frac{t}{27}\left(5t + 3(xy + yz + zx)\right)^2$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$(x^2 + t)(y^2 + t)(z^2 + t) \geq \frac{t}{27}\left(5t + 3(xy + yz + zx)\right)^2$$

$$\Leftrightarrow 27x^2y^2z^2 + 27t^2\sum_{cyc} x^2 + 27t\sum_{cyc} x^2y^2 + 27t^3 \geq$$

$$25t^3+30t^2\sum_{cyc}xy+9t\left(\sum_{cyc}xy\right)^2\Leftrightarrow 27x^2y^2z^2+2t^3+$$

$$27t^2\sum_{cyc}x^2-30t^2\sum_{cyc}xy+9t\left(3\sum_{cyc}x^2y^2-\left(\sum_{cyc}xy\right)^2\right)\overset{(*)}{\geq}0$$

$$\text{Now, } 27x^2y^2z^2+t^3+t^3\overset{AM-GM}{\geq}3.\sqrt[3]{27x^2y^2z^2t^6}\ (\because x,y,z,t\geq 0)$$

$$=9t^2.\sqrt[3]{x^2y^2z^2}\ \text{and}\ \because 9t\left(3\sum_{cyc}x^2y^2-\left(\sum_{cyc}xy\right)^2\right)\geq 0\ \therefore\ \text{in order to prove}\ (*),$$

$$\text{it sufices to prove}: 9t^2.\sqrt[3]{x^2y^2z^2}+27t^2\sum_{cyc}x^2-30t^2\sum_{cyc}xy\geq 0\Leftrightarrow$$

$$F(a,b,c)=9\left(\sum_{cyc}a^6\right)+3a^2b^2c^2-10\sum_{cyc}a^3b^3\ \boxed{\overset{(**)}{\geq}}\ 0\ \left(a=\sqrt[3]{x},b=\sqrt[3]{y},c=\sqrt[3]{z}\right)$$

$$\Leftrightarrow 9\left(3a^2b^2c^2+\left(\sum_{cyc}a^2\right)\left(\left(\sum_{cyc}a^2\right)^2-3\sum_{cyc}a^2b^2\right)\right)+3a^2b^2c^2-$$

$$10\left(\left(\sum_{cyc}ab\right)^3-3abc\left(\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)-abc\right)\right)\geq 0$$

$$\Leftrightarrow 9(p^2-2q)^3-27(p^2-2q)(q^2-2rp)-10q^3+30rpq\geq 0$$

$$\left(p=\sum_{cyc}a\,,q=\sum_{cyc}ab\,,r=abc\right)$$

$$\Leftrightarrow 9(p^2-2q)^3-27q^2(p^2-2q)-10q^3+r(54p^3-78pq)\geq 0$$

$$\Leftrightarrow Ar^2+r.G(p,q)+H(p,q)\geq 0$$

$$\left(\begin{matrix}\text{Here } A=0, G(p,q)=54p^3-78pq\ and\\ H(p,q)=9(p^2-2q)^3-27q^2(p^2-2q)-10q^3\end{matrix}\right)and\ \because F(a,b,c)=$$

$Ar^2+r.G(p,q)+H(p,q)\geq 0$ for $A\leq 0\leq a,b,c$ holds iff $F(a,1,1)\geq 0$ and $F(0,b,c)\geq 0\ \therefore (**)$ is true iff $F(a,1,1)\geq 0$ and $F(0,b,c)\geq 0\ \forall a,b,c\geq 0$;

and $F(0,b,c)=9(b^6+c^6)-10b^3c^3=9(b^3-c^3)^2+8b^3c^3\geq 0\ (\because b,c\geq 0)$

and $F(a,1,1)=9(a^6+2)+3a^2-10(2a^3+1)$

$=(a-1)^2(9a^4+18a^3+27a^2+16a+8)\geq 0\ (\because a\geq 0)$

$\therefore F(a,1,1)\geq 0$ and $F(0,b,c)\geq 0\ \forall a,b,c\geq 0\Rightarrow (**)\Rightarrow (*)$ is true

$$\therefore (x^2+t)(y^2+t)(z^2+t)\geq \frac{t}{27}\left(5t+3(xy+yz+zx)\right)^2\ \forall x,y,z,t\geq 0,$$

$$''='' \text{ iff } x=y=z=\sqrt{\frac{t}{3}}\ (\text{QED})$$

1933. For $x,y,z,k,t\geq 0$ and $t\geq k.\sqrt{2}$ prove that :

$$(x^2+t^2)(y^2+t^2)(z^2+t^2)\geq (k^2+t^2)(k(x+y+z)+t^2-2k^2)^2$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$(x^2+t^2)(y^2+t^2)(z^2+t^2)\geq (k^2+t^2)(k(x+y+z)+t^2-2k^2)^2$$

$$\Leftrightarrow (a^2+s)(b^2+s)(c^2+s)\geq (1+s)\left(\sum_{cyc}a+s-2\right)^2$$

$$\left(a=\frac{x}{k},b=\frac{y}{k},c=\frac{z}{k}\text{ and }s=\frac{t^2}{k^2}\geq 2\right)\Leftrightarrow (a^2+2+m)(b^2+2+m)(c^2+2+m)$$

$$\geq (3+m)\left(\sum_{cyc}a+m\right)^2\ (m=s-2\geq 0)\Leftrightarrow$$

$$m^2\left(\sum_{cyc}a^2-2\sum_{cyc}a+3\right)+m\left(\sum_{cyc}a^2b^2+3\sum_{cyc}a^2-2\sum_{cyc}ab-6\sum_{cyc}a+12\right)+$$

$$a^2b^2c^2+2\sum_{cyc}a^2b^2+\sum_{cyc}a^2-6\sum_{cyc}ab+8\overset{(*)}{\geq}0$$

$$\text{Now, } a^2b^2c^2+2\sum_{cyc}a^2b^2+\sum_{cyc}a^2-6\sum_{cyc}ab+8=$$

$$\sum_{cyc} a^2 + (1 + a^2b^2c^2) + 1 - 2\sum_{cyc} ab + 2\left(\sum_{cyc} a^2b^2 - 2\sum_{cyc} ab + 3\right) \overset{AM-GM}{\geq}$$

$$\left(\sum_{cyc} a^2 + 2abc + 1 - 2\sum_{cyc} ab\right) + 2\sum_{cyc}(ab-1)^2 \overset{Darij\ Grinberg}{\geq} 2\sum_{cyc}(ab-1)^2 \geq 0$$

$$\therefore a^2b^2c^2 + 2\sum_{cyc} a^2b^2 + \sum_{cyc} a^2 - 6\sum_{cyc} ab + 8 \overset{①}{\geq} 0 \therefore \text{LHS of } (*) \overset{via\ ①}{\geq}$$

$$m^2\sum_{cyc}(a-1)^2 + m\left(\sum_{cyc}(ab-1)^2 + 3\sum_{cyc}(a-1)^2\right) \overset{m\geq 0}{\geq} 0 \Rightarrow (*) \text{ is true and so,}$$

$$(x^2+t^2)(y^2+t^2)(z^2+t^2) \geq (k^2+t^2)(k(x+y+z)+t^2-2k^2)^2$$

$$\forall x, y, z, k, t \geq 0 \text{ and } t \geq k.\sqrt{2}, \text{"}=\text{"} \text{ iff } x = y = z = k \text{ (QED)}$$

1934. If $a, b, c > 0,\ a + b + c = 3abc$ then:

$$\frac{1}{1+a+2bc} + \frac{1}{1+b+2ac} + \frac{1}{1+c+2ab} \leq \frac{3}{4}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$a + b + c = 3abc \text{ or } 3\sqrt[3]{abc} \overset{AM-GM}{\leq} 3abc$$

$$(abc)^{\frac{2}{3}} \geq 1 \text{ or } abc \geq 1\ (1)$$

$$1 + a + 2bc = 1 + (a + bc) + bc \overset{AM-GM}{\geq} 1 + 2\sqrt{abc} + bc \overset{(1)}{\geq}$$
$$\geq 1 + 2.1 + b = 3 + bc\ (2)$$

$$\frac{1}{1+a+2bc} + \frac{1}{1+b+2ac} + \frac{1}{1+c+2ab} = \sum \frac{1}{1+a+2bc} \overset{(2)}{\leq}$$

$$\leq \sum \frac{1}{3+bc} = \sum \frac{1}{1+1+1+bc} \overset{AM-HM}{\leq}$$

$$\leq \frac{1}{16}\left(\sum 1 + \sum 1 + \sum 1 + \sum \frac{1}{bc}\right) = \frac{1}{16}\left(9 + \frac{a+b+c}{abc}\right) \overset{a+b+c=3abc}{=}$$

$$=\frac{1}{16}\left(9+\frac{3abc}{abc}\right)=\frac{1}{16}(9+3)=\frac{3}{4}$$

$Equality\ holds\ for\ a=b=c=1.$

1935. If $a,b,c>0, abc=1$ then:

$$(b+c)^{b+c-a}.(c+a)^{c+a-b}.(a+b)^{a+b-c}\geq 8$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$a+b+c\overset{AM-GM}{\geq}3\sqrt[3]{abc}\overset{abc=1}{=}3\ (1)$$

$$(b+c)^{b+c-a}.(c+a)^{c+a-b}.(a+b)^{a+b-c}\overset{GM-HM}{\geq}\left(\frac{\sum(b+c-a)}{\frac{b+c-a}{b+c}+\frac{c+a-b}{c+a}+\frac{a+b-c}{a+b}}\right)^{\sum(b+c-a)}=$$

$$=\left(\frac{a+b+c}{3-\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)}\right)^{a+b+c}\overset{Nesbitt}{\geq}\left(\frac{a+b+c}{3-\frac{3}{2}}\right)^{a+b+c}\overset{(1)}{\geq}\left(\frac{3}{\frac{3}{2}}\right)^{3}=8$$

Equality holds for a=b=c=1.

1936. If $a,b,c>0\ , abc=1$ then:

$$\frac{1}{2a+b+4}+\frac{1}{2b+c+4}+\frac{1}{2c+a+4}\leq\frac{3}{7}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$We\ need\ to\ show$:

$$\frac{1}{2a+b+4}+\frac{1}{2b+c+4}+\frac{1}{2c+a+4}\leq\frac{3}{7}$$

$$\sum\frac{1}{2a+b+4}\leq\frac{3}{7},\quad\sum\frac{4}{2a+b+4}\leq\frac{12}{7}$$

$$3-\sum\frac{2a+b}{2a+b+4}\leq\frac{12}{7},\quad\sum\frac{2a+b}{2a+b+4}\geq\frac{9}{7}$$

$$\sum\frac{2a+b}{2a+b+4}=\sum\frac{(2a+b)^2}{(2a+b)^2+4(2a+b)}\overset{Bergstrom}{\geq}$$

$$\geq\frac{(2a+b+2b+c+2c+a)^2}{(2a+b)^2+(2b+c)^2+(2c+a)^2+4(2a+b+2b+c+2c+a)}=$$

$$= \frac{9(a+b+c)^2}{4(a^2+b^2+c^2)+(a^2+b^2+c^2)+4(ab+bc+ca)+12(a+b+c)} =$$
$$= \frac{9(a+b+c)^2}{5(a^2+b^2+c^2)+4(ab+bc+ca)+12(a+b+c)} =$$
$$= \frac{9(a+b+c)^2}{5(a+b+c)^2-10(ab+bc+ca)+4(ab+bc+ca)+12(a+b+c)} =$$
$$= \frac{9(a+b+c)^2}{5(a+b+c)^2-6(ab+bc+ca)+12(a+b+c)}$$

$$\textit{We need to show:}$$
$$\frac{9(a+b+c)^2}{5(a+b+c)^2-6(ab+bc+ca)+12(a+b+c)} \geq \frac{9}{7}$$
$$7(a+b+c)^2 \geq 5(a+b+c)^2-6(ab+bc+ca)+12(a+b+c)$$
$$2(a+b+c)^2-12(a+b+c)+6(ab+bc+ca) \geq 0$$
$$2(a+b+c)^2-12(a+b+c)+6\times 3\sqrt[3]{a^2b^2c^2} \overset{AM-GM}{\geq} 0$$
$$2(a+b+c)^2-12(a+b+c)+18 \overset{abc=1}{\geq} 0$$
$$2(a+b+c-3)^2 \geq 0 \ true$$

$$\textit{Equality holds for } a=b=c=1.$$

1937. If $a,b,c>0, a+b+c=3$ then:

$$\frac{a^3}{27-a^3}+\frac{b^3}{27-b^3}+\frac{c^3}{27-c^3} \geq \frac{3}{26}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$a^3+b^3+c^3 \overset{CBS}{\geq} \frac{1}{9}(a+b+c)^3 \overset{a+b+c=3}{=} 3$$
$$let\ \ a^3=x, b^3=y, c^3=z\ \ then\ a^3+b^3+c^3=x+y+z \geq 3\ (1)$$

Lemma:
$$\forall\, m>0, \frac{m}{27-m} \geq \frac{27m-1}{676}$$

Proof:
$$\textit{We need to show:}$$
$$\frac{m}{27-m} \geq \frac{27m-1}{676}$$
$$676m \geq 729m-27-27m^2+m$$
$$27m^2-54m+27 \geq 0$$
$$27(m-1)^2 \geq 0 \ true$$

$$\frac{a^3}{27-a^3}+\frac{b^3}{27-b^3}+\frac{c^3}{27-c^3}=\sum\frac{a^3}{27-a^3}=\sum\frac{x}{27-x}\overset{Lemma}{\geq}$$

$$\geq\sum\frac{27x-1}{676}=\frac{27(x+y+z)-3}{676}\overset{(1)}{\geq}\frac{27\times3-3}{676}=\frac{78}{676}=\frac{3}{26}$$

Equality holds for a=b=c=1.

1938. If $a,b,c>0$, $ab+bc+ca=3$ **then:**

$$(a+b)^{ab}(b+c)^{bc}(c+a)^{ca}\geq 8$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\frac{ab}{a+b}+\frac{bc}{b+c}+\frac{ca}{c+a}=\sum\frac{ab}{a+b}\overset{AM-GM}{\leq}\sum\frac{ab}{2\sqrt{ab}}=$$

$$=\frac{1}{2}\sum\sqrt{ab}\overset{CBS}{\leq}\frac{1}{2}\sqrt{3(ab+bc+ca)}\overset{ab+bc+ca=3}{=}\frac{3}{2}(1)$$

$$(a+b)^{ab}(b+c)^{bc}(c+a)^{ca}\overset{GM-HM}{\geq}\left(\frac{ab+bc+ca}{\frac{ab}{a+b}+\frac{bc}{b+c}+\frac{ca}{c+a}}\right)^{ab+bc+ca}\geq$$

$$\overset{(1)\ \&\ ab+bc+ca=3}{\geq}\left(\frac{3}{\frac{3}{2}}\right)^3=8$$

Equality holds for a=b=c=1.

1939. If $a,b,c>0$ **then:**

$$\log_{a+1}(b^2+c^2+2)+\log_{b+1}(c^2+a^2+2)+\log_{c+1}(a^2+b^2+2)\geq 6$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$Using\ the\ rule\ \log_x y=\frac{\log x}{\log y}\ (1)$$

$$let\ \log(a+1)=p,\log(b+1)=q,\log(c+1)=r\quad(A)$$

$$b^2+c^2+2=(b^2+1^2)+(c^2+1^2)\overset{CBS}{\geq}$$

$$\geq \frac{(b+1)^2}{2}+\frac{(c+1)^2}{2} \overset{AM-GM}{\geq} 2\sqrt{\frac{(b+1)^2}{2}\cdot\frac{(c+1)^2}{2}} = (b+1)(c+1) \overset{(A)}{=} qr$$

$$Similarly: (c^2+a^2+2) \geq (c+1)(a+1) \overset{(A)}{=} pr \text{ and}$$

$$(a^2+b^2+2) \geq (a+1)(b+1) \overset{(A)}{=} pq$$

$$\log_{a+1}(b^2+c^2+2) \geq \log_{a+1}(b+1)(c+1) \overset{(1)}{=} \frac{\log(c+1)(b+1)}{\log(a+1)} =$$

$$= \frac{\log(c+1)+\log(b+1)}{\log(a+1)} \overset{(A)}{=} \frac{q+r}{p} = \left(\frac{q}{p}+\frac{r}{p}\right)$$

$$Similarly:$$

$$\log_{b+1}(c^2+a^2+2) \geq \left(\frac{p}{q}+\frac{r}{q}\right) \text{ and } \log_{c+1}(a^2+b^2+2) \geq \left(\frac{p}{r}+\frac{q}{r}\right)$$

$$\log_{a+1}(b^2+c^2+2) + \log_{b+1}(c^2+a^2+2) + \log_{c+1}(a^2+b^2+2) \geq$$

$$\geq \left(\frac{q}{p}+\frac{r}{p}\right)+\left(\frac{p}{q}+\frac{r}{q}\right)+\left(\frac{p}{r}+\frac{q}{r}\right) \overset{AM-GM}{\geq} 6\sqrt[6]{\frac{p^2q^2r^2}{p^2q^2r^2}} = 6$$

Equality holds for a=b=c=1.

1940. If $x,y,z>0, \frac{1}{x}+\frac{1}{y}+\frac{1}{z} \leq \lambda$ and $n \in \mathbb{N}, \lambda > 0$ then :

$$\sum_{cyc} \frac{1}{\sqrt{(n^2+n+1)x^2-2xy+(n+2)y^2}} \leq \frac{\lambda}{n+1}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{For } n=0, LHS = \sum_{cyc}\frac{1}{\sqrt{x^2-2xy+2y^2}} = \sum_{cyc}\frac{1}{\sqrt{(x-y)^2+y^2}} \leq \sum_{cyc}\frac{1}{\sqrt{y^2}}$$

$$= \sum_{cyc}\frac{1}{x} \leq \lambda = \frac{\lambda}{n+1} \ (\because n=0) \text{ and we now consider the case when } n \in \mathbb{N}^* \text{ and}$$

$$\text{then : } nx^2+y^2 \overset{?}{\geq} \frac{1}{n+1}(nx+y)^2 \Leftrightarrow n^2x^2+ny^2+nx^2+y^2 \overset{?}{\geq} n^2x^2+y^2+2nxy$$

$$\Leftrightarrow n(x-y)^2 \overset{?}{\geq} 0 \rightarrow \text{true} \therefore nx^2+y^2 \overset{①}{\geq} \frac{1}{n+1}(nx+y)^2 \text{ and analogs and}$$

$$\sum_{cyc}\frac{1}{\sqrt{(n^2+n+1)x^2-2xy+(n+2)y^2}}$$

$$=\sum_{cyc}\frac{1}{\sqrt{n(n+1)x^2+(n+1)y^2+(x-y)^2}}\le\frac{1}{\sqrt{n+1}}\cdot\sum_{cyc}\frac{1}{\sqrt{nx^2+y^2}}\overset{via\ (1)}{\le}$$

$$\frac{\sqrt{n+1}}{\sqrt{n+1}}\cdot\sum_{cyc}\frac{1}{nx+y}\overset{\text{Weighted AM}\ge\text{Weighted HM}}{\le}\frac{1}{(n+1)^2}\cdot\sum_{cyc}\left(\frac{n}{x}+\frac{1}{y}\right)$$

$$=\frac{1}{(n+1)^2}\cdot\left(n\sum_{cyc}\frac{1}{x}+\sum_{cyc}\frac{1}{x}\right)=\frac{n+1}{(n+1)^2}\cdot\sum_{cyc}\frac{1}{x}\overset{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\le\lambda}{\le}\frac{\lambda}{n+1}\ \text{and so,}$$

$$\sum_{cyc}\frac{1}{\sqrt{(n^2+n+1)x^2-2xy+(n+2)y^2}}\le\frac{\lambda}{n+1}\ \forall x,y,z>0\ \Big|\ \frac{1}{x}+\frac{1}{y}+\frac{1}{z}\le\lambda$$

$$\text{and } n\in\mathbb{N},\lambda>0,''=''\text{ iff } x=y=z=\frac{3}{\lambda}\ (\text{QED})$$

1941. If $a,b,c>0, ab+bc+ca=3$ and $\lambda\ge\frac{1}{2}$ then :

$$\sum_{cyc}\sqrt{\frac{bc}{a^4+2(\lambda-1)a^3+10\lambda-3}}\le\frac{3}{2.\sqrt{3\lambda-1}}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$a^4+2(\lambda-1)a^3+10\lambda-3=$$

$$a^4-2a^3+a^2+\left(2\lambda a^3-a^2+10\lambda-3-(3\lambda-1)(a^2+3)\right)+(3\lambda-1)(a^2+3)$$

$$=a^2(a-1)^2+\lambda(2a^3-3a^2+1)+(3\lambda-1)(a^2+3)$$

$$=a^2(a-1)^2+\lambda(a-1)^2(2a+1)+(3\lambda-1)(a^2+3)\ge(3\lambda-1)(a^2+3)$$

$$\therefore\sqrt{\frac{bc}{a^4+2(\lambda-1)a^3+10\lambda-3}}\le\sqrt{\frac{bc}{(3\lambda-1)(a^2+3)}}\ \text{and analogs}$$

$$\therefore\sum_{cyc}\sqrt{\frac{bc}{a^4+2(\lambda-1)a^3+10\lambda-3}}\le\sum_{cyc}\sqrt{\frac{bc}{(3\lambda-1)(a^2+3)}}\overset{CBS}{\le}$$

$$\frac{1}{\sqrt{3\lambda-1}}\sqrt{\sum_{cyc}ab}\cdot\sqrt{\sum_{cyc}\frac{1}{a^2+3}}\overset{ab+bc+ca=3}{=}\frac{\sqrt{3}}{\sqrt{3\lambda-1}}\cdot\sqrt{\sum_{cyc}\frac{1}{a^2+3}}\overset{?}{\le}\frac{3}{2.\sqrt{3\lambda-1}}$$

$$\Leftrightarrow\sum_{cyc}\frac{1}{a^2+3}\overset{?}{\le}\frac{3}{4}\Leftrightarrow3\prod_{cyc}(a^2+3)\overset{?}{\ge}4\sum_{cyc}\left((b^2+3)(c^2+3)\right)$$

$$\Leftrightarrow3a^2b^2c^2+5\sum_{cyc}a^2b^2+3\sum_{cyc}a^2\overset{?}{\ge}27\Leftrightarrow$$

$$3a^2b^2c^2+\frac{5}{3}\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}ab\right)+\frac{3}{9}\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}ab\right)^2\overset{?}{\geq}\frac{27}{27}\left(\sum_{cyc}ab\right)^3$$

$$\Leftrightarrow\sum_{cyc}a^4b^2+\sum_{cyc}a^2b^4+2\sum_{cyc}a^3b^3+2abc\sum_{cyc}a^3\overset{?}{\geq}$$

$$2abc\left(\sum_{cyc}a^2b+\sum_{cyc}ab^2\right)+6a^2b^2c^2\rightarrow\text{true}\because$$

$$\sum_{cyc}a^4b^2+\sum_{cyc}a^2b^4+2\sum_{cyc}a^3b^3+2abc\sum_{cyc}a^3$$

$$=\sum_{cyc}\left(a^4(b^2+c^2)\right)+2abc\sum_{cyc}a^3+2\sum_{cyc}a^3b^3\overset{AM-GM}{\geq}4abc\sum_{cyc}a^3+6a^2b^2c^2$$

$$\overset{Schur+AM-GM}{\geq}2abc\left(\sum_{cyc}a^2b+\sum_{cyc}ab^2\right)+6a^2b^2c^2$$

$$\therefore\sum_{cyc}\sqrt{\frac{bc}{a^4+2(\lambda-1)a^3+10\lambda-3}}\leq\frac{3}{2.\sqrt{3\lambda-1}}\ \forall a,b,c>0\,|\,ab+bc+ca=3$$

$$and\ \lambda\geq\frac{1}{2},''='' iff\ a=b=c=1\ (QED)$$

1942. If $x,y,z>0$, $\lambda\geq 0$ then:

$$\sum\sqrt{\lambda x^2+2(\lambda+2)xy+\lambda y^2}\leq 2\sqrt{\lambda+1}(x+y+z)$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

Lemma:

$$\forall a,b>0\ and\ \lambda\geq 0\ ,\sqrt{\lambda a^2+2(\lambda+2)ab+\lambda b^2}\leq\sqrt{\lambda+1}(a+b)$$

Proof:

We need to show :

$$\sqrt{\lambda a^2+2(\lambda+2)ab+\lambda b^2}\leq\sqrt{\lambda+1}(a+b)$$

$$\sqrt{\lambda(a+b)^2+4ab}\leq\sqrt{\lambda+1}(a+b)$$

$$\lambda(a+b)^2+4ab\overset{squaring}{\leq}(\lambda+1)(a+b)^2$$

$$or\ ,(a+b)^2(\lambda+1-\lambda)-4ab\geq 0$$

$$(a+b)^2-4ab\geq 0\ or,(a-b)^2\geq 0\ true$$

$$\sum\sqrt{\lambda x^2+2(\lambda+2)xy+\lambda y^2}\overset{Lemma}{\leq}\sum\sqrt{\lambda+1}(x+y)=$$
$$=\sqrt{\lambda+1}\sum x+y\ =2\sqrt{\lambda+1}(x+y+z)$$

Equality holds for x=y=z.

1943. If $a,b,c>0,abc=1$ then:

$$(a^2+1)^a(b^2+1)^b(c^2+1)^c\geq 8$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\frac{a}{a^2+1}+\frac{b}{b^2+1}+\frac{c}{c^2+1}\overset{AM-GM}{\leq}\frac{a}{2a}+\frac{b}{2b}+\frac{c}{2c}=\frac{3}{2}\ (1)$$

$$a+b+c\overset{AM-GM}{\geq}3\sqrt[3]{abc}\overset{abc=1}{=}3\ (2)$$

$$(a^2+1)^a(b^2+1)^b(c^2+1)^c\overset{GM-HM}{\geq}\left(\frac{a+b+c}{\frac{a}{a^2+1}+\frac{b}{b^2+1}+\frac{c}{c^2+1}}\right)^{a+b+c}\overset{(1)}{\geq}$$
$$\geq\left(\frac{a+b+c}{\frac{3}{2}}\right)^{a+b+c}\overset{(2)}{\geq}\left(\frac{3}{\frac{3}{2}}\right)^3=8$$

Equality holds for a=b=c=1.

1944. If $x,y,z\geq 0$ and $xy+yz+zx+xyz=4$ then prove that :

$$x^2+y^2+z^2+3xyz\geq 2(x+y+z)$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

If one variable equals zero (WLOG $x=0$), then : $yz=4$ and LHS $=$

$y^2+z^2 \geq \frac{1}{2}(y+z)^2 \overset{AM-GM}{\geq} \sqrt{yz}.(y+z) = 2(y+z) = RHS$ and we now consider the case when $x,y,z>0$ and we have,

$$\sum_{cyc}((2+y)(2+z)) - (2+x)(2+y)(2+z) = 4-(xy+yz+zx+xyz) = 0$$

$$\therefore \sum_{cyc}\frac{1}{2+x} = 1 \to (m) \text{ and } \because \frac{1}{2+x} \overset{x>0}{<} \frac{1}{2} \therefore \text{we can set : } \frac{1}{2+x} = \frac{1}{2}-a$$

$$\left(a>0 \text{ and } a<\frac{1}{2}\right) \therefore x+2 = \frac{2}{1-2a} \Rightarrow x = \frac{2}{1-2a}-2 = \frac{2a}{\frac{1}{2}-a} \to (1)$$

$$\text{Similarly, we set : } \frac{1}{2+y} = \frac{1}{2}-b \text{ and } \frac{1}{2+z} = \frac{1}{2}-c \therefore 1 \overset{\text{via (m)}}{=}$$

$$\frac{1}{2+x}+\frac{1}{2+y}+\frac{1}{2+z} = \frac{1}{2}-a+\frac{1}{2}-b+\frac{1}{2}-c \Rightarrow a+b+c = \frac{1}{2} \to (i)$$

$$\therefore (1) \text{ and } (i) \Rightarrow x = \frac{2a}{b+c} \text{ and analogously, } y = \frac{2b}{c+a} \text{ and } z = \frac{2c}{a+b}$$

$\left(a,b,c \in \left(0,\frac{1}{2}\right)\right)$ Assigning $b+c=X, c+a=Y, a+b=Z \Rightarrow X+Y-Z = 2c>0$,

$Y+Z-X = 2a>0$ and $Z+X-Y = 2b>0 \Rightarrow X+Y>Z, Y+Z>X, Z+X>Y$

$\Rightarrow X,Y,Z$ form sides of a triangle XYZ with semiperimeter, circumradius & inradius $= s,R,r$ (say); and then : $a = s-X, b = s-Y, c = s-Z \therefore abc = r^2s$ and so,

$$\sum_{cyc}x^2+3xyz \geq 2\left(\sum_{cyc}x\right) \Leftrightarrow \sum_{cyc}\frac{a^2}{(b+c)^2}+\frac{6abc}{(b+c)(c+a)(a+b)} \geq \sum_{cyc}\frac{a}{b+c}$$

$$\Leftrightarrow \sum_{cyc}\frac{s^2-2sX+X^2}{X^2}+\frac{6r^2s}{4Rrs} \geq \sum_{cyc}\frac{s-X}{X}$$

$$\Leftrightarrow \frac{s^2\left((s^2+4Rr+r^2)-16Rrs^2\right)}{16R^2r^2s^2}-\frac{3(s^2+4Rr+r^2)}{4Rr}+6+\frac{6r}{4R} \geq 0$$

$$\Leftrightarrow s^4-(20Rr-2r^2)s^2+r^2(64R^2+20Rr+r^2) \overset{(*)}{\geq} 0 \text{ and } \because P =$$

$$(s^2-16Rr+5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } (*), \text{it suffices to prove :}$$

$$\text{LHS of } (*) \overset{?}{\geq} P \Leftrightarrow (3R-2r)s^2 \overset{(**)}{\geq} r(48R^2-45Rr+6r^2)$$

$$\text{Now, } (R-r)(s^2-16Rr+5r^2) \overset{\text{Rouche}}{\geq}$$

$$(R-r)\left(2R^2-6Rr+4r^2-2(R-2r)\sqrt{R^2-2Rr}\right)$$

$$= (R-2r)\left(\left(R-r-\sqrt{R^2-2Rr}\right)^2+r^2\right) \overset{\text{Euler}}{\geq} r^2(R-2r) \Rightarrow$$

$$(R-r)(s^2-16Rr+5r^2) \geq r^2(R-2r) \Rightarrow s^2 \geq 16Rr-5r^2+\frac{r^2(R-2r)}{R-r}$$

$$\text{and so, LHS of } (**) \geq (3R-2r)\left(16Rr-5r^2+\frac{r^2(R-2r)}{R-r}\right) \overset{?}{\geq}$$

$$r(48R^2-45Rr+6r^2) \Leftrightarrow 4Rr^3(R-2r) \overset{?}{\geq} 0 \rightarrow \text{true via Euler} \Rightarrow (**) \Rightarrow (*) \text{ is true}$$

$$\therefore x^2+y^2+z^2+3xyz \geq 2(x+y+z)\ \forall x,y,z \geq 0 \,|\, xy+yz+zx+xyz=4,$$

$$'' ='' \text{ iff } \begin{pmatrix} x=0, \\ y=z=2 \end{pmatrix} \text{ or } \begin{pmatrix} y=0, \\ x=z=2 \end{pmatrix} \text{ or } \begin{pmatrix} z=0, \\ x=y=2 \end{pmatrix} \text{ or } (x=y=z=1)\ (\text{QED})$$

1945. If $x,y,z>0$, $x+y+z=3$, $\lambda>1$ then:

$$\sum \frac{x^3}{\lambda-x^3} \geq \frac{3}{\lambda-1}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$x^3+y^3+z^3=\frac{x^3}{(1)^2}+\frac{y^3}{(1)^2}+\frac{z^3}{(1)^2} \overset{Radon}{\geq} \frac{(x+y+z)^3}{9} \overset{x+y+z=3}{=} 3\ (1)$$

$$\sum \frac{x^3}{\lambda-x^3} \overset{Holder}{\geq} \frac{1}{3}\cdot\frac{(x+y+z)^3}{3\lambda-(x^3+y^3+z^3)} \overset{(1)\&\,(a+b+c)=3}{\geq} \frac{1}{3}\cdot\frac{27}{3\lambda-3}=\frac{3}{\lambda-1}$$

$$Equality\ holds\ for\ x=y=z=1.$$

1946. If $a,b>0$ then:

$$3(a^{n+3}-a^n+3)(b^{n+3}-b^n+3) \geq (a+b+1)^3, n \in N$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

For every positive real number x, we have $(x^n-1),(x^3-1)$ have the same signs and because of this :

$$(x^n-1)(x^3-1) \geq 0 \ \ or\ x^{n+3}-x^n-x^3+1 \geq 0$$

$$x^{n+3}-x^n \geq x^3-1\ or\ x^{n+3}-x^n+3 \geq x^3+2 \ \ (1)$$

$$3(a^{n+3}-a^n+3)(b^{n+3}-b^n+3) \overset{(1)}{\geq} (1+1+1)(a^3+2)(b^3+2)=$$

$$=(1+1+1)(a^3+1+1)(1+b^3+1)\overset{Holder}{\geq}(a+b+1)^3$$

$$Equality\ holds\ for\ a=b=1$$

1947. If $a,b,c>0$ and $abc=1$ then prove that :

$$\sqrt{\frac{a^2+3}{(a^3+b^3)(a^3+c^3)}}+\sqrt{\frac{b^2+3}{(b^3+c^3)(b^3+a^3)}}+\sqrt{\frac{c^2+3}{(c^3+a^3)(c^3+b^3)}}\leq 3$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}a^3b^3\overset{AM-GM}{\geq}3a^2b^2c^2\overset{abc=1}{\geq}3\ and\ so,\sum_{cyc}\sqrt{\frac{a^2+3}{(a^3+b^3)(a^3+c^3)}}\leq$$

$$\sum_{cyc}\sqrt{\frac{a^2+3}{a^6+3}}\overset{AM-GM}{\leq}\sum_{cyc}\frac{\frac{a^2+3}{a^6+3}+1}{2}\overset{?}{\leq}3\Leftrightarrow\sum_{cyc}\frac{a^2+3}{a^6+3}\overset{?}{\leq}3\Leftrightarrow\sum_{cyc}\frac{\frac{x^2}{y^2}+3}{\frac{x^6}{y^6}+3}\overset{?}{\leq}3$$

$$\left(a=\frac{x}{y},b=\frac{y}{z},c=\frac{z}{x}\right)\Leftrightarrow\sum_{cyc}\frac{3y^6+x^6-(x^6-x^2y^4)}{3y^6+x^6}\overset{?}{\leq}3$$

$$\Leftrightarrow\sum_{cyc}\frac{x^6}{x^6+3y^6}\overset{?\ (*)\ AM\geq}{\geq}\sum_{cyc}\frac{x^2y^4}{x^6+3y^6}$$

$$Now,\sum_{cyc}\frac{x^2y^4}{x^6+3y^6}\leq\sum_{cyc}\frac{x^2y^4}{x^2y^2(x^2+y^2)+2y^6}=\sum_{cyc}\frac{x^2y^2}{x^4+y^4+x^2y^2+y^4}\overset{AM-GM}{\leq}$$

$$\sum_{cyc}\frac{x^2y^2}{3x^2y^2+y^4}=\frac{1}{3}\cdot\sum_{cyc}\frac{3x^2+y^2-y^2}{3x^2+y^2}=1-\frac{1}{3}\cdot\sum_{cyc}\frac{y^4}{y^4+3x^2y^2}\overset{Bergstrom}{\leq}$$

$$1-\frac{1}{3}\cdot\frac{\left(\sum_{cyc}x^2\right)^2}{\sum_{cyc}x^4+3\sum_{cyc}x^2y^2}=1-\frac{1}{3}\cdot\frac{\left(\sum_{cyc}x^2\right)^2}{\left(\sum_{cyc}x^2\right)^2+\sum_{cyc}x^2y^2}\leq$$

$$1-\frac{1}{3}\cdot\frac{\left(\sum_{cyc}x^2\right)^2}{\left(\sum_{cyc}x^2\right)^2+\frac{\left(\sum_{cyc}x^2\right)^2}{3}}\therefore\sum_{cyc}\frac{x^2y^4}{x^6+3y^6}\overset{①}{\leq}\frac{3}{4}\ and\sum_{cyc}\frac{x^6}{x^6+3y^6}=$$

$$\sum_{cyc}\frac{x^{12}}{x^{12}+3x^6y^6}\overset{\text{Bergstrom}}{\geq}\frac{\left(\sum_{cyc}x^6\right)^2}{\left(\sum_{cyc}x^6\right)^2+\sum_{cyc}x^6y^6}=\frac{\left(\sum_{cyc}x^6\right)^2}{\left(\sum_{cyc}x^6\right)^2+\frac{\left(\sum_{cyc}x^6\right)^2}{3}}$$

$$\therefore\sum_{cyc}\frac{x^6}{x^6+3y^6}\overset{②}{\geq}\frac{3}{4}\therefore ①\ \textbf{and}\ ②\Rightarrow(*)\ \textbf{is true}$$

$$\therefore\sqrt{\frac{a^2+3}{(a^3+b^3)(a^3+c^3)}}+\sqrt{\frac{b^2+3}{(b^3+c^3)(b^3+a^3)}}+\sqrt{\frac{c^2+3}{(c^3+a^3)(c^3+b^3)}}\leq 3$$

$$\forall\, a,b,c>0\,|\,abc=1,''='' \text{ iff } a=b=c=1\ (\text{QED})$$

1948. If $a,b,c\geq 0$ and $a+b+c=ab+bc+ca>0$ then prove that :

$$\frac{1}{a+2}+\frac{1}{b+2}+\frac{1}{c+2}>\frac{4}{5}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{1}{a+2}>\frac{4}{5}\Leftrightarrow 5\sum_{cyc}((b+2)(c+2))>4\prod_{cyc}(a+2)\Leftrightarrow 28+4p-3q>$$

$$4r\left(p=\sum_{cyc}a=\sum_{cyc}ab=q>0\ \text{and}\ r=abc\right)\overset{p=q>0}{\Leftrightarrow}$$

$$28\left(\frac{q}{p}\right)^2+q>\frac{4rp}{q}\Leftrightarrow 28q^3+p^2q^2-4rp^3\overset{(*)}{>}0\Leftrightarrow Ar^2+r.G(p,q)+H(p,q)>0$$

$$(\text{Here } A=0, G(p,q)=-4p^3 \text{ and } H(p,q)=28q^3+p^2q^2) \text{ and } \because F(a,b,c)=$$

$$28\left(\sum_{cyc}ab\right)^3+\left(\sum_{cyc}a\right)^2\left(\sum_{cyc}ab\right)^2-4abc\left(\sum_{cyc}a\right)^3=Ar^2+r.G(p,q)+H(p,q)$$

$$\geq 0 \text{ for } A\leq 0\leq a,b,c \text{ holds iff } F(a,1,1)\geq 0 \text{ and } F(0,b,c)\geq 0\ \therefore (*) \text{ is true iff}$$

$$F(a,1,1)\geq 0 \text{ and } F(0,b,c)\geq 0\ \forall\, a,b,c\geq 0; \text{and } F(0,b,c)=$$

$$28b^3c^3+b^2c^2(b+c)^2\overset{b,c\geq 0}{>}0\left(\because 28b^3c^3+b^2c^2(b+c)^2=0 \text{ iff } b=c=0 \Rightarrow \sum_{cyc} ab=0\rightarrow \text{impossible}\right)$$

$$\text{and } F(a,1,1)=28(2a+1)^3+(2a+1)^2(a+2)^2-4a(a+2)^3$$

$$=220a^3+321a^2+156a+32\overset{a\geq 0}{>}0\therefore F(a,1,1)>0 \text{ and } F(0,b,c)>0$$

$$\Rightarrow (*) \text{ is true } \therefore \frac{1}{a+2}+\frac{1}{b+2}+\frac{1}{c+2}>\frac{4}{5}$$

$$\forall a,b,c\geq 0 \text{ and } a+b+c=ab+bc+ca>0 \text{ (QED)}$$

1949. If $x,y,z\in\left[1,\sqrt{2}\right]$ then:

$$\frac{\sqrt{x^2-1}}{y}+\frac{\sqrt{y^2-1}}{z}+\frac{\sqrt{z^2-1}}{x}\leq\frac{3\sqrt{2}}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$let\ u=\sqrt{x^2-1},v=\sqrt{y^2-1},w=\sqrt{z^2-1}\ then$$
$$x=\sqrt{u^2+1},y=\sqrt{v^2+1},z=\sqrt{w^2+1}$$
$$as\ x,y,z\in\left[1,\sqrt{2}\right]\ then\ u,v,w\in[0,1]$$
$$\left(since,1\leq x\leq\sqrt{2}\ or\ 1\leq\sqrt{u^2+1}\leq\sqrt{2}\ or\ 1\leq u^2+1\leq 2 \atop 0\leq u^2\leq 1\ or\ 0\leq u\leq 1\right)$$

Lemma:

$$\forall t\in[0,1],\frac{1}{1+t}\leq 1-\frac{t}{2}\ \ (1)$$

Proof:

$$We\ need\ to\ show:$$
$$\frac{1}{1+t}\leq 1-\frac{t}{2}$$
$$or,\frac{1}{1+t}\leq\frac{2-t}{2}\ \ or\ \ 2\leq(2-t)(1+t)\ or,t^2-t\leq 0$$
$$t(t-1)\leq 0\ true\ as\ \ t\in[0,1]$$

$$\frac{\sqrt{x^2-1}}{y}+\frac{\sqrt{y^2-1}}{z}+\frac{\sqrt{z^2-1}}{x}=\frac{u}{\sqrt{v^2+1}}+\frac{v}{\sqrt{w^2+1}}+\frac{w}{\sqrt{v^2+1}}=$$

$$\overset{CBS}{\leq}\sqrt{(u^2+v^2+w^2)\left(\frac{1}{u^2+1}+\frac{1}{v^2+1}+\frac{1}{w^2+1}\right)}=$$

$$=\sqrt{(a+b+c)\left(\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}\right)}\leq$$

$$(we\ take\ a=u^2,b=v^2,c=w^2\ as\ \ u,v,w\in[0,1]\ then\ a,b,c\in[0,1])$$

$$\overset{Lemma\ (1)}{\leq}\sqrt{(a+b+c)\left(1-\frac{a}{2}+1-\frac{b}{2}+1-\frac{c}{2}\right)}=$$

$$=\sqrt{(a+b+c)\left(3-\frac{a+b+c}{2}\right)}\overset{a+b+c=x}{=}\sqrt{3x-\frac{x^2}{2}}\ \ (2)$$

$$Let\ m=3x-\frac{x^2}{2}\ or\ x^2-6x+2m=0$$

$$this\ is\ the\ quadratic\ equation\ in\ x\ \ and\ x\ is\ real\ then\ discriminant\geq 0$$

$$(-6)^2-4.1.2m\geq 0\ or\ m\leq\frac{36}{8}=\frac{9}{2}$$

$$so,m=3x-\frac{x^2}{2}\leq\frac{9}{2}$$

$$From(2)we\ get\ \frac{\sqrt{x^2-1}}{y}+\frac{\sqrt{y^2-1}}{z}+\frac{\sqrt{z^2-1}}{x}\leq\sqrt{3x-\frac{x^2}{2}}\leq\sqrt{\frac{9}{2}}=\frac{3\sqrt{2}}{2}$$

$$Equality\ holds\ for\ x=y=z=\sqrt{2}$$

1950. If $x,y,z>0, x+y+z=1$ and $3\lambda\geq 2n\geq 0$ then :

$$\lambda\left(\sum_{cyc}x^3+\sum_{cyc}xy\right)+nxyz\ \geq\frac{12\lambda+n}{27}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\lambda\left(\sum_{cyc}x^3+\sum_{cyc}xy\right)+nxyz\geq\frac{12\lambda+n}{27}$$

$$\Leftrightarrow\lambda\left(27\sum_{cyc}x^3+27\left(\sum_{cyc}xy\right)\left(\sum_{cyc}x\right)-12\left(\sum_{cyc}x\right)^3\right)\geq n\left(\left(\sum_{cyc}x\right)^3-27xyz\right)$$

$$\left(\because \sum_{cyc} x = 1\right) \Leftrightarrow$$

$$\lambda\left(\begin{matrix} 27\sum_{cyc} x^3 + 81xyz + 27\left(\sum_{cyc} x^2y + \sum_{cyc} xy^2\right) - \\ 12\left(\sum_{cyc} x^3 + 6xyz + 3\sum_{cyc} x^2y + 3\sum_{cyc} xy^2\right) \end{matrix}\right) \geq n\left(\left(\sum_{cyc} x\right)^3 - 27xyz\right)$$

$$\Leftrightarrow 3\lambda\left(5\sum_{cyc} x^3 + 3xyz - 3\sum_{cyc} x^2y - 3\sum_{cyc} xy^2\right) \overset{(*)}{\geq} n\left(\left(\sum_{cyc} x\right)^3 - 27xyz\right)$$

$$\text{Now, } 5\sum_{cyc} x^3 + 15xyz \overset{Schur}{\geq} 5\sum_{cyc} x^2y + 5\sum_{cyc} xy^2 \text{ and}$$

$$2\sum_{cyc} x^2y + 2\sum_{cyc} xy^2 \overset{AM-GM}{\geq} 12xyz \text{ and so,}$$

$$5\sum_{cyc} x^3 + 15xyz + 2\sum_{cyc} x^2y + 2\sum_{cyc} xy^2 \geq 5\sum_{cyc} x^2y + 5\sum_{cyc} xy^2 + 12xyz$$

$$\Rightarrow 5\sum_{cyc} x^3 + 3xyz - 3\sum_{cyc} x^2y - 3\sum_{cyc} xy^2 \geq 0 \text{ and } \because 3\lambda \geq 2n \therefore \text{LHS of } (*) \geq$$

$$2n\left(5\sum_{cyc} x^3 + 3xyz - 3\sum_{cyc} x^2y - 3\sum_{cyc} xy^2\right) \overset{?}{\geq} n\left(\left(\sum_{cyc} x\right)^3 - 27xyz\right)$$

$$\Leftrightarrow n\left(\begin{matrix} 10\sum_{cyc} x^3 + 6xyz - 6\sum_{cyc} x^2y - 6\sum_{cyc} xy^2 - \\ \left(\sum_{cyc} x^3 + 6xyz + 3\sum_{cyc} x^2y + 3\sum_{cyc} xy^2\right) + 27xyz \end{matrix}\right) \overset{?}{\geq} 0$$

$$\Leftrightarrow 9n\left(\sum_{cyc} x^3 + 3xyz - \sum_{cyc} x^2y - \sum_{cyc} xy^2\right) \overset{?}{\geq} 0 \rightarrow \text{true via Schur } (\because n \geq 0) \Rightarrow$$

$$(*)\ \text{is true} \Rightarrow \lambda\left(\sum_{cyc} x^3 + \sum_{cyc} xy\right) + nxyz \geq \frac{12\lambda + n}{27}\ \forall\, x,y,z > 0 \,|\, x+y+z = 1$$

$$\text{and } 3\lambda \geq 2n \geq 0, \text{''} = \text{'' iff } x = y = z = \frac{1}{3}\ (\text{QED})$$

1951.

If $0 \leq a,b,c \leq 1 \leq x,y,z$ and $a+b+c+x+y+z = 6$ then prove that

$$a^2 + b^2 + c^2 + x^2 + y^2 + z^2 \leq 18$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$0 \leq a,b,c \leq 1 \Rightarrow a(a-1), b(b-1), c(c-1) \leq 0 \Rightarrow \sum_{cyc} a(a-1) \leq 0$$

$$\Rightarrow \sum_{cyc} a^2 \leq \sum_{cyc} a \overset{a+b+c+x+y+z=6}{=} 6 - \sum_{cyc} x \Rightarrow \sum_{cyc} a^2 + \sum_{cyc} x^2 \leq$$

$$6 - \sum_{cyc} x + \left(\sum_{cyc} x\right)^2 - 2\sum_{cyc} xy \leq 6 - \sum_{cyc} x + \left(\sum_{cyc} x\right)^2 - 2\sum_{cyc} (x+y-1)$$

$(\because x,y,z \geq 1 \Rightarrow (x-1)(y-1) \geq 0$ and analogs $\Rightarrow xy \geq x+y-1$ and analogs$)$

$$= 6 - \sum_{cyc} x + \left(\sum_{cyc} x\right)^2 - 4\sum_{cyc} x + 6 = \left(\sum_{cyc} x\right)^2 - 5\sum_{cyc} x - 6 + 18$$

$$= \left(\sum_{cyc} x - 6\right)\left(\sum_{cyc} x + 1\right) + 18 \leq 18$$

$$\left(\because \sum_{cyc} x = 6 - \sum_{cyc} a \overset{a,b,c \geq 0}{\leq} 6 \text{ and } \sum_{cyc} x \overset{x,y,z \geq 1}{>} 0\right) \therefore a^2 + b^2 + c^2 + x^2 + y^2 + z^2$$

≤ 18 whenever $0 \leq a,b,c \leq 1 \leq x,y,z$ and $a+b+c+x+y+z = 6$ (QED)

1952. If $a,b,c > 0, (ab)^c (bc)^a (ca)^b \geq 1$ then:

$$a + b + c \geq 3$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$(ab)^c(bc)^a(ca)^b \geq 1$$

$$a^{b+c}.b^{c+a}.c^{a+b} \overset{AM-GM}{\geq} \frac{(b+c)a+(c+a)b+(a+b)c}{2(a+b+c)} \geq 1 \Leftrightarrow$$

$$\Leftrightarrow \frac{2(ab+bc+ca)}{2(a+b+c)} \geq 1 \Leftrightarrow \frac{(a+b+c)^2}{3(a+b+c)} \geq 1$$

$$a+b+c \geq 3$$

Equality holds for a=b=c=1.

1953. If $a,b,c>0$ then:

$$(a-b)(b^2+c^2)+(b-c)(c^2+a^2)+(c-a)(a^2+b^2) \leq 0$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$Schur's\ inequality:$$
$$a^3+b^3+c^3+3abc \geq a^2(b+c)+b^2(c+a)+c^2(a+b)$$

$$(a-b)(b^2+c^2)+(b-c)(c^2+a^2)+(c-a)(a^2+b^2)=$$

$$=a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)-(a^3+b^3+c^3+ab^2+bc^2+ca^2) \overset{AM-GM}{\leq}$$

$$\leq a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)-(a^3+b^3+c^3+3abc) \overset{Schur}{\leq} 0$$

Equality holds for a=b=c.

1954. If $a,b,c>0$ then:

$$(a-b)\sqrt{b+c}+(b-c)\sqrt{c+a}+(c-a)\sqrt{a+b} \leq 0$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$Schur's\ inequality:$$
$$a^3+b^3+c^3+3abc \geq a^2(b+c)+b^2(c+a)+c^2(a+b)$$

$$a^3+b^3+c^3 \geq a^2(b+c)+b^2(c+a)+c^2(a+b)-3abc\ (1)$$

$$Let\ x=\sqrt{a+b}, y=\sqrt{b+c}, z=\sqrt{c+a}\ then:$$

$$b-c=x^2-z^2, c-a=y^2-x^2, a-b=z^2-y^2$$

$$(a-b)\sqrt{b+c}+(b-c)\sqrt{c+a}+(c-a)\sqrt{a+b}=$$

$$=(a-b)y+(b-c)z+(c-a)x=$$

$$=(z^2-y^2)y+(x^2.-z^2)z+(y^2-x^2)x=$$

$$=(z^2y+x^2z+y^2x)-(x^3+y^3+z^3)\overset{(1)}{\leq}$$

$$\leq(z^2y+x^2z+y^2x)+3xyz-x^2(y+z)-y^2(z+x)-z^2(x+y)=$$

$$=3xyz+\sum\left(x^2z-x^2(y+z)\right)=3xyz-\sum x^2y\overset{AM-GM}{\leq}3xyz-3xyz=0$$

Equality holds for a=b=c.

1955. If $a,b>0, a+b\leq 2ab$ then:

$$\frac{a^3}{a^2+1}+\frac{b^3}{b^2+1}\geq 1$$

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Solution by Tapas Das-India

$$a+b\leq 2ab\ or\ 2\sqrt{ab}\overset{AM-GM}{\leq}2ab\ or\ \sqrt{ab}\geq 1\ or\ ab\geq 1\ (1)$$

$$\frac{a^3}{a^2+1}=a-\frac{a}{a^2+1}\overset{AM-GM}{\geq}a-\frac{a}{2a}=a-\frac{1}{2}$$

$$Similarly:\frac{b^3}{b^2+1}\geq b-\frac{1}{2}$$

$$\frac{a^3}{a^2+1}+\frac{b^3}{b^2+1}\geq(a+b)-1\overset{AM-GM}{\geq}2\sqrt{ab}-1\overset{(1)}{\geq}2-1=1$$

Equality holds for a=b=1.

1956. If $a,b,c>0,\ abc=1$ then:

$$\frac{a}{(1+a+ab)^2}+\frac{b}{(1+b+bc)^2}+\frac{c}{(1+c+ac)^2}\geq\frac{1}{a+b+c}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\frac{a}{(1+a+ab)^2}\overset{abc=1}{=}\frac{a}{(abc+a+ab)^2}=\frac{1}{a(bc+1+b)^2}\ (1)$$

$$Similarly:\frac{b}{(1+b+bc)^2}=\frac{1}{b(1+ac+c)^2}\ (2)$$

$$and\ \frac{c}{(1+c+ac)^2}=\frac{1}{c(a+1+ab)^2}\ (3)$$

$$\frac{1}{bc+b+1}+\frac{1}{ac+c+1}+\frac{1}{ab+a+1}\overset{abc=1}{=}$$
$$=\frac{abc}{bc+b+abc}+\frac{1}{ac+c+1}+\frac{c}{abc+ac+c}=$$

$$\overset{abc=1}{=}\frac{ac}{c+1+ac}+\frac{1}{ac+c+1}+\frac{c}{1+ac+c}=\frac{1+ac+c}{1+ac+c}=1\ (4)$$

$$\frac{a}{(1+a+ab)^2}+\frac{b}{(1+b+bc)^2}+\frac{c}{(1+c+ac)^2}\overset{(1),(2),(3)}{=}$$
$$=\frac{1}{a(bc+1+b)^2}+\frac{1}{b(1+ac+c)^2}+\frac{1}{c(a+1+ab)^2}=$$
$$=\frac{\left(\frac{1}{bc+b+1}\right)^2}{a}+\frac{\left(\frac{1}{ac+c+1}\right)^2}{b}+\frac{\left(\frac{1}{ab+a+1}\right)^2}{c}\overset{Bergstrom}{\geq}$$
$$\geq\frac{\left(\frac{1}{bc+b+1}+\frac{1}{ac+c+1}+\frac{1}{ab+a+1}\right)^2}{a+b+c}\overset{(4)}{=}\frac{1}{a+b+c}$$

Equality holds for a=b=c=1.

1957. If $a,b,c>0,\ abc=1$ then:

$$\frac{1}{a^4-2a+2b^2+5}+\frac{1}{b^4-2b+2c^2+5}+\frac{1}{c^4-2c+2a^2+5}\leq\frac{1}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$a^4-2a+2b^2+5=(a^4+1)+(b^2+1)+b^2-2a+3\overset{AM-GM}{\geq}$$

$$\geq 2a^2-2a+2b+b^2+3=(a^2+b^2)+2b+(a^2+1)-2a+2 \overset{AM-GM}{\geq}$$
$$\geq 2ab+2b+2a-2a+2=2(1+b+ab)$$

$$Similarly:$$
$$b^4-2b+2c^2+5\geq 2(1+c+bc) \text{ and}$$
$$c^4-2c+2a^2+5\geq 2(1+a+ac)$$
$$\frac{1}{a^4-2a+2b^2+5}+\frac{1}{b^4-2b+2c^2+5}+\frac{1}{c^4-2c+2a^2+5}\leq$$
$$\leq \frac{1}{2(1+b+ab)}+\frac{1}{2(1+c+bc)}+\frac{1}{2(1+a+ac)}=$$
$$=\frac{1}{2}\left(\frac{1}{(1+b+ab)}+\frac{1}{(1+c+bc)}+\frac{1}{(1+a+ac)}\right)\overset{abc=1}{=}$$
$$=\frac{1}{2}\left(\frac{abc}{(abc+b+ab)}+\frac{a}{(a+ac+abc)}+\frac{1}{(1+a+ac)}\right)=$$
$$\overset{abc=1}{=}\frac{1}{2}\left(\frac{ac}{(ac+1+a)}+\frac{a}{(a+ac+1)}+\frac{1}{(1+a+ac)}\right)=\frac{1}{2}\frac{1+a+ac}{1+a+ac}=\frac{1}{2}$$

Equality holds for a=b=c=1.

1958. If $a,b,c>0$ and $a^2+b^2+c^2\geq a+b+c$ then prove that :

$$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}\geq\frac{3}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$1\leq\frac{\sum_{cyc}a^2}{\sum_{cyc}a}\Rightarrow\frac{3}{2}\leq\frac{3}{2}\cdot\frac{\sum_{cyc}a^2}{\sum_{cyc}a}\overset{?}{\leq}\sum_{cyc}\frac{a^2}{b+c}$$

$$\Leftrightarrow\frac{(\sum_{cyc}a^2)(\sum_{cyc}ab)+\sum_{cyc}a^4}{\prod_{cyc}(b+c)}\overset{?}{\geq}\frac{3}{2}\cdot\frac{\sum_{cyc}a^2}{\sum_{cyc}a}$$

$$\Leftrightarrow 2\sum_{cyc}a^5+\sum_{cyc}a^4b+\sum_{cyc}ab^4\overset{?}{\underset{(*)}{\geq}}\sum_{cyc}a^3b^2+\sum_{cyc}a^2b^3+2abc\sum_{cyc}ab$$

$$2\sum_{cyc} a^5+\sum_{cyc} a^4b+\sum_{cyc} ab^4=$$

$$\left(\sum_{cyc} a^5+\sum_{cyc} ab^4\right)+\left(\sum_{cyc} b^5+\sum_{cyc} a^4b\right)\overset{AM-GM}{\geq} 2\sum_{cyc} a^3b^2+2\sum_{cyc} a^2b^3$$

$$=\sum_{cyc} a^3b^2+\sum_{cyc} a^2b^3+abc\sum_{cyc}\frac{a^2b^2}{bc}+abc\sum_{cyc}\frac{a^2b^2}{ac}\overset{Bergstrom}{\geq}$$

$$\sum_{cyc} a^3b^2+\sum_{cyc} a^2b^3+2.\frac{abc\left(\sum_{cyc} ab\right)^2}{\sum_{cyc} ab}=\sum_{cyc} a^3b^2+\sum_{cyc} a^2b^3+2abc\sum_{cyc} ab$$

$$\Rightarrow (*)\ \text{is true}\ \therefore \frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}\geq\frac{3}{2}\ \forall a,b,c>0\,\big|\,a^2+b^2+c^2\geq a+b+c,$$
$$''='' \text{ iff } a=b=c=1\ (\text{QED})$$

1959. If $x,y,z>0, xy+yz+zx=3$ and $\lambda\geq 2$ then :

$$\sum_{cyc}\sqrt{\lambda x^2+y+\frac{1}{x}}\geq\sqrt{3(\lambda+2)(x+y+z)}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\left(\sum_{cyc}\sqrt{\lambda x^2+y+\frac{1}{x}}\right)^2=$$

$$=\lambda\sum_{cyc} x^2+\sum_{cyc} x+\sum_{cyc}\frac{1}{x}+2\sum_{cyc}\left(\sqrt{\lambda x^2+y+\frac{1}{x}}.\sqrt{\lambda y^2+z+\frac{1}{y}}\right)\overset{Reverse\ CBS}{\geq}$$

$$\geq\lambda\sum_{cyc} x^2+\sum_{cyc} x+\sum_{cyc}\frac{1}{x}+2\lambda\sum_{cyc} xy+2\left(\sum_{cyc}\sqrt{xy}+\sum_{cyc}\frac{1}{\sqrt{xy}}\right)\overset{AM-GM}{\geq}$$

$$\geq\lambda\left(\sum_{cyc} x^2+2\sum_{cyc} xy\right)+6+2(3.2)=\lambda\left(\sum_{cyc} x\right)^2+18\overset{?}{\geq}3(\lambda+2)\left(\sum_{cyc} x\right)$$

$$\Leftrightarrow\lambda\left(\sum_{cyc} x\right)^2-3\lambda\left(\sum_{cyc} x\right)\overset{?}{\geq}6\sum_{cyc} x-18\Leftrightarrow\left(\lambda\left(\sum_{cyc} x\right)-6\right)\left(\sum_{cyc} x-3\right)\overset{?}{\geq}0$$

$$\rightarrow \text{true} \because \sum_{cyc} x \geq \sqrt{3\sum_{cyc} xy} \overset{xy+yz+zx=3}{=} 3 \text{ and so, } \lambda\left(\sum_{cyc} x\right) \overset{\lambda \geq 2}{\geq} 6$$

$$\therefore \sum_{cyc} \sqrt{\lambda x^2 + y + \frac{1}{x}} \geq \sqrt{3(\lambda+2)(x+y+z)}\ \forall\, x,y,z>0 \,\big|\, xy+yz+zx=3$$
$$\wedge\ \lambda \geq 2, '' = '' \text{ iff } x=y=z=1 \text{ (QED)}$$

1960. If $a,b,c>0$ then prove that :

$$\frac{a^3-b^3}{b+c}+\frac{b^3-c^3}{c+a}+\frac{c^3-a^3}{a+b}\geq 0$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{a^3-b^3}{b+c}+\frac{b^3-c^3}{c+a}+\frac{c^3-a^3}{a+b}\geq 0 \Leftrightarrow \sum_{cyc}\left((a^3-b^3)(c+a)(a+b)\right)\geq 0$$

$$\Leftrightarrow \sum_{cyc}\left((a^3-b^3)\left(a^2+\sum_{cyc} ab\right)\right)\geq 0$$

$$\Leftrightarrow \left(\sum_{cyc} ab\right)\left(\sum_{cyc}(a^3-b^3)\right)+\sum_{cyc} a^2(a^3-b^3)\geq 0 \Leftrightarrow \sum_{cyc} a^5 \overset{(*)}{\geq} \sum_{cyc} a^2b^3$$

$$\text{Now, } a^5+a^5+b^5+b^5+b^5 \underbrace{\geq}_{①}{}^{AM-GM} a^2b^3, b^5+b^5+c^5+c^5+c^5 \underbrace{\geq}_{②}{}^{AM-GM} 2b^2c^3 \text{ and}$$

$$c^5+c^5+a^5+a^5+a^5 \underbrace{\geq}_{③}{}^{AM-GM} 2c^2a^3 \text{ and } ①+②+③ \Rightarrow (*) \text{ is true}$$

$$\therefore \frac{a^3-b^3}{b+c}+\frac{b^3-c^3}{c+a}+\frac{c^3-a^3}{a+b}\geq 0\ \forall\, a,b,c>0, '' = '' \text{ iff } a=b=c \text{ (QED)}$$

1961. If $a,b>0, a+b=2$ then:

$$a^{a+1}+b^{b+1}\geq 2$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

Lemma :

$$\forall x>0, x^x\geq x$$

Proof:

$$\textbf{\textit{We need to show}}: x^x \geq x \text{ or } x^{x-1} \geq 1$$

$$\textbf{\textit{Let}}\ f(x) = (x-1)\ln x, x > 0$$

$$\textbf{\textit{If}}\ x = 1\ \textbf{\textit{then}}\ f(x) = (x-1)\ln x = 0$$

$$\textbf{\textit{If}}\ 0 < x < 1\ \textit{then}\ (x-1) < 0, \ln x < 0\ \textit{so}\ f(x) > 0$$

$$\textbf{\textit{If}}\ x > 1, (x-1) > 0, \ln x > 0\ \textit{then}\ f(x) > 0$$

$$\textbf{\textit{thus}}\ (x-1)\ln x \geq 0\ \textbf{\textit{or}}\ \ln x^{(x-1)} \geq \ln 1\ \ \textbf{\textit{or}}\ \ x^{x-1} \geq 1$$

$$a^{a+1} + b^{b+1} = a^a.a + b^b.b \overset{lemma}{\geq} a^2 + b^2 \overset{CBS}{\geq} \frac{(a+b)^2}{2} \overset{a+b=2}{=} 2$$

Equality holds for a=b=1.

1962. If $a, b, c > 0$ then:

$$\frac{a-b}{\sqrt{b+c}} + \frac{b-c}{\sqrt{c+a}} + \frac{c-a}{\sqrt{a+b}} \geq 0$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\textbf{\textit{Let}}\ x^2 = a+b, y^2 = b+c, z^2 = c+a\ \textbf{\textit{then}}\ a+b+c = \frac{1}{2}(x^2+y^2+z^2)$$

$$\textbf{\textit{and}}\ a = (a+b+c) - (b+c) = \frac{1}{2}(x^2+y^2+z^2) - y^2 = \frac{x^2+z^2-y^2}{2}$$

$$\textbf{\textit{Similarly}}:\ b = \frac{x^2+y^2-z^2}{2}, c = \frac{y^2+z^2-x^2}{2}$$

$$\textbf{\textit{Using above relation we get}}:$$

$$\frac{a-b}{\sqrt{b+c}} = \frac{\frac{x^2+z^2-y^2}{2} - \frac{x^2+y^2-z^2}{2}}{y} = \frac{z^2-y^2}{y} = \frac{z^2}{y} - y$$

$$\textbf{\textit{Similarly}}: \frac{b-c}{\sqrt{c+a}} = \frac{x^2}{z} - z, \qquad \frac{c-a}{\sqrt{a+b}} = \frac{y^2}{x} - x$$

$$\frac{a-b}{\sqrt{b+c}} + \frac{b-c}{\sqrt{c+a}} + \frac{c-a}{\sqrt{a+b}} = \frac{z^2}{y} - y + \frac{x^2}{z} - z + \frac{y^2}{x} - x =$$

$$=\left(\frac{x^2}{z}+\frac{y^2}{x}+\frac{z^2}{y}\right)-(x+y+z)\overset{Bergstrom}{\geq}\frac{(x+y+z)^2}{x+y+z}-(x+y+z)=0$$

Equality holds for x=y=z.

1963. If $a,b,c>0,\ a+b+c=3$ then:

$$\frac{a^3+b^2+c}{a^4+b^3+c^2}\leq 1$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$(a^4-a^3)-(a-1)=(a-1)(a^3-1)=(a-1)^2(a^2+a+1)\geq 0$$
$$(a^4-a^3)\geq(a-1)$$

$$(b^3-b^2)-(b-1)=(b-1)(b^2-1)=(b-1)^2(b+1)\geq 0$$
$$(b^3-b^2)\geq b-1$$

$$(c^2-c)-(c-1)=(c-1)^2\geq 0$$
$$(c^2-c)\geq c-1$$

$$We\ need\ to\ show:$$
$$\frac{a^3+b^2+c}{a^4+b^3+c^2}\leq 1\ or\ a^3+b^2+c\leq a^4+b^3+c^2$$

$$or\ (a^4-a^3)+(b^3-b^2)+(c^2-c)\geq 0$$

$$(a^4-a^3)+(b^3-b^2)+(c^2-c)\geq a-1+b-1+c-1=$$
$$=(a+b+c)-3\overset{a+b+c=3}{=}3-3=0$$

Equality holds for a=b=c=1.

1964. If $x,y,z>0,\ xyz=2$ then:

$$x^3+y^3+z^3\geq x\sqrt{y+z}+y\sqrt{z+x}+z\sqrt{x+y}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\sum x\sqrt{y+z}\overset{AM-GM}{\geq}3\sqrt[3]{xyz\sqrt{(x+y)(y+z)(z+x)}}\overset{Cesor}{\geq}3\sqrt[3]{xyz\sqrt{8xyz}}\overset{xyz=2}{=}6$$

$$so, x\sqrt{y+z}+y\sqrt{z+x}+z\sqrt{x+y}\geq 6\ (1)$$

$$\textbf{Schur's inequality}:$$

$$x^3+y^3+z^3+3xyz\geq x^2(y+z)+z^2(x+y)+y^2(z+x)=$$

$$=\left(x\sqrt{y+z}\right)^2+\left(y\sqrt{z+x}\right)^2+\left(z\sqrt{x+y}\right)^2\overset{CBS}{\geq}\frac{1}{3}\left(x\sqrt{y+z}+y\sqrt{z+x}+z\sqrt{x+y}\right)^2=$$

$$=\frac{1}{3}\left(x\sqrt{y+z}+y\sqrt{z+x}+z\sqrt{x+y}\right)\left(x\sqrt{y+z}+y\sqrt{z+x}+z\sqrt{x+y}\right)\overset{(1)}{\geq}$$

$$\geq\frac{1}{3}\times 6\left(x\sqrt{y+z}+y\sqrt{z+x}+z\sqrt{x+y}\right)=2\left(x\sqrt{y+z}+y\sqrt{z+x}+z\sqrt{x+y}\right)$$

$$x^3+y^3+z^3\geq 2\left(x\sqrt{y+z}+y\sqrt{z+x}+z\sqrt{x+y}\right)-3xyz\overset{xyz=2}{=}$$

$$=2\left(x\sqrt{y+z}+y\sqrt{z+x}+z\sqrt{x+y}\right)-6\overset{(1)}{\geq}$$

$$\geq 2\left(x\sqrt{y+z}+y\sqrt{z+x}+z\sqrt{x+y}\right)-\left(x\sqrt{y+z}+y\sqrt{z+x}+z\sqrt{x+y}\right)=$$

$$=\left(x\sqrt{y+z}+y\sqrt{z+x}+z\sqrt{x+y}\right)$$

$$\textbf{Equality holds for } x=y=z=\sqrt[3]{2}.$$

1965. If $a,b,c>0$ then:

$$(a-b)\left(\frac{1}{b}+\frac{1}{c}\right)+(b-c)\left(\frac{1}{c}+\frac{1}{a}\right)+(c-a)\left(\frac{1}{a}+\frac{1}{b}\right)\geq 0$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$(a-b)\left(\frac{1}{b}+\frac{1}{c}\right)+(b-c)\left(\frac{1}{c}+\frac{1}{a}\right)+(c-a)\left(\frac{1}{a}+\frac{1}{b}\right)=$$

$$=\frac{(a-b)+(c-a)}{b}+\frac{(a-b)+(b-c)}{c}+\frac{(b-c)+(c-a)}{a}=$$

$$=\frac{c-b}{b}+\frac{a-c}{c}+\frac{b-a}{a}=$$

$$=\left(\frac{c}{b}-1\right)+\left(\frac{a}{c}-1\right)+\left(\frac{b}{a}-1\right)=\left(\frac{c}{b}+\frac{a}{c}+\frac{b}{a}\right)-3\overset{AM-GM}{\geq}3-3=0$$

Equality holds for a=b=c.

1966. If $a,b,c>0, a+b+c=3$ then:

$$\frac{a^3b^3}{c^3}+\frac{b^3c^3}{a^3}+\frac{c^3a^3}{b^3}\geq 3$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\frac{a^3b^3}{c^3}+\frac{b^3c^3}{a^3}+\frac{c^3a^3}{b^3}=\left(\frac{ab}{c}\right)^3+\left(\frac{bc}{a}\right)^3+\left(\frac{ca}{b}\right)^3\overset{CBS}{\geq}\frac{1}{9}\left(\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b}\right)^3=$$

$$=\frac{1}{9}\left(\frac{a^2b^2+b^2c^2+c^2a^2}{abc}\right)^3\geq\frac{1}{9}\left(\frac{ab.bc+bc.ca+ca.ab}{abc}\right)^3=$$

$$=\frac{1}{9}\left(\frac{abc(a+b+c)}{abc}\right)^3\overset{a+b+c=3}{=}\frac{27}{9}=3$$

Equality holds for a=b=c=1.

1967. If $x,y,z>0$ and $\lambda\geq 2$ then :

$$\sum_{cyc}\frac{yz}{x^2+\lambda yz}\leq\frac{3}{\lambda+1}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{yz}{x^2+\lambda yz}=\frac{1}{\lambda}\cdot\sum_{cyc}\frac{\lambda yz+x^2-x^2}{x^2+\lambda yz}=\frac{3}{\lambda}-\frac{3}{\lambda+1}+\frac{3}{\lambda+1}-\frac{1}{\lambda}\cdot\sum_{cyc}\frac{x^2}{x^2+\lambda yz}$$

$$\overset{Bergstrom}{\leq}\frac{3}{\lambda+1}+\frac{3}{\lambda(\lambda+1)}-\frac{1}{\lambda}\cdot\frac{\left(\sum_{cyc}x\right)^2}{\sum_{cyc}x^2+\lambda\sum_{cyc}xy}\overset{?}{\leq}\frac{3}{\lambda+1}\Leftrightarrow\frac{m+2n}{m+\lambda n}\overset{?}{\geq}\frac{3}{\lambda+1}$$

$$\left(m=\sum_{cyc}x^2\ and\ n=\sum_{cyc}xy\right)\Leftrightarrow m(\lambda+1-3)\overset{?}{\geq}n(3\lambda-2\lambda-2)$$

$$\Leftrightarrow(\lambda-2)(m-n)\overset{?}{\geq}0\rightarrow true\ \because m=\sum_{cyc}x^2\geq n=\sum_{cyc}xy\ and\ \lambda\geq 2$$

$$\therefore\sum_{cyc}\frac{yz}{x^2+\lambda yz}\leq\frac{3}{\lambda+1}\ \forall\ x,y,z>0\ and\ \lambda\geq 2,''='' iff\ x=y=z\ (QED)$$

1968. If $a,b>0$ and $ab=1$ then prove that :

$$(a+1)^{a^2+1}(b+1)^{b^2+1}\geq 16$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$(a+1)^{a^2+1}(b+1)^{b^2+1} = (a+1)(b+1)(a+1)^{a^2}(b+1)^{b^2} \overset{ab=1}{=}$$
$$(1+1+a+b)(a+1)^{a^2}(b+1)^{b^2} \overset{AM-GM}{\geq} (1+1+2\sqrt{ab})(a+1)^{a^2}(b+1)^{b^2}$$
$$\overset{ab=1}{=} 4(a+1)^{a^2}(b+1)^{b^2} \overset{?}{\geq} 16 \Leftrightarrow a^2\ln(a+1)+b^2\ln(b+1) \underset{(*)}{\overset{?}{\geq}} 2\ln 2$$
$$\text{Let } f(x) = x^2\ln(x+1)\ \forall x>0 \text{ and then : } f''(x) =$$
$$2\ln(x+1)+\frac{4x}{x+1}-\frac{x^2}{(x+1)^2} = 2\ln(x+1)+\frac{3x^2+4x}{(x+1)^2} > 0\ (\because x>0)$$
$$\Rightarrow f(x) \text{ is convex} \Rightarrow a^2\ln(a+1)+b^2\ln(b+1) \overset{Jensen}{\geq} 2\left(\frac{a+b}{2}\right)^2.\ln\left(\frac{a+b}{2}+1\right)$$
$$\overset{AM-GM}{\geq} 2\left(\frac{2\sqrt{ab}}{2}\right)^2.\ln\left(\frac{2\sqrt{ab}}{2}+1\right) \overset{ab=1}{=} 2\ln 2 \Rightarrow (*) \text{ is true}$$
$$\therefore (a+1)^{a^2+1}(b+1)^{b^2+1} \geq 16\ \forall a,b>0\,|\,ab=1, ''='' \text{ iff } a=b=1 \text{ (QED)}$$

1969. If $a,b,c>0$ then:

$$\frac{a^2-b^2}{(b+c)^3}+\frac{b^2-c^2}{(c+a)^3}+\frac{c^2-a^2}{(a+b)^2} \geq 0$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$Let\ x=b+c, y=c+a, z=a+b\ then\ a+b+c=\frac{1}{2}(x+y+z)$$
$$a=(a+b+c)-(b+c)=\frac{1}{2}(x+y+z)-x=\frac{y+z-x}{2}$$

$$Similarly:$$
$$b=\frac{x+z-y}{2}, c=\frac{x+y-z}{2}$$

$$Vasc's\ Inequality:$$
$$u,v,w>0\ then\ (u^2+v^2+w^2)^2 \geq 3(u^3v+v^3w+w^3u)$$

$$If\ \ we\ put\ u=xy, v=yz, w=zx\ then:$$

$$(x^2y^2+y^2z^2+z^2x^2)^2 \geq 3((xy)^3yz+(yz)^3zx+(zx)^3xy)\ (1)$$
$$a^2-b^2=\frac{1}{4}((y+z-x)^2-(x+z-y)^2)=z(y-x)\ \ (2)$$

Using above transformation we get:

$$\frac{a^2-b^2}{(b+c)^3}+\frac{b^2-c^2}{(c+a)^3}+\frac{c^2-a^2}{(a+b)^2}=\sum\frac{a^2-b^2}{(b+c)^3}=\sum\frac{z(y-x)}{x^3}=$$

$$=\frac{zy^3z^3(y-x)+xx^3z^3(z-y)+yx^3y^3(x-z)}{x^3y^3z^3}=\frac{\sum z^4y^3(y-x)}{x^3y^3z^3}=$$

$$=\frac{\sum z^4y^4-\sum(yz)^3(xz)}{x^3y^3z^3}\overset{CBS}{\geq}\frac{\frac{1}{3}(x^2y^2+y^2z^2+z^2x^2)^2-\sum(yz)^3(xz)}{x^3y^3z^3}\overset{(1)}{\geq}$$

$$=\frac{((xy)^3yz+(yz)^3zx+(zx)^3xy)\ -\sum(yz)^3(xz)}{x^3y^3z^3}=0$$

Equality holds for x=y=z.

1970. If $x,y,z>0$ then prove that :

$$\sum_{cyc}\sqrt{13x^2-8xy+20y^2}\geq 5(x+y+z)$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\left(\sum_{cyc}\sqrt{13x^2-8xy+20y^2}\right)^2=\sum_{cyc}(13x^2-8xy+20y^2)+$$

$$2.\sum_{cyc}\sqrt{(13x^2-8xy+20y^2)(13y^2-8yz+20z^2)}=33\sum_{cyc}x^2-8\sum_{cyc}xy+$$

$$2\sum_{cyc}\sqrt{(4(x-y)^2+9x^2+16y^2)(4(y-z)^2+9y^2+16z^2)}\geq$$

$$33\sum_{cyc}x^2-8\sum_{cyc}xy+2\sum_{cyc}\sqrt{(9x^2+16y^2)(9y^2+16z^2)}\overset{\text{Reverse CBS}}{\geq}$$

$$33\sum_{cyc}x^2-8\sum_{cyc}xy+2.\sum_{cyc}(9xy+16yz)=33\sum_{cyc}x^2-8\sum_{cyc}xy+50\sum_{cyc}xy$$

$$= 33\sum_{cyc} x^2 + 42\sum_{cyc} xy \overset{?}{\geq} \left(5\sum_{cyc} x\right)^2 = 25\sum_{cyc} x^2 + 50\sum_{cyc} xy$$

$$\Leftrightarrow 8\sum_{cyc} x^2 \overset{?}{\geq} 8\sum_{cyc} xy \to \text{true} \therefore \sum_{cyc} \sqrt{13x^2 - 8xy + 20y^2} \geq 5(x + y + z)$$

$$\forall x, y, z > 0, '' = '' \text{ iff } x = y = z \text{ (QED)}$$

1971. If $a, b, c > 0$, $abc = 1$ then:

$$\frac{1}{\sqrt{a^3 + 2b^3 + 6}} + \frac{1}{\sqrt{b^3 + 2c^3 + 6}} + \frac{1}{\sqrt{c^3 + 2a^3 + 6}} \leq 1$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$a^3 + 2b^3 + 6 = (a^3 + b^3 + 1) + (b^3 + 1 + 1) + 3 \overset{AM-GM}{\geq}$$
$$\geq 3ab + 3b + 3 = 3(ab + b + 1)$$

$$Similarly: b^3 + 2c^3 + 6 \geq 3(bc + c + 1), c^3 + 2a^3 + 6 \geq 3(ac + a + 1)$$

$$\frac{1}{a^3 + 2b^3 + 6} + \frac{1}{a^3 + 2b^3 + 6} + \frac{1}{b^3 + 2c^3 + 6} \leq$$

$$\leq \frac{1}{3(ab + b + 1)} + \frac{1}{3(bc + c + 1)} + \frac{1}{3(ac + a + 1)} =$$

$$= \frac{1}{3}\left(\frac{1}{(ab + b + 1)} + \frac{1}{(bc + c + 1)} + \frac{1}{(ac + a + 1)}\right) \overset{abc=1}{=}$$

$$= \frac{1}{3}\left(\frac{1}{(ab + b + 1)} + \frac{ab}{(abc.b + abc + ab)} + \frac{b}{(abc + ab + b)}\right) =$$

$$\overset{abc=1}{=} \frac{1}{3}\left(\frac{1}{(ab + b + 1)} + \frac{ab}{(b + 1 + ab)} + \frac{b}{(1 + ab + b)}\right) = \frac{1}{3}\frac{1 + ab + b}{(ab + b + 1)} = \frac{1}{3} \ (1)$$

$$\frac{1}{\sqrt{a^3 + 2b^3 + 6}} + \frac{1}{\sqrt{b^3 + 2c^3 + 6}} + \frac{1}{\sqrt{c^3 + 2a^3 + 6}} \overset{CBS}{\leq}$$

$$\leq \sqrt{3\left(\frac{1}{a^3 + 2b^3 + 6} + \frac{1}{a^3 + 2b^3 + 6} + \frac{1}{b^3 + 2c^3 + 6}\right)} \overset{(1)}{\leq} \sqrt{3 \times \frac{1}{3}} = 1$$

$$Equality\ holds\ for\ a=b=c=1.$$

1972. If $a,b,c>0$, $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{abc}$ then:

$$\left(\frac{bc+1}{a^2+1}\right)^2+\left(\frac{ca+1}{b^2+1}\right)^2+\left(\frac{ab+1}{c^2+1}\right)^2\geq 3$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{abc}\Rightarrow ab+bc+ca=1\ (1)$$

$$bc+1\overset{(1)}{=}bc+ab+bc+ca=b(a+c)+c(a+b)$$

$$a^2+1\overset{(1)}{=}a^2+ab+bc+ca=(a+b)(a+c)$$

$$\frac{bc+1}{a^2+1}=\frac{b(a+c)+c(a+b)}{(a+b)(a+c)}=\frac{b}{a+b}+\frac{c}{a+c}$$

$$Similarly:$$

$$\frac{ca+1}{b^2+1}=\frac{c}{b+c}+\frac{a}{b+a},\qquad \frac{ab+1}{a^2+1}=\frac{a}{c+a}+\frac{b}{c+b}$$

$$\frac{bc+1}{a^2+1}+\frac{ca+1}{b^2+1}+\frac{ab+1}{a^2+1}=$$

$$=\left(\frac{b}{a+b}+\frac{c}{a+c}\right)+\left(\frac{c}{b+c}+\frac{a}{b+a}\right)+\left(\frac{a}{c+a}+\frac{b}{c+b}\right)=$$

$$=\sum\left(\frac{b}{a+b}+\frac{a}{a+b}\right)=\sum\frac{a+b}{a+b}=3\ (2)$$

$$\left(\frac{bc+1}{a^2+1}\right)^2+\left(\frac{ca+1}{b^2+1}\right)^2+\left(\frac{ab+1}{c^2+1}\right)^2\overset{CBS}{\geq}\frac{1}{3}\left(\frac{bc+1}{a^2+1}+\frac{ca+1}{b^2+1}+\frac{ab+1}{a^2+1}\right)^2\overset{(2)}{\geq}\frac{1}{3}\times 9=3$$

$$Equality\ holds\ for\ a=b=c=\frac{1}{\sqrt{3}}.$$

1973. If $a,b,c>0$ then:

$$(a-b)\left(\frac{1}{b^2}+\frac{1}{c^2}\right)+(b-c)\left(\frac{1}{c^2}+\frac{1}{a^2}\right)+(c-a)\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\geq 0$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$(a-b)\left(\frac{1}{b^2}+\frac{1}{c^2}\right)+(b-c)\left(\frac{1}{c^2}+\frac{1}{a^2}\right)+(c-a)\left(\frac{1}{a^2}+\frac{1}{b^2}\right)=$$

$$=\left(\frac{a-b}{b^2}+\frac{c-a}{b^2}\right)+\left(\frac{a-b}{c^2}+\frac{b-c}{c^2}\right)+\left(\frac{b-c}{a^2}+\frac{c-a}{a^2}\right)=$$

$$=\frac{c-b}{b^2}+\frac{a-c}{c^2}+\frac{b-a}{a^2}=\left(\frac{c}{b^2}+\frac{a}{c^2}+\frac{b}{a^2}\right)-\left(\frac{1}{b}+\frac{1}{c}+\frac{1}{a}\right)=$$

$$=\frac{\left(\frac{1}{b}\right)^2}{\frac{1}{c}}+\frac{\left(\frac{1}{c}\right)^2}{\frac{1}{a}}+\frac{\left(\frac{1}{a}\right)^2}{\frac{1}{b}}-\left(\frac{1}{b}+\frac{1}{c}+\frac{1}{a}\right)\overset{Bergstrom}{\geq}\frac{\left(\frac{1}{b}+\frac{1}{c}+\frac{1}{a}\right)^2}{\left(\frac{1}{b}+\frac{1}{c}+\frac{1}{a}\right)}-\left(\frac{1}{b}+\frac{1}{c}+\frac{1}{a}\right)=$$

$$=\left(\frac{1}{b}+\frac{1}{c}+\frac{1}{a}\right)-\left(\frac{1}{b}+\frac{1}{c}+\frac{1}{a}\right)=0$$

$Equality\ holds\ for\ a=b=c.$

1974. If $a,b>1, ab=4$ then:

$$\log_a(a^2+4)+\log_b(b^2+4)\geq 6$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$a^2+4\overset{AM-GM}{\geq}4a\ (1)$$
$$b^2+4\overset{AM-GM}{\geq}4b\ (2)$$

$$\log_a(a^2+4)+\log_b(b^2+4)=\frac{\log(a^2+4)}{\log a}+\frac{\log(b^2+4)}{\log b}\overset{(1)\&(2)}{\geq}$$

$$\geq \frac{\log(4a)}{\log a}+\frac{\log(4b)}{\log b}=\frac{log4+loga}{loga}+\frac{log4+logb}{logb}=$$

$$=2+log4\left(\frac{1^2}{loga}+\frac{1^2}{logb}\right)\overset{Bergstrom}{\geq}$$

$$\geq 2+\frac{log4(1+1)^2}{loga+logb}=2+\frac{4log4}{\log(ab)}\overset{ab=4}{=}2+\frac{4log4}{log4}=2+4=6$$

$$Equality\ holds\ for\ a=b=2.$$

1975. If $a,b,c\geq 0$, $a+b+c=1$ then:

$$(1-5ab)^2+(1-5bc)^2+(1-5ca)^2\leq 3$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$Let\ p=\ (1-5ab)^2+(1-5bc)^2+(1-5ca)^2=$$
$$=3-10(ab+bc+ca)+25(a^2b^2+b^2c^2+c^2a^2)=3-10s_1+25s_2$$
$$\left(s_1=ab+bc+ca, s_2=(a^2b^2+b^2c^2+c^2a^2)\right)$$
$$s_1^2=(ab+bc+ca)^2=(a^2b^2+b^2c^2+c^2a^2)+2abc(a+b+c)\overset{a+b+c=1}{=}s_2+2abc$$
$$s_2=s_1^2-2abc\leq s_1^2(as\ a,b,c\geq 0)\ \ again:$$
$$s_1=ab+bc+ca\leq\frac{(a+b+c)^2}{3}\overset{a+b+c=1}{=}\frac{1}{3}$$
$$so\ \ s_1\in\left[0,\frac{1}{3}\right], we\ can\ say\ s_1\leq\frac{1}{3}<\frac{2}{5}, s_1^2=s_1.s_1<s_1.\frac{2}{5}=\frac{2s_1}{5}$$
$$so\ s_2=s_1^2-2abc\leq s_1^2<\frac{2s_1}{5}\ (2)$$
$$p=\ (1-5ab)^2+(1-5bc)^2+(1-5ca)^2=$$
$$=3-10s_1+25s_2\overset{(2)}{\leq}\ 3-10s_1+25\times\frac{2s_1}{5}=3$$

$$Equality\ holds\ for\ (a,b,c)=(1,0,0),(0,1,0),(0,0,1).$$

1976. *If* $x,y,z,r,s,t,u,v,w>0$ *then:*

$$(x^2+ru^2)(y^2+sv^2)(z^2+tw^2)\geq\frac{3}{4}\left(x\sqrt{s}.\sqrt{t}.vw+y\sqrt{r}.\sqrt{t}.uw+z\sqrt{r}.\sqrt{s}.uv\right)^2$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

Solution by Mirsadix Muzefferov-Azerbaijan

$$(x^2+ru^2)(y^2+sv^2)(z^2+tw^2)=$$
$$=(rst)(uvw)^2\left(\frac{x^2}{ru^2}+1\right)\left(\frac{y^2}{sv^2}+1\right)\left(\frac{z^2}{tw^2}+1\right)\overset{Arkady\ Alt's}{\geq}$$
$$\geq\frac{3}{4}\cdot 1^4\cdot\left(\frac{x}{\sqrt{ru}}+\frac{y}{\sqrt{sv}}+\frac{z}{\sqrt{tw}}\right)^2\cdot(rst)(uvw)^2=$$
$$=\frac{3}{4}\left(x\sqrt{s}\cdot\sqrt{t}\cdot vw+y\sqrt{r}\cdot\sqrt{t}\cdot uw+z\sqrt{r}\cdot\sqrt{s}\cdot uv\right)^2$$
$$Equality\ holds\ for:\frac{x}{\sqrt{ru}}=\frac{y}{\sqrt{sv}}=\frac{z}{\sqrt{tw}}=\frac{1}{\sqrt{2}}$$

1977. If $a,b,c>0,\ \lambda\geq 0$ then:

$$\prod\left(\lambda+\frac{a}{b}\right)\geq\lambda^3+1+\lambda(\lambda+1)\frac{a+b+c}{\sqrt[3]{abc}}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$3\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)=\left(\frac{a}{b}+\frac{a}{b}+\frac{b}{c}\right)+\left(\frac{b}{c}+\frac{b}{c}+\frac{c}{a}\right)+\left(\frac{c}{a}+\frac{c}{a}+\frac{a}{b}\right)\overset{AM-GM}{\geq}$$

$$\geq 3\sqrt[3]{\frac{a^2}{bc}}+3\sqrt[3]{\frac{b^2}{ac}}+\sqrt[3]{\frac{c^2}{ab}}=3\sqrt[3]{\frac{a^3}{abc}}+3\sqrt[3]{\frac{b^3}{abc}}+\sqrt[3]{\frac{c^3}{abc}}=3\frac{a+b+c}{\sqrt[3]{abc}}$$

$$\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\geq\frac{a+b+c}{\sqrt[3]{abc}}\ (1)$$

$$\prod\left(\lambda+\frac{a}{b}\right)=\lambda^3+\lambda^2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+\lambda\left(\frac{a}{c}+\frac{b}{a}+\frac{c}{b}\right)+1\overset{(1)}{\geq}$$

$$\geq\lambda^3+1+\lambda^2\frac{a+b+c}{\sqrt[3]{abc}}+\lambda\frac{a+b+c}{\sqrt[3]{abc}}=\lambda^3+1+\lambda(\lambda+1)\frac{a+b+c}{\sqrt[3]{abc}}$$

$$Equality\ holds\ for\ a=b=c.$$

1978. If $x,y,z>0\ ,x+y+z=3\ ,\lambda\geq 0$ then:

$$\prod\left(\lambda+\frac{x}{y}\right)\geq\frac{9\lambda(\lambda+1)}{\sum xy}+\lambda^3+1$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sum\frac{x}{y}=\sum\frac{x^2}{xy}\overset{Bergstrom}{\geq}\frac{(x+y+z)^2}{\sum xy}\overset{x+y+z=3}{=}\frac{9}{\sum xy}\ (1)$$

$$Similarly:\ \sum\frac{x}{z}\geq\frac{9}{\sum xy}\ (2)$$

$$\prod\left(\lambda+\frac{x}{y}\right)=\lambda^3+\lambda^2\sum\frac{x}{y}+\lambda\sum\frac{x}{z}+1\overset{(1)\ \&\ (2)}{\geq}$$
$$\geq\lambda^3+1+\lambda^2\frac{9}{\sum xy}+\lambda\frac{9}{\sum xy}=\frac{9\lambda(\lambda+1)}{\sum xy}+\lambda^3+1$$

$$Equality\ holds\ for\ a=b=c=1.$$

1979. If $a,b>0$ and $a+b=2$, then prove that:

$$\left(\frac{a+1}{b^2}\right)^{a^2}\cdot\left(\frac{b+1}{a^2}\right)^{b^2}\geq 4$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Via Weighted GM − Weighted HM inequality,

$$\sqrt[a^2+b^2]{\left(\frac{a+1}{b^2}\right)^{a^2}\cdot\left(\frac{b+1}{a^2}\right)^{b^2}}\geq\frac{a^2+b^2}{\frac{a^2b^2}{a+1}+\frac{a^2b^2}{b+1}}=\frac{(a^2+b^2)(ab+1+a+b)}{a^2b^2(a+b+2)}$$
$$\overset{a+b=2}{=}\frac{(a^2+b^2)(ab+3)}{4a^2b^2}=(a^2+b^2)\left(\frac{1}{4ab}+\frac{3}{4a^2b^2}\right)\overset{AM-GM}{\geq}$$
$$(a^2+b^2)\left(\frac{1}{(a+b)^2}+\frac{12}{(a+b)^4}\right)\geq\frac{1}{2}(a+b)^2.\left(\frac{1}{(a+b)^2}+\frac{12}{(a+b)^4}\right)$$
$$=\frac{1}{2}+\frac{6}{(a+b)^2}\overset{a+b=2}{=}\frac{1}{2}+\frac{3}{2}=2\Rightarrow\left(\frac{a+1}{b^2}\right)^{a^2}\cdot\left(\frac{b+1}{a^2}\right)^{b^2}\geq 2^{a^2+b^2}\geq 2^{\frac{(a+b)^2}{2}}$$
$$(\because 2>1)\overset{a+b=2}{=}2^2=4\therefore\left(\frac{a+1}{b^2}\right)^{a^2}\cdot\left(\frac{b+1}{a^2}\right)^{b^2}\geq 4\ \forall\ a,b>0\,|\,a+b=2,$$
$$''='' \text{ iff } a=b=1\ (QED)$$

1980. If $a,b,c>0$ and $a^3b^3+b^3c^3+c^3a^3\le a^2b^2+b^2c^2+c^2a^2$

then prove that : $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\ge 3$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\overset{?}{\ge}3\Leftrightarrow 1\overset{?}{\ge}\frac{3abc}{\sum_{cyc}ab}\Leftrightarrow 1\overset{?}{\ge}\frac{9a^2b^2c^2}{\left(\sum_{cyc}ab\right)^2}\ \text{and}\ \because 1\ge\frac{\sum_{cyc}a^3b^3}{\sum_{cyc}a^2b^2}$$

$$\therefore \text{it suffices to prove}: \frac{\sum_{cyc}a^3b^3}{\sum_{cyc}a^2b^2}\overset{?}{\ge}\frac{9a^2b^2c^2}{\left(\sum_{cyc}ab\right)^2}\ \text{and}\ \because \sum_{cyc}a^3b^3\overset{\text{Chebyshev}}{\ge}$$

$$\frac{1}{3}\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}ab\right)\therefore \text{it suffices to prove}: \frac{\frac{1}{3}\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}ab\right)}{\sum_{cyc}a^2b^2}\overset{?}{\ge}\frac{9a^2b^2c^2}{\left(\sum_{cyc}ab\right)^2}$$

$$\Leftrightarrow\left(\sum_{cyc}ab\right)^3\overset{?}{\ge}27a^2b^2c^2\Leftrightarrow\sum_{cyc}ab\overset{?}{\ge}3.\sqrt[3]{abc}\rightarrow\text{true via } AM-GM$$

$$\therefore \frac{1}{a}+\frac{1}{b}+\frac{1}{c}\ge 3\ \forall\ a,b,c>0\ \big|\ a^3b^3+b^3c^3+c^3a^3\le a^2b^2+b^2c^2+c^2a^2,$$

$$''='' \text{ iff } a=b=c=1\ (QED)$$

1981. If $a,b,c>0$ and $\frac{1}{2}\le\lambda\le 2$ then :

$$\sum_{cyc}\frac{a^2}{b^2+\lambda ac}\ge\frac{3}{\lambda+1}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{a^2}{b^2+\lambda ac}=\sum_{cyc}\frac{a^4}{a^2b^2+\lambda a^3c}\overset{\text{Bergstrom}}{\ge}\frac{\left(\sum_{cyc}a^2\right)^2}{\sum_{cyc}a^2b^2+\lambda\sum_{cyc}a^3c}$$

$$\overset{Vasc}{\ge}\frac{\left(\sum_{cyc}a^2\right)^2}{\sum_{cyc}a^2b^2+\lambda.\frac{\left(\sum_{cyc}a^2\right)^2}{3}}\ge\frac{\left(\sum_{cyc}a^2\right)^2}{\frac{\left(\sum_{cyc}a^2\right)^2}{3}+\lambda.\frac{\left(\sum_{cyc}a^2\right)^2}{3}}=\frac{3}{\lambda+1}$$

$$\therefore \sum_{cyc} \frac{a^2}{b^2+\lambda ac} \geq \frac{3}{\lambda+1} \ \forall\, a,b,c>0 \ and \ \frac{1}{2} \leq \lambda \leq 2, \text{"}=\text{"} \ iff \ a=b=c \ (QED)$$

1982. If $a,b,c \geq 0$ and $a+b+c=1$ then prove that :

$$\sqrt{3a+1}+\sqrt{3b+1}+\sqrt{3c+1} \geq 4$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$(LHS)^2 = 3\sum_{cyc} a + 3 + 2\sum_{cyc} \sqrt{(3a+1)(3b+1)}$$
$$\overset{a+b+c=1}{=} 6 + 2\sum_{cyc} \sqrt{(4a+b+c)(4b+c+a)}$$
$$= 6 + 2\sum_{cyc} \sqrt{(2a+2b+c)^2 + 9a + c(a+b)} \overset{a,b,c\geq 0}{\geq} 6 + 2\sum_{cyc} (2a+2b+c)$$
$$= 6 + 10\sum_{cyc} a \overset{a+b+c=1}{=} 16 \therefore \sum_{cyc} \sqrt{3a+1} \geq 4 \ \forall\, a,b,c \geq 0 \,|\, a+b+c=1,$$

$$\text{"}=\text{"} \ iff \begin{pmatrix} a=1 \\ b=c=0 \end{pmatrix} or \begin{pmatrix} b=1 \\ c=a=0 \end{pmatrix} or \begin{pmatrix} c=1 \\ a=b=0 \end{pmatrix} (QED)$$

1983. If $a,b,c>0$ and $abc=1$ then prove that :

$$\frac{a}{a^4+b+c} + \frac{b}{b^4+c+a} + \frac{c}{c^4+a+b} \leq \frac{3(ab+bc+ca)}{(a+b+c)^2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$Via \ Weighted \ AM - Weighted \ HM \ inequality, a^4+b+c = \frac{a^3.a+1.b+1.c}{a+b+c} \geq \frac{a+b+c}{\frac{a}{a^3}+\frac{b}{1}+\frac{c}{1}} \Rightarrow \frac{a}{a^4+b+c} \leq \frac{1}{(a+b+c)^2}.\left(\frac{1}{a}+ab+ac\right)$$
$$\overset{abc=1}{=} \frac{1}{(a+b+c)^2}.(bc+ab+ac) \ and \ analogs$$
$$\Rightarrow \sum_{cyc} \frac{a}{a^4+b+c} \leq \frac{3(ab+bc+ca)}{(a+b+c)^2} \forall\, a,b,c>0 \,|\, abc=1,$$
$$\text{"}=\text{"} \ iff \ a=b=c=1 \ (QED)$$

1984. If $a,b,c>0$, $a+b+c=3$ then:

$$\left(\frac{a+1}{b^2}\right)^3+\left(\frac{b+1}{c^2}\right)+\left(\frac{c+1}{a^2}\right)\geq 24$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\sum\frac{a+1}{b^2}=\sum\frac{a}{b^2}+\sum\frac{1}{b^2}=\sum\frac{\left(\frac{1}{b}\right)^2}{\frac{1}{a}}+\sum\frac{1^3}{b^2}\overset{Bergstrom\ \&\ Radon}{\geq}$$

$$\geq\frac{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}+\frac{(1+1+1)^3}{a+b+c}=\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)+\frac{3^3}{a+b+c}\overset{a+b+c=3}{=}$$

$$=\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)+\frac{(a+b+c)^3}{a+b+c}=$$

$$=\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)+(a+b+c)\overset{AM-GM}{\geq}2\sqrt{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)(a+b+c)}\overset{C-S}{\geq}2\sqrt{9}=6\ (1)$$

$$\left(\frac{a+1}{b^2}\right)^3+\left(\frac{b+1}{c^2}\right)+\left(\frac{c+1}{a^2}\right)=\sum\left(\frac{a+1}{b^2}\right)^3\overset{CBS}{\geq}\frac{1}{9}\left(\sum\frac{a+1}{b^2}\right)^3\overset{(1)}{\geq}\frac{1}{9}\times 6^3=24$$

$$Equality\ holds\ for\ a=b=c=1.$$

1985. If $a,b,c>0$, $a+b+c=3$ then:

$$\left(\frac{a^3+2}{b}\right)^a\left(\frac{b^3+2}{c}\right)^b\left(\frac{c^3+2}{a}\right)^c\geq 27$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\sum\frac{ab}{a^3+2}=\sum\frac{ab}{a^3+1+1}\overset{AM-GM}{\leq}\sum\frac{ab}{3a}=\frac{1}{3}(a+b+c)\ (1)$$

$$\left(\frac{a^3+2}{b}\right)^a\left(\frac{b^3+2}{c}\right)^b\left(\frac{c^3+2}{a}\right)^c \overset{GM-HM}{\geq} \left(\frac{a+b+c}{\frac{a}{\frac{a^3+2}{b}}+\frac{b}{\frac{b^3+2}{c}}+\frac{c}{\frac{c^3+2}{a}}}\right)^{a+b+c} =$$

$$= \left(\frac{a+b+c}{\sum\frac{ab}{a^3+2}}\right)^{a+b+c} \overset{(1)}{\geq} = \left(\frac{a+b+c}{\frac{1}{3}(a+b+c)}\right)^{a+b+c} = 3^{a+b+c} \overset{a+b+c=3}{=} 27$$

$$\textbf{Equality holds for } a=b=c=1.$$

1986.

$$\textit{If } a,b,c>0 \textit{ and } a+b+c=3 \textit{ then :}$$

$$\sqrt{\frac{a}{3+6c}}+\sqrt{\frac{b}{3+6a}}+\sqrt{\frac{c}{3+6b}}\geq\sqrt[3]{abc}$$

Proposed by Gheorghe Crăciun-Romania

Solution by Mirsadix Muzefferov-Azerbaijan

$$LHS^2=\left(\sum_{cyc}\sqrt{\frac{a}{3+6c}}\right)^2=\sum_{cyc}\frac{a}{3+6c}+2\sum_{cyc}\sqrt{\frac{ab}{(3+6c)(3+6a)}}$$

$$LHS_1=\sum_{cyc}\frac{a}{3+6c}=\sum_{cyc}\frac{a^2}{3a+6ac}\overset{Bergstrom}{\geq}\frac{\left(\sum_{cyc}a\right)^2}{3\sum_{cyc}a+6\sum_{cyc}ab}\overset{(*)}{\geq}$$

$$a+b+c=3\ \rightarrow\ (a+b+c)^2=\sum_{cyc}a^2+2\sum_{cyc}ab\geq$$

$$\sum_{cyc}ab+2\sum_{cyc}ab=3\sum_{cyc}ab\rightarrow\sum_{cyc}ab\leq 3\ \ (*)$$

$$\overset{(*)}{\geq}\frac{9}{9+18}=\frac{1}{3}\ \rightarrow\ LHS_1\geq\frac{1}{3}$$

$$\sqrt{(3+6c)(3+6a)}\overset{A-G}{\leq}\frac{(3+6c)+(3+6a)}{2}=3+3(a+c)=12-3b$$

$$LHS_2=2.\sum_{cyc}\sqrt{\frac{ab}{(3+6c)(3+6a)}}\overset{A-G}{\geq}2\sum_{cyc}\frac{\left(\sqrt[4]{ab}\right)^2}{12-3b}\overset{Bergstrom}{\geq}2.\frac{\left(\sum_{cyc}\sqrt[4]{ab}\right)^2}{36-3\sum_{cyc}a}=$$

$$=\frac{2.\left(\sum_{cyc}\sqrt[4]{ab}\right)^2}{27}\overset{A-G}{\geq}\frac{2\left(3\sqrt[12]{a^2b^2c^2}\right)^2}{27}=\frac{2}{3}\sqrt[3]{abc}$$

$$LHS_2\geq\frac{2}{3}\sqrt[3]{abc}\ ;\ a+b+c=3\geq 3\sqrt[3]{abc}\rightarrow\sqrt[3]{abc}\leq 1\ \ (**)$$

$$LHS_1 \geq \frac{1}{3} \overset{(**)}{\geq} \frac{1}{3}\sqrt[3]{abc}$$

$$LHS^2 = LHS_1 + LHS_2 \geq \frac{1}{3}\sqrt[3]{abc} + \frac{2}{3}\sqrt[3]{abc} = \sqrt[3]{abc}$$

$$LHS \geq \sqrt{\sqrt[3]{abc}} = \sqrt[6]{abc} \geq \sqrt[3]{abc}$$

$$Equality\ holds\ for:\ a = b = c = 1.$$

1987. If $x, y, z > 0, x + y + z = 3$ and $\lambda \geq 2$ then :

$$\sum_{cyc} \frac{\lambda x^2 + 1}{\sqrt{x^2 - x + \lambda^2}} \geq \frac{3(\lambda + 1)}{\lambda}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

We shall prove that : $\frac{\lambda x^2 + 1}{\sqrt{x^2 - x + \lambda^2}} \overset{?}{\geq} \frac{\lambda + 1}{\lambda} + (x - 1).\frac{4\lambda^3 - \lambda - 1}{2\lambda^3}$

$= \frac{(4\lambda^3 - \lambda - 1)x - (2\lambda^3 - 2\lambda^2 - \lambda - 1)}{2\lambda^3}$ and it's trivially true if :

$x \leq \frac{2\lambda^3 - 2\lambda^2 - \lambda - 1}{4\lambda^3 - \lambda - 1}$ and if : $x > \frac{2\lambda^3 - 2\lambda^2 - \lambda - 1}{4\lambda^3 - \lambda - 1}$, it's equivalent to :

$$\frac{(\lambda x^2 + 1)^2}{x^2 - x + \lambda^2} \overset{?}{\geq} \frac{4\lambda^4(\lambda + 1)^2 + (x - 1)^2(4\lambda^3 - \lambda - 1)^2 + 4\lambda^2(\lambda + 1)(4\lambda^3 - \lambda - 1)(x - 1)}{4\lambda^6}$$

$$\Leftrightarrow (x - 1).\begin{pmatrix} 4\lambda^4(\lambda^4(x^2 + 1)(x + 1) + 2\lambda^3(x + 1) - x(\lambda + 1)^2) - \\ 4\lambda^2(\lambda + 1)(4\lambda^3 - \lambda - 1)(x^2 - x + \lambda^2) - \\ (x - 1)(4\lambda^3 - \lambda - 1)^2(x^2 - x + \lambda^2) \end{pmatrix} \overset{?}{\geq} 0$$

$$\Leftrightarrow (x - 1)^2.\left(4\lambda^2\begin{pmatrix} \lambda^6(x^2 + 2x + 3) + 2\lambda^5 - \lambda^4(4x + 1) - \\ \lambda^3(4x + 2) + \lambda^2(x - 1) + 2\lambda x + x \\ -(4\lambda^3 - \lambda - 1)^2(x^2 - x + \lambda^2) \end{pmatrix}\right) \overset{?}{\geq} 0$$

$$\Leftrightarrow (x - 1)^2.T \overset{?}{\underset{(*)}{\geq}} 0; \text{ where } T = (4\lambda^8 - 16\lambda^6 + 8\lambda^4 + 8\lambda^3 - \lambda^2 - 2\lambda - 1)x^2 +$$

$$(8\lambda^8 - 16\lambda^5 - 4\lambda^4 + 5\lambda^2 + 2\lambda + 1)x - (4\lambda^8 - 8\lambda^7 - 4\lambda^6 + 5\lambda^4 + 2\lambda^3 + \lambda^2)$$

Now, since $\lambda \geq 2$, we have : $4\lambda^8 - 16\lambda^6 + 8\lambda^4 + 8\lambda^3 - \lambda^2 - 2\lambda - 1 =$

$$4\lambda^6(\lambda^2 - 4) + 4\lambda^4 + 8\lambda^3 + (2\lambda^2 + \lambda + 1)(\lambda - 1)(2\lambda + 1) > 0$$

and $8\lambda^8 - 16\lambda^5 - 4\lambda^4 + 5\lambda^2 + 2\lambda + 1 =$

$$(\lambda - 2)(8\lambda^7 + 16\lambda^6 + 32\lambda^5 + 48\lambda^4 + 92\lambda^3 + 184\lambda^2 + 373\lambda + 748) + 1497 > 0$$

and so, $x > \frac{2\lambda^3 - 2\lambda^2 - \lambda - 1}{4\lambda^3 - \lambda - 1} \Rightarrow$

$$T > (4\lambda^8 - 16\lambda^6 + 8\lambda^4 + 8\lambda^3 - \lambda^2 - 2\lambda - 1).\left(\frac{2\lambda^3 - 2\lambda^2 - \lambda - 1}{4\lambda^3 - \lambda - 1}\right)^2 +$$

$$(8\lambda^8 - 16\lambda^5 - 4\lambda^4 + 5\lambda^2 + 2\lambda + 1).\frac{2\lambda^3 - 2\lambda^2 - \lambda - 1}{4\lambda^3 - \lambda - 1} -$$

$$(4\lambda^8 - 8\lambda^7 - 4\lambda^6 + 5\lambda^4 + 2\lambda^3 + \lambda^2) \overset{?}{>} 0$$

$$\Leftrightarrow 16\lambda^{12} + 32\lambda^{11} - 16\lambda^{10} - 64\lambda^9 - 8\lambda^8 + 56\lambda^7 +$$

$$32\lambda^6 - 16\lambda^5 - 23\lambda^4 - 4\lambda^3 + 6\lambda^2 + 4\lambda + 1 \overset{?}{>} 0 \Leftrightarrow$$

$$(\lambda - 2)\left(\begin{matrix}16\lambda^{11} + 64\lambda^{10} + 112\lambda^9 + 160\lambda^8 + 312\lambda^7 + 680\lambda^6 + 1392\lambda^5 + \\ 2768\lambda^4 + 5513\lambda^3 + 11022\lambda^2 + 22050\lambda + 44104\end{matrix}\right) + 88209$$

$$\overset{?}{>} 0 \to \text{true} \because \lambda \geq 2 \Rightarrow T > 0 \Rightarrow (*) \text{ is true}$$

$$\therefore \frac{\lambda x^2 + 1}{\sqrt{x^2 - x + \lambda^2}} \geq \frac{\lambda + 1}{\lambda} + (x - 1).\frac{4\lambda^3 - \lambda - 1}{2\lambda^3} \text{ and analogs} \Rightarrow$$

$$\sum_{cyc} \frac{\lambda x^2 + 1}{\sqrt{x^2 - x + \lambda^2}} \geq \frac{3(\lambda + 1)}{\lambda} + \left(\sum_{cyc} x - 3\right).\frac{4\lambda^3 - \lambda - 1}{2\lambda^3} \overset{x+y+z=3}{=} \frac{3(\lambda + 1)}{\lambda}$$

$$\forall x, y, z > 0 \,|\, x + y + z = 3 \text{ and } \lambda \geq 2, '' = '' \text{ iff } x = y = z = 1 \text{ (QED)}$$

1988.

If $a, b, c > 0$ and $\lambda \geq 0$ then :

$$\sum_{cyc} \frac{a^3}{a^2 + \lambda ab + b^2} \geq \frac{3\sqrt[3]{abc}}{\lambda + 2}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chaktraborty-Kolkata-India

$$\sum_{cyc} \frac{a^3}{a^2 + \lambda ab + b^2} \overset{AM-GM}{\geq} \sum_{cyc} \frac{a^3}{a^2 + \lambda.\frac{a^2 + b^2}{2} + b^2} = \sum_{cyc} \frac{2a^3}{(\lambda + 2)(a^2 + b^2)}$$

$$= \sum_{cyc} \frac{2a\left((a^2 + b^2) - b^2\right)}{(\lambda + 2)(a^2 + b^2)} = \frac{2}{\lambda + 2}.\sum_{cyc} a - \frac{2}{\lambda + 2}.\sum_{cyc} \frac{ab^2}{a^2 + b^2}$$

$$\overset{AM-GM}{\geq} \frac{2}{\lambda + 2}.\sum_{cyc} a - \frac{2}{\lambda + 2}.\sum_{cyc} \frac{ab^2}{2ab} = \frac{2}{\lambda + 2}.\sum_{cyc} a - \frac{1}{\lambda + 2}.\sum_{cyc} b = \frac{1}{\lambda + 2}.\sum_{cyc} a$$

$$\overset{AM-GM}{\geq} \frac{3.\sqrt[3]{abc}}{\lambda + 2} \text{ and so, } \sum_{cyc} \frac{a^3}{a^2 + \lambda ab + b^2} \geq \frac{3.\sqrt[3]{abc}}{\lambda + 2} \; \forall a, b, c > 0 \text{ and } \lambda \geq 0,$$

$''='' \text{ iff } a=b=c \text{ (QED)}$

1989. If $a,b,c>0$ and $\lambda\geq 1$ then :

$$\sum_{cyc}\frac{b+c}{\lambda a+b+c}\geq\frac{6}{\lambda+2}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{b+c}{\lambda a+b+c}\geq\frac{6}{\lambda+2}\Leftrightarrow$$

$$(\lambda+2)\sum_{cyc}\big((b+c)(\lambda b+c+a)(\lambda c+a+b)\big)\geq 6\prod_{cyc}(\lambda a+b+c)$$

$$\Leftrightarrow 2\lambda(\lambda-1)\left(\sum_{cyc}a^3\right)+\lambda(\lambda-1)^2\left(\sum_{cyc}a^2b+\sum_{cyc}ab^2\right)-6(\lambda^3-\lambda^2)abc\overset{(*)}{\geq}0$$

$$\text{Now, } \lambda\geq 1 \text{ and } \sum_{cyc}a^3\overset{A-G}{\geq}3abc \text{ and } \sum_{cyc}a^2b+\sum_{cyc}ab^2\overset{A-G}{\geq}6abc \text{ and so,}$$

$$\text{LHS of } (*)\geq\Big(6\lambda(\lambda-1)+6\lambda(\lambda-1)^2-6(\lambda^3-\lambda^2)\Big)abc=0 \text{ as desired and so,}$$

$$\sum_{cyc}\frac{b+c}{\lambda a+b+c}\geq\frac{6}{\lambda+2}\ \forall\, a,b,c>0 \text{ and } \lambda\geq 1,'' ='' \text{ iff } a=b=c \text{ (QED)}$$

1990. If $a,b>0, a+b=2$ and $0\leq\lambda\leq\frac{3}{2}$ then :

$$(\lambda+a^2)(\lambda+b^2)\leq\left(\lambda+\frac{1}{\sqrt{ab}}\right)^2$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$(\lambda+a^2)(\lambda+b^2)\leq\left(\lambda+\frac{1}{\sqrt{ab}}\right)^2\Leftrightarrow\lambda((a+b)^2-2ab)+a^2b^2\leq\frac{1}{ab}+\frac{2\lambda}{\sqrt{ab}}$$

$$\overset{a+b=2}{\Leftrightarrow}2\lambda\left(\frac{1}{\sqrt{ab}}+ab-2\right)+\frac{1}{ab}-a^2b^2\overset{(*)}{\geq}0$$

$$\text{Now, } 2=a+b\overset{A-G}{\geq}2\sqrt{ab}\Rightarrow ab\leq 1\Rightarrow 1\geq a^3b^3\Rightarrow\frac{1}{ab}-a^2b^2\geq 0 \text{ and if :}$$

$\frac{1}{\sqrt{ab}}+ab-2\geq 0$, then, as $\lambda\geq 0\therefore(*)$ is true and if : $\frac{1}{\sqrt{ab}}+ab-2<0$,

then, as $\lambda\leq\frac{3}{2}$, $\therefore$ LHS of $(*)\geq 3\left(\frac{1}{x}+x^2-2\right)+\frac{1}{x^2}-x^4\overset{?}{\geq}0\ \left(x=\sqrt{ab}\right)$

$$\Leftrightarrow 1+3x-6x^2+3x^4-x^6\overset{?}{\geq}0$$

$$\Leftrightarrow (1-x)\left(1+2x(1-x)+2x(1-x^2)+1+x^5+x^4\right)\overset{?}{\geq}0\rightarrow\text{true}$$

$\because ab\leq 1\Rightarrow 1-x\geq 0$ and $a,b>0\Rightarrow x>0\Rightarrow(*)$ is true and so,

$$(\lambda+a^2)(\lambda+b^2)\leq\left(\lambda+\frac{1}{\sqrt{ab}}\right)^2\ \forall a,b>0\,\big|\,a+b=2\text{ and }0\leq\lambda\leq\frac{3}{2},$$

$''=''$ iff $a=b=1$ (QED)

1991. If $a,b,c>=\sum_{cyc}ab^{2n}$ and $n\in\mathbb{N}^*$ then :

$$\sum_{cyc}\frac{a^{2n}}{b}+\sum_{cyc}a\geq 2\sum_{cyc}a^{2n}\,0,\sum_{cyc}a^{6n-1}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

We first consider $n=1$ and then : $\sum_{cyc}a^5=\sum_{cyc}ab^2\overset{AM-GM}{\leq}\sum_{cyc}a^3\Rightarrow 1\geq$

$$\sqrt{\frac{\sum_{cyc}a^5}{\sum_{cyc}a^3}}\rightarrow(i);\ \text{LHS}=\sum_{cyc}\frac{a^2}{b}+\sum_{cyc}a\overset{\text{Bergstrom}}{\geq}2\sum_{cyc}a\overset{\text{via (i)}}{\geq}2\sqrt{\frac{\left(\sum_{cyc}a\right)^2\left(\sum_{cyc}a^5\right)}{\sum_{cyc}a^3}}$$

$$\overset{\text{Reverse CBS}}{\geq}2\sqrt{\frac{\left(\sum_{cyc}a^3\right)^2\left(\sum_{cyc}a\right)}{\sum_{cyc}a^3}}=2\sqrt{\left(\sum_{cyc}a^3\right)\left(\sum_{cyc}a\right)}$$

$$\overset{\text{Reverse CBS}}{\geq}2\sum_{cyc}a^2=\text{RHS and we now focus on the case when : }n\in(\mathbb{N}^*-\{1\})$$

Now, $\frac{b^{3n}+b^{3n}+a^3}{3}\overset{AM-GM}{\geq}ab^{2n}$ and analogs $\Rightarrow\frac{2\sum_{cyc}a^{3n}+\sum_{cyc}a^3}{3}\geq\sum_{cyc}ab^{2n}$

$$=\sum_{cyc}a^{6n-1}=\sum_{cyc}\frac{a^{6n}}{a}\overset{\text{Bergstrom}}{\geq}\frac{\left(\sum_{cyc}a^{3n}\right)^2}{\sum_{cyc}a}\Rightarrow 3t^2-2\left(\sum_{cyc}a\right)t-\left(\sum_{cyc}a\right)\left(\sum_{cyc}a^3\right)\leq 0$$

$$\left(t=\sum_{cyc}a^{3n}\right)\Rightarrow t\leq\frac{2\left(\sum_{cyc}a\right)+\sqrt{4\left(\sum_{cyc}a\right)^2+12\left(\sum_{cyc}a\right)\left(\sum_{cyc}a^3\right)}}{6}$$

$$\Rightarrow \sum_{cyc} a^{3n} \overset{(*)}{\leq} \frac{\sum_{cyc} a + \sqrt{\left(\sum_{cyc} a\right)^2 + 3\left(\sum_{cyc} a\right)\left(\sum_{cyc} a^3\right)}}{3} \text{ \& via Power Mean Inequality \&}$$

$$\because n \geq 2 \therefore x = \left(\frac{\sum_{cyc} a^{3n}}{3}\right)^{\frac{1}{3n}} \geq \left(\frac{\sum_{cyc} a^3}{3}\right)^{\frac{1}{3}} \Rightarrow 3x^3 \geq \sum_{cyc} a^3 \geq \frac{\left(\sum_{cyc} a\right)^3}{9} \Rightarrow \sum_{cyc} a \leq 3x$$

$$\Rightarrow (*) \Rightarrow 3x^{3n} \leq x + \sqrt{x^2 + 3x^4} \Rightarrow x\left(x^{3n-1} - 1\right) + \frac{3x^4\left(x^{6n-4} - 1\right) + x^2\left(x^{6n-2} - 1\right)}{2x^{3n} + \sqrt{x^2 + 3x^4}}$$

$$\leq 0 \text{ and } \because n \in (\mathbb{N}^* - \{1\}), x^{3n-1} - 1 = (x-1)(A), x^{6n-4} - 1 = (x-1)(B) \text{ and}$$

$$x^{6n-2} - 1 = (x-1)(C) \text{ (where } A, B, C > 0) \text{ and so, } x \leq 1 \Rightarrow \sum_{cyc} a^{3n} \overset{①}{\leq} 3 \Rightarrow \sum_{cyc} a^3 \overset{②}{\leq} 3$$

$$\text{Via PMI \& } \because n \geq 2, \left(\frac{\sum_{cyc} a^{3n}}{3}\right)^{\frac{1}{3n}} \geq \left(\frac{\sum_{cyc} a^{2n-3}}{3}\right)^{\frac{1}{2n-3}} \text{ \& } \left(\frac{\sum_{cyc} a^3}{3}\right)^{\frac{1}{3}} \geq \left(\frac{\sum_{cyc} a^2}{3}\right)^{\frac{1}{2}}$$

$$\text{\& ① \& ②} \Rightarrow \sum_{cyc} a^{2n-3} \overset{③}{\leq} 3 \text{ \& } \sum_{cyc} a^2 \overset{④}{\leq} 3 \text{ \& } \sum_{cyc} \frac{a^{2n}}{b} + \sum_{cyc} a \geq \frac{\left(\sum_{cyc} a^{2n}\right)^2}{\sum_{cyc} ba^{2n}} + \sum_{cyc} a$$

$$\overset{?}{\geq} 2\sum_{cyc} a^{2n} \Leftrightarrow m^2 - 2\left(\sum_{cyc} ba^{2n}\right)m + \left(\sum_{cyc} a\right)\left(\sum_{cyc} ba^{2n}\right) \overset{?}{\geq} 0 \left(m = \sum_{cyc} a^{2n}\right)$$

$$\text{\& suffices to prove : } 4\left(\sum_{cyc} ba^{2n}\right)^2 \overset{?}{\leq} 4\left(\sum_{cyc} a\right)\left(\sum_{cyc} ba^{2n}\right) \text{ and } \because \sum_{cyc} ba^{2n} \overset{CBS}{\leq}$$

$$\sqrt{\left(\sum_{cyc} a^{4n-2}\right)} \therefore \text{ suffices prove : } \left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a^{4n-2}\right) \overset{?}{\geq} \left(\sum_{cyc} a\right)^2 \quad (\bullet)$$

$$\text{Via CBS \& } Bergstrom, \sqrt{\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a^{4n-2}\right)} \geq \sum_{cyc} ab^{2n} = \sum_{cyc} \frac{a^{8n-4}}{a^{2n-3}} \geq \frac{\left(\sum_{cyc} a^{4n-2}\right)^2}{\sum_{cyc} a^{2n-3}}$$

$$\geq \frac{\left(\sum_{cyc} a^{4n-2}\right)^2}{3}\left(\sum_{cyc} a^2b^2\right) \Rightarrow \sum_{cyc} a^{4n-2} \leq \sqrt[3]{9\sum_{cyc} a^2b^2} \overset{?}{\leq} \frac{\left(\sum_{cyc} a\right)^2}{\sum_{cyc} a^2b^2}$$

$$\Leftrightarrow \left(\sum_{cyc} a\right)^3 \overset{?}{\geq} 3\left(\sum_{cyc} a^2b^2\right)^2 \quad (\bullet\bullet)$$

$$\text{and } \because 1 \overset{\text{via ② and ④}}{\geq} \frac{\left(\sum_{cyc} a^3\right)\left(\sum_{cyc} a^2\right)}{9} \therefore \text{ in order to prove } (\bullet\bullet), \text{ it suffices to prove :}$$

$$\left(\sum_{cyc} a^3\right)\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a\right)^3 \overset{?}{\geq} 27\left(\sum_{cyc} a^2b^2\right)^2 \quad (\bullet\bullet\bullet) \text{ \& assigning } b + c = X, c + a = Y,$$

$a+b=Z\Rightarrow X,Y,Z$ form sides of ΔXYZ with s,R,r (say); then : $\sum_{cyc} a^2=s^2-8Rr-2r^2$,

$\sum_{cyc} a^3=s^3-12Rrs, \sum_{cyc} a^2b^2=r^2\left((4R+r)^2-2s^2\right)\Rightarrow(\bullet\bullet\bullet)\Leftrightarrow s^8-\left(20Rr+2r^2\right)s^6+$

$r^2\left(96R^2+24Rr-108r^2\right)s^4+r^4\left(1728R^2+864Rr+108r^2\right)s^2-27r^4(4R+r)^4\overset{?}{\geq}0$ &

$U=s^2-16Rr+5r^2\Rightarrow P=U^4+\left(44Rr-22r^2\right)U^3+2r^2\left(336R^2-366Rr+36r^2\right)U^2+$

$2r^3\left(2048R^3-2880R^2r+714Rr^2+269r^3\right)U\overset{\text{Gerretsen}}{\geq}0\therefore$ to prove $(\bullet\bullet\bullet\bullet)$, suffices :

$\text{LHS of }(\bullet\bullet\bullet\bullet)\overset{?}{\geq}P\Leftrightarrow 320t^4-448t^3-1104t^2+1739t-598\overset{?}{\geq}0\left(t=\frac{R}{r}\right)$

$\Leftrightarrow(t-2)\left(320t^3+192t^2-720t+299\right)\overset{?}{\geq}0\rightarrow\text{true}\because t\overset{\text{Euler}}{\geq}2\Rightarrow(\bullet\bullet\bullet\bullet)\Rightarrow(\bullet\bullet\bullet)\Rightarrow(\bullet\bullet)$

$\Rightarrow(\bullet)$ is true $\therefore \sum_{cyc}\frac{a^{2n}}{b}+\sum_{cyc} a\geq 2\sum_{cyc} a^{2n}\ \forall a,b,c>0,''=''$ iff $a=b=c$ (QED)

1992. If $x,y,z\geq 0, x+y+z=1$ and $0\leq\lambda\leq\frac{9}{4}$ then :

$$0\leq xy+yz+zx-\lambda xyz\leq\frac{9-\lambda}{27}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

When exactly 2 variables equal to zero, then : $xy+yz+zx-\lambda xyz=0\overset{?}{<}\frac{9-\lambda}{27}\Leftrightarrow\lambda\overset{?}{<}9\rightarrow\text{true}\because\lambda\leq\frac{9}{4}$ and when exactly 1 variable equals to zero, then WLOG we may assume $x=0\ (y+z=1\wedge y,z>0)$ and then :

$xy+yz+zx-\lambda xyz=yz\overset{AM-GM}{\leq}\frac{(y+z)^2}{4}=\frac{1}{4}\overset{?}{\leq}\frac{9-\lambda}{27}\Leftrightarrow\lambda\overset{?}{\leq}\frac{9}{4}\rightarrow\text{true}$

$\Rightarrow\sum_{cyc} xy-\lambda xyz\leq\frac{9-\lambda}{27}$ and when $x,y,z>0$, then : $\sum_{cyc} xy-\lambda xyz\overset{?}{\leq}\frac{9-\lambda}{27}$

$\Leftrightarrow\lambda\left(\frac{\left(\sum_{cyc} x\right)^3}{27}-xyz\right)\overset{?}{\leq}\left(\frac{\left(\sum_{cyc} x\right)^3}{3}-\left(\sum_{cyc} xy\right)\left(\sum_{cyc} x\right)\right)\left(\because\sum_{cyc} x=1\right)$ and

$\because\lambda\leq\frac{9}{4}\wedge\frac{\left(\sum_{cyc} x\right)^3}{27}-xyz\overset{AM-GM}{\geq}0\therefore$ it suffices to prove :

$$\frac{9}{4}\left(\frac{\left(\sum_{cyc} x\right)^3}{27}-xyz\right)\overset{?}{\leq}\left(\frac{\left(\sum_{cyc} x\right)^3}{3}-\left(\sum_{cyc} xy\right)\left(\sum_{cyc} x\right)\right)$$

$$\Leftrightarrow\left(\sum_{cyc}x\right)^3+9xyz\overset{?}{\geq}4\left(\sum_{cyc}xy\right)\left(\sum_{cyc}x\right)\Leftrightarrow\sum_{cyc}x^3+3xyz\overset{?}{\geq}\sum_{cyc}x^2y+\sum_{cyc}xy^2$$

$$\rightarrow \text{true via Schur}\therefore\sum_{cyc}xy-\lambda xyz\leq\frac{9-\lambda}{27}\ \text{and also,}\sum_{cyc}xy-\lambda xyz\geq$$

$$\left(\sum_{cyc}xy\right)\left(\sum_{cyc}x\right)-\frac{9xyz}{4}\left(\because\sum_{cyc}x=1\ \text{and}\ \lambda\leq\frac{9}{4}\right)\overset{AM-GM}{\geq}9xyz-\frac{9xyz}{4}>0\ \text{and}$$

$$\text{so},0\leq xy+yz+zx-\lambda xyz\leq\frac{9-\lambda}{27}\ \forall x,y,z\geq 0\,\big|\,x+y+z=1\ \wedge\ 0\leq\lambda\leq\frac{9}{4},$$

$$''='' \text{ for}\sum_{cyc}xy-\lambda xyz\geq 0\ \text{iff}\begin{pmatrix}x=0\\y=0\\z=1\end{pmatrix}\text{or}\begin{pmatrix}x=0\\y=1\\z=0\end{pmatrix}\text{or}\begin{pmatrix}x=1\\y=0\\z=0\end{pmatrix}\text{and}$$

$$''='' \text{ for}\sum_{cyc}xy-\lambda xyz\leq\frac{9-\lambda}{27}\ \text{iff}\begin{pmatrix}x=0\\y=\frac{1}{2}\\z=\frac{1}{2}\\\lambda=\frac{9}{4}\end{pmatrix}\text{or}\begin{pmatrix}x=\frac{1}{2}\\y=0\\z=\frac{1}{2}\\\lambda=\frac{9}{4}\end{pmatrix}\text{or}\begin{pmatrix}x=\frac{1}{2}\\y=\frac{1}{2}\\z=0\\\lambda=\frac{9}{4}\end{pmatrix}$$

$$\text{or}\left(x=y=z=\frac{1}{3}\right)\ (\text{QED})$$

1993. *If* $a,b,c>0$ *then prove :*

$$\frac{\sqrt[3]{ab}+\sqrt[3]{c^2}}{\sqrt[3]{ab}+\sqrt[3]{bc}}+\frac{\sqrt[3]{bc}+\sqrt[3]{a^2}}{\sqrt[3]{bc}+\sqrt[3]{ac}}+\frac{\sqrt[3]{ac}+\sqrt[3]{b^2}}{\sqrt[3]{ac}+\sqrt[3]{ab}}\geq 3$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Mirsadix Muzefferov-Azerbaijan

$$\textit{Let}:\ \sqrt[3]{a}=x>0,\quad \sqrt[3]{b}=y>0,\quad \sqrt[3]{c}=z>0\ \textit{Then:}$$

$$\frac{xy+z^2}{y(x+z)}+\frac{zy+x^2}{z(x+y)}+\frac{xz+y^2}{x(y+z)}\geq 3$$

$$\left(\frac{xy+z^2}{y(x+z)}-1\right)+\left(\frac{zy+x^2}{z(x+y)}-1\right)+\left(\frac{xz+y^2}{x(y+z)}-1\right)\geq 0$$

$$\frac{z^2-yz}{y(x+z)}+\frac{x^2-xz}{z(x+y)}+\frac{y^2-xy}{x(y+z)}\geq 0$$

$$\frac{z^2}{y(x+z)}+\frac{x^2}{z(x+y)}+\frac{y^2}{x(y+z)}\geq\frac{yz}{y(x+z)}+\frac{xz}{z(x+y)}+\frac{xy}{x(y+z)}$$

$$z^3x(x+y)(y+z)+x^3y(x+z)(y+z)+y^3z(x+z)(x+y)\geq$$
$$xyz^2(x+y)(y+z)+yzx^2(x+z)(y+z)+xzy^2(x+z)(x+y)\geq$$
$$(x^2z^4+z^4xy+z^3x^2y+z^3y^2x)+(x^4y^2+x^4yz+x^3y^2z+x^3z^2y)+$$
$$(y^4z^2+y^4xz+y^3x^2z+y^3z^2x)\geq(z^3y^2x+z^3x^2y+y^3z^2x+x^2y^2z^2)+$$
$$(x^3y^2z+x^3z^2y+z^3x^2y+x^2y^2z^2)+(x^3y^2z+y^3z^2x+y^3x^2z+x^2y^2z^2)$$
$$Let's\ simplify\ :$$
$$(x^4y^2+y^4z^2+z^4x^2)+(x^4yz+y^4xz+z^4xy)\geq$$
$$(x^3y^2z+y^3z^2x+z^3x^2y)+(x^2y^2z^2+x^2y^2z^2+x^2y^2z^2)$$
$$According\ \ to\ \ Muirhead's\ \ inequality:$$
$$x^4y^2+y^4z^2+z^4x^2\geq x^3y^2z+y^3z^2x+z^3x^2y$$
$$(4\,;2\,;0)\ >\ (3\,;2\,;1)$$
$$4+2+0=3+2+1$$
$$x^4yz+y^4xz+z^4xy\geq x^2y^2z^2+x^2y^2z^2+x^2y^2z^2$$
$$(4\,;1\,;1)\ >\ (2\,;2\,;2)$$
$$4+1+1=2+2+2$$
$$Equality\ \ holds\ \ iff:\ a=b=c$$

1994. If $x,y,z>0, x+y+z=1$ and $\lambda\geq 2$ then :

$$1+3\lambda\sum_{cyc}x^3\geq(\lambda+3)\left(\sum_{cyc}x^2\right)$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$1+3\lambda\sum_{cyc}x^3\geq(\lambda+3)\left(\sum_{cyc}x^2\right)\overset{x+y+z=1}{\Leftrightarrow}$$
$$\left(\sum_{cyc}x\right)^3+3\lambda\sum_{cyc}x^3\geq(\lambda+3)\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}x\right)$$
$$\Leftrightarrow\lambda\left(3\sum_{cyc}x^3-\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}x\right)\right)\geq\left(\sum_{cyc}x\right)\left(3\left(\sum_{cyc}x^2\right)-\left(\sum_{cyc}x\right)^2\right)\ and$$
$$\because\lambda\geq 2\ \wedge\ 3\sum_{cyc}x^3-\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}x\right)\overset{Chebyshev}{\geq}0\ \therefore\ it\ suffices\ to\ prove:$$

$$2\left(3\sum_{cyc}x^3-\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}x\right)\right)\geq\left(\sum_{cyc}x\right)\left(3\left(\sum_{cyc}x^2\right)-\left(\sum_{cyc}x\right)^2\right)$$

$$\Leftrightarrow\sum_{cyc}x^3+3xyz\geq\sum_{cyc}x^2y+\sum_{cyc}xy^2\rightarrow\text{true via a Schur}$$

$$\therefore 1+3\lambda\sum_{cyc}x^3\geq(\lambda+3)\left(\sum_{cyc}x^2\right)\ \forall x,y,z>0\,|\,x+y+z=1\ \wedge\ \lambda\geq 2,$$

$$''=''\text{ iff }x=y=z=\frac{1}{3}\ (\text{QED})$$

1995. If $a,b,c>0,\ a+b+c\leq 6$ then:

$$\frac{a^2-1}{a^2+1}+\frac{b^2-1}{b^2+1}+\frac{c^2-1}{c^2+1}\leq 0$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$(a^2-1)-(a-1)(a^2+1)=-a^3-a+2a^2=-a(a-1)^2\leq 0$$
$$so\ a^2-1\leq(a-1)(a^2+1)\ (1)$$

$$\frac{a^2-1}{a^2+1}+\frac{b^2-1}{b^2+1}+\frac{c^2-1}{c^2+1}=\sum\frac{a^2-1}{a^2+1}\overset{(1)}{\leq}\sum\frac{(a-1)(a^2+1)}{a^2+1}=\sum(a-1)$$
$$=(a+b+c)-3\overset{a+b+c\leq 3}{\leq}3-3=0$$

Equality holds for a=b=c=1.

1996. If $a,b,c>0,\ a^2+b^2+c^2\leq 3$ then:

$$\frac{a^3-1}{a^3+2}+\frac{b^3-1}{b^3+2}+\frac{c^3-1}{c^3+2}\leq 0$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

Lemma:

$$\forall x>0,\frac{1}{x^3+2}\geq\frac{1}{3}-\frac{x^2-1}{6}$$

Proof:

$$\frac{1}{x^3+2}-\left(\frac{1}{3}-\frac{x^2-1}{6}\right)=\frac{x^2(x^3-3x+2)}{6(x^3+2)}=\frac{x^2(x-1)^2(x+2)}{6(x^3+2)}\geq 0$$

$$true\ as\ x>0, so\ \frac{1}{x^3+2}\geq\frac{1}{3}-\frac{x^2-1}{6}$$

$$\frac{a^3-1}{a^3+2}+\frac{b^3-1}{b^3+2}+\frac{c^3-1}{c^3+2}=\sum\frac{a^3-1}{a^3+2}=\sum\left(1-\frac{3}{a^3+2}\right)=$$

$$=3-3\sum\frac{1}{a^3+2}\overset{lemma}{\leq}3-3\sum\left(\frac{1}{3}-\frac{a^2-1}{6}\right)=$$

$$=3-3+\frac{1}{2}(a^2+b^2+c^2-3)\overset{a^2+b^2+c^2\leq 3}{\leq}0$$

Equality holds for a=b=c=1.

1997. If $a,b,c>1, a+b+c\leq 6$ then:

$$\frac{1}{(\log_2 a+\log_2 b)^2}+\frac{1}{(\log_2 b+\log_2 c)^2}+\frac{1}{(\log_2 c+\log_2 a)^2}\geq\frac{3}{4}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$abc\overset{AM-GM}{\leq}\left(\frac{a+b+c}{3}\right)^3\overset{a+b+c\leq 6}{\leq}\left(\frac{6}{3}\right)^3=8\ (1)$$

$$\frac{1}{(\log_2 a+\log_2 b)^2}+\frac{1}{(\log_2 b+\log_2 c)^2}+\frac{1}{(\log_2 c+\log_2 a)^2}=$$

$$=\frac{1^3}{(\log_2 a+\log_2 b)^2}+\frac{1^3}{(\log_2 b+\log_2 c)^2}+\frac{1^3}{(\log_2 c+\log_2 a)^2}\geq$$

$$\overset{Radon}{\geq}\frac{(1+1+1)^3}{\left(2(\log_2 a+\log_2 b+\log_2 c)\right)^2}=\frac{27}{4\left(\log_2(abc)\right)^2}\overset{(1)}{\geq}\frac{27}{4(\log_2 8)^2}=$$

$$=\frac{27}{4(3\log_2 2)^2}=\frac{27}{4\times 9}=\frac{3}{4}$$

$$Equality\ holds\ for\ a=b=c=2.$$

1998. If $a,b,c>1, a+b+c\leq 6$ then:

$$\sum_{cyc}\frac{\log_2(a^2+4)}{log_2 a}\geq 9$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\log(a^2+4)\overset{AM-GM}{\geq}\log 4a\ (1)$$

$$\frac{\log_2(a^2+4)}{log_2 a}=\log_a(a^2+4)\overset{(1)}{\geq}\log_a 4a=\frac{\log 4a}{\log a}=\frac{\log a+\log 4}{\log a}=$$

$$=1+\log 4.\frac{1}{\log a}\ (2)$$

$$abc\overset{AM-GM}{\leq}\left(\frac{a+b+c}{3}\right)^3\overset{a+b+c\leq 6}{\leq}8\ (3)$$

$$\sum\frac{\log_2(a^2+4)}{log_2 a}\overset{(2)}{\geq}\sum\left(1+\log 4.\frac{1}{\log a}\right)=$$

$$=3+\log 4\sum\frac{1^2}{\log a}\overset{Bergstrom}{\geq}3+log4.\frac{(1+1+1)^2}{loga+logb+logc}$$

$$=3+log4.\frac{9}{\log abc}\overset{(3)}{\geq}3+2log2.\frac{9}{\log 8}=3+2log2.\frac{9}{3\log 2}=3+6=9$$

Equality holds for a=b=c=2.

1999. If $a,b,c>0$ then:

$$\sum_{cyc}\frac{\sqrt{(a+b)(b+c)}}{b}\geq 6$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$\sum_{cyc}\frac{\sqrt{(a+b)(b+c)}}{b}\overset{AM-GM}{\geq}3\sqrt[3]{\frac{(a+b)(b+c)(c+a)}{abc}}\overset{CESARO}{\geq}3\sqrt[3]{\frac{8abc}{abc}}=3\sqrt[3]{8}=6$$

Equality holds for: $a=b=c$.

2000.

If $0<a,b,c\leq 1$ and $\lambda\geq 2$ then :

$$\frac{1}{\lambda a+\lambda b}+\frac{1}{a+(\lambda+1)b}+\frac{1}{b+(\lambda+1)a}\geq\frac{3}{2\lambda}\left(\frac{1}{\lambda a+1}+\frac{1}{\lambda b+1}+\frac{1}{a+b+1}\right)$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{b+(\lambda+1)a}\overset{?}{\geq}\frac{3}{2\lambda}\cdot\frac{1}{\lambda a+1}\Leftrightarrow 2\lambda^2a+2\lambda\overset{?}{\geq}3b+3\lambda a+3a$$

$$\Leftrightarrow a(2\lambda^2-3\lambda-2)+2\lambda-3b-a\underset{(*)}{\overset{?}{\geq}}0\ and\ \because -3b-a\geq -4\ (as\ a,b\leq 1)$$

$\therefore$ in order to prove $(*)$, it suffices to prove : $a(\lambda-2)(2\lambda+1)+2(\lambda-2)\overset{?}{\geq}0$

$\rightarrow$ true $\because \lambda\geq 2\wedge a>0\ \therefore \frac{1}{b+(\lambda+1)a}\overset{①}{\geq}\frac{3}{2\lambda}\cdot\frac{1}{\lambda a+1}$ and similarly,

since $-3a-b\geq -4\ \therefore$ in order to prove : $\frac{1}{a+(\lambda+1)b}\overset{?}{\geq}\frac{3}{2\lambda}\cdot\frac{1}{\lambda b+1}$,

it suffices to prove : $b(\lambda-2)(2\lambda+1)+2(\lambda-2)\overset{?}{\geq}0\rightarrow$ true $\because \lambda\geq 2\wedge b>0$

$\therefore \frac{1}{a+(\lambda+1)b}\overset{②}{\geq}\frac{3}{2\lambda}\cdot\frac{1}{\lambda b+1}$ and again, $\frac{3}{2\lambda}\cdot\frac{1}{a+b+1}\overset{1\geq\frac{a+b}{2}}{\leq}\frac{3}{2\lambda}\cdot\frac{1}{a+b+\frac{a+b}{2}}$

$$=\frac{1}{\lambda(a+b)}\therefore\frac{1}{\lambda a+\lambda b}\overset{③}{\geq}\frac{3}{2\lambda}\cdot\frac{1}{a+b+1}\ and\ so,\ ①+②+③\Rightarrow$$

$$\frac{1}{\lambda a+\lambda b}+\frac{1}{a+(\lambda+1)b}+\frac{1}{b+(\lambda+1)a}\geq\frac{3}{2\lambda}\left(\frac{1}{\lambda a+1}+\frac{1}{\lambda b+1}+\frac{1}{a+b+1}\right)$$

$\forall a,b,c\in(0,1]\ \wedge\ \lambda\geq 2,''='' \text{ iff } a=b=c=1\ \wedge\ \lambda=2$ (QED)

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

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