THE PENTAGON JOURNAL CHALLENGES-(II)

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780. Prove that if $a, b, c \in [1, \infty)$ then:

$$ab + bc + ca \geq 3 + 2\ln(a^b \cdot b^c \cdot c^a)$$

Proof. Let be $f : [1, \infty) \to \mathbb{R}; f(x) = \ln x - \frac{x}{2} + \frac{1}{2x}$

$$f'(x) = \frac{1}{x} - \frac{1}{2} - \frac{1}{2x^2} = \frac{2x - x^2 - 1}{2x^2} = -\frac{(x-1)^2}{2x^2} \leq 0$$

$$\max f(x) = f(1) = 0; f \text{ decreasing;}$$

$$\text{It follows: } \ln x - \frac{x}{2} + \frac{1}{2x} \leq 0; (\forall) x \geq 1$$

Let be $x = a \Rightarrow \ln a \leq \frac{a}{2} - \frac{1}{2a}$
We multiply with b. It follows: $b \ln a \leq \frac{ba^2 - b}{2a}$
Analogous:

$$c \ln b \leq \frac{cb^2 - c}{2b}; a \ln c \leq \frac{ac^2 - a}{2c}$$

By adding:

$$b \ln a + c \ln b + a \ln c \leq \frac{1}{2}(ab + bc + ca) - \frac{1}{2}\left(\frac{b}{a} + \frac{c}{b} + \frac{a}{c}\right)$$

$$\ln(a^b \cdot b^c \cdot c^a) \leq \frac{1}{2}(ab + bc + ca) - \frac{1}{2}\left(\frac{b}{a} + \frac{c}{b} + \frac{a}{c}\right) \tag{1}$$

From means inequality:

$$\frac{b}{a} + \frac{c}{b} + \frac{a}{c} \geq 3\sqrt[3]{\frac{b}{a} \cdot \frac{c}{b} \cdot \frac{a}{c}} = 3$$

$$-\frac{1}{2}\left(\frac{b}{a} + \frac{c}{b} + \frac{a}{c}\right) \leq -\frac{3}{2} \tag{2}$$

From (1) and (2):

$$\ln(a^b \cdot b^c \cdot c^a) \leq \frac{1}{2}(ab + bc + ca) - \frac{3}{2}$$

$$2\ln(a^b \cdot b^c \cdot c^a) \leq ab + bc + ca - 3$$

$$ab + bc + ca \geq 3 + 2\ln(a^b \cdot b^c \cdot c^a)$$

□

790. Prove that if $a, b \in \mathbb{R}; a < b$ then:

$$\left|\ln\left(\frac{2 + \sin 2b}{2 + \sin 2a}\right)\right| \leq \frac{2\sqrt{3}}{3}(b - a)$$

Proof. The inequality can be written successively:

$$|\ln(2+\sin 2b)-\ln(2+\sin 2a)|\leq \frac{2\sqrt{3}}{3}(b-a)$$
$$\Big|\int_a^b \frac{(2+\sin 2x)'}{2+\sin 2x}dx\Big|\leq \frac{2\sqrt{3}}{3}(b-a)$$
$$\Big|\frac{2\cos 2x}{2+\sin 2x}\Big|\leq \frac{2\sqrt{3}}{3}$$
$$\Big|\frac{\cos 2x}{2+\sin 2x}\Big|\leq \frac{1}{\sqrt{3}}$$

It remains to prove $(1);(2)$:

$$\frac{\cos 2x}{2+\sin 2x}\geq -\frac{1}{\sqrt{3}} \tag{1}$$

$$\frac{2x}{2+\sin 2x}\leq \frac{1}{\sqrt{3}} \tag{2}$$

Relationship (1) can be written:

$$-2-\sin 2x\leq \sqrt{3}\cos 2x \Leftrightarrow \frac{\sqrt{3}}{2}\cos 2x+\frac{1}{2}\sin 2x+1\geq 0\Leftrightarrow$$
$$\Leftrightarrow \cos\Big(\frac{\pi}{6}-2x\Big)\geq -1 \text{ which is true}$$

Relationship (2) can be written:

$$\frac{\cos 2x}{2+\sin 2x}\leq \frac{1}{\sqrt{3}}\Leftrightarrow \sqrt{3}\cos 2x\leq 2+\sin 2x\Leftrightarrow$$
$$\Leftrightarrow \frac{\sqrt{3}}{2}\cos 2x-\frac{1}{2}\sin 2x\leq 1\Leftrightarrow \cos\Big(\frac{\pi}{6}+2x\Big)\leq 1$$

□

799. Prove that if $a,b,c\in(0,2]$ then:

$$3\sqrt{2}\leq \sum \frac{b(\sqrt{a}+\sqrt{2-a})}{c}\leq 2\Big(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\Big)$$

Proof. Let be $f:[0,2]\to\mathbb{R}; f(x)=\sqrt{x}+\sqrt{2-x}$

$$f'(x)=\frac{1}{2\sqrt{x}}-\frac{1}{2\sqrt{2-x}}=\frac{\sqrt{2-x}-\sqrt{x}}{2\sqrt{x(2-x)}}$$
$$f'(x)=0\Rightarrow x=1;\min f(x)=f(0)=\sqrt{2}$$
$$\max f(x)=f(1)=2;$$
$$\sqrt{2}\leq\sqrt{x}+\sqrt{2-x}\leq 2$$

Let be $x=a\Rightarrow \sqrt{2}\leq\sqrt{a}+\sqrt{2-a}\leq 2$
By multiplying with $\frac{b}{c}$ we obtain:

$$\sqrt{2}\frac{b}{c}\leq\frac{b}{c}(\sqrt{a}+\sqrt{2-a})\leq 2\cdot\frac{b}{c}$$

Analogous:

$$\sqrt{2}\frac{c}{a} \leq \frac{c}{a}(\sqrt{b}+\sqrt{2-b}) \leq 2 \cdot \frac{c}{a}$$
$$\sqrt{2}\frac{a}{b} \leq \frac{a}{b}(\sqrt{c}+\sqrt{2-c}) \leq 2 \cdot \frac{a}{b}$$

(1)
$$\sqrt{2}\sum \frac{a}{b} \leq \sum \frac{b(\sqrt{a}+\sqrt{2-a})}{c} \leq 2\sum \frac{a}{b}$$

From means inequality:

(2)
$$\sum \frac{a}{b} \geq 3\sqrt[3]{\frac{a}{b}\cdot\frac{b}{c}\cdot\frac{c}{a}} = 3$$

From $(1);(2)$ it follows:

$$3\sqrt{2} \leq \sum \frac{b(\sqrt{a}+\sqrt{2-a})}{c} \leq 2\sum \frac{a}{b}$$

□

800. Prove that if $a \in \mathbb{R}$ then:

$$\int_{a+3}^{a+5} \ln(1+e^x)dx + \int_{a+6}^{a+8} \ln(1+e^x)dx \leq \int_{a}^{a+2} \ln(1+e^x)dx + \int_{a+9}^{a+11} \ln(1+e^x)dx$$

Proof. Let be $f:[a,a+11] \to \mathbb{R}; f(x) = \ln(1+e^x)$

$$f'(x) = \frac{e^x}{e^x+1}; f''(x) = \frac{e^x}{(e^x+1)^2} > 0 \Rightarrow f \text{ convex}$$

Let be the divisions:

$$\Delta_1^n = (a < x_0^n < x_1^n < \ldots < x_n^n < a+2); x_k^n = a + \frac{2k}{n}; k \in \overline{0,n}$$
$$\Delta_2^n = (a+3 < y_0^n < y_1^n < \ldots < y_n^n < a+5); y_k^n = a + 3 + \frac{2k}{n}; k \in \overline{0,n}$$
$$\Delta_3^n = (a+6 < z_0^n < z_1^n < \ldots < z_n^n < a+8); z_k^n = a + 6 + \frac{2k}{n}; k \in \overline{0,n}$$
$$\Delta_4^n = (a+9 < t_0^n < t_1^n < \ldots < t_n^n < a+11); t_k^n = a + 9 + \frac{2k}{n}; k \in \overline{0,n}$$
$$||\Delta_1^n|| = ||\Delta_2^n|| = ||\Delta_3^n|| = ||\Delta_4^n|| = \frac{2}{n} \to 0;$$

We find $p, q \in (0,1)$ such that:

$$y_k^n = px_k^n + (1-p)t_k^n; z_k^n = qx_k^n + (1-q)t_k^n; k \in \overline{0,n}$$
$$\begin{cases} a+3+\frac{2k}{n} = p\left(a+\frac{2k}{n}\right) + (1-p)\left(a+9+\frac{2k}{n}\right) \\ a+6+\frac{2k}{n} = q\left(a+\frac{2k}{n}\right) + (1-q)\left(a+9+\frac{2k}{n}\right) \end{cases} \Rightarrow$$
$$\Rightarrow \begin{cases} 3 = 9-9p \\ 6 = 9-9q \end{cases} \Rightarrow \begin{cases} p = \frac{2}{3} \\ q = \frac{1}{3} \end{cases}$$

Because f convex:

$$\begin{cases}\ln\left(1+e^{y_k^n}\right) \le p\ln\left(1+e^{x_k^n}\right)+(1-p)\ln\left(1+e^{t_k^n}\right)\\ \ln\left(1+e^{z_k^n}\right) \le q\ln\left(1+e^{x_k^n}\right)+(1-q)\ln\left(1+e^{t_k^n}\right)\end{cases}$$

$$\begin{cases}\ln\left(1+e^{y_k^n}\right) \le \frac{2}{3}\ln\left(1+e^{x_k^n}\right)+\frac{1}{3}\ln\left(1+e^{t_k^n}\right)\\ \ln\left(1+e^{z_k^n}\right) \le \frac{1}{3}\ln\left(1+e^{x_k^n}\right)+\frac{2}{3}\ln\left(1+e^{t_k^n}\right)\end{cases}$$

$$\ln\left(1+e^{y_k^n}\right)+\ln\left(1+e^{z_k^n}\right) \le \ln\left(1+e^{x_k^n}\right)+\ln\left(1+e^{t_k^n}\right)$$

$$\frac{2}{n}\sum_{k=1}^{n}\ln\left(1+e^{y_k^n}\right)+\frac{2}{n}\sum_{k=1}^{n}\ln\left(1+e^{z_k^n}\right) \le \frac{2}{n}\sum_{k=1}^{n}\ln\left(1+e^{x_k^n}\right)+\frac{2}{n}\sum_{k=1}^{n}\ln\left(1+e^{t_k^n}\right)$$

$$\sigma_{\Delta_1^n}(f,(y^n))+\sigma_{\Delta_3^n}(f,(z^n)) \le \sigma_{\Delta_1^n}(f,(x^n))+\sigma_{\Delta_4^n}(f,(t^n))$$

$$\lim_{n\to\infty}\sigma_{\Delta_2^n}+\lim_{n\to\infty}\sigma_{\Delta_3^n} \le \lim_{n\to\infty}\sigma_{\Delta_1^n}+\lim_{n\to\infty}\sigma_{\Delta_4^n}$$

$$\int_{a+3}^{a+5} f(x)dx+\int_{a+6}^{a+8} f(x)dx \le \int_{a}^{a+2} f(x)dx+\int_{a+9}^{a+11} f(x)dx$$

□

808. Prove that if $a,b,c \in [1,\infty)$ then:

$$\frac{e^{a+b+c}}{e^{\frac{b}{a}+\frac{c}{b}+\frac{a}{c}}} \le a^b b^c c^a \le \frac{e^{ab+bc+ca}}{e^{a+b+c}}$$

Proof. If $x \in [1,\infty)$ then: $1-\frac{1}{x} \le \ln x \le x-1 \Leftrightarrow$

$$\Leftrightarrow \begin{cases}\ln x+\frac{1}{x}-1 \ge 0\\ x-1-\ln x \ge 0\end{cases} \tag{1}$$

Let be: $f,g:[1,\infty)\to\mathbb{R}; f(x)=\ln x+\frac{1}{x}-1; g(x)=x-1-\ln x$ □

$$f'(x)=\frac{1}{x}-\frac{1}{x^2}=\frac{x-1}{x^2}; f'(x)=0 \Rightarrow x=1$$

$$g'(x)=1-\frac{1}{x}=\frac{x-1}{x}; g'(x)=0 \Rightarrow x=1$$

$$\min_{x\ge 1} f(x)=f(1)=0; \min_{x\ge 1} g(x)=g(1)=0$$

$$\Rightarrow f(x)\ge 0; g(x)\ge 0; (\forall)x\in[1,\infty)$$

Let be $x=a$ in $1 \Rightarrow 1-\frac{1}{a} \le \ln a \le a-1$.
By multiply with $b: b-\frac{b}{a} \le b\ln a \le ab-b$

$$\ln e^{b-\frac{b}{a}} \le \ln a^b \le \ln e^{ab-b} \Rightarrow \frac{e^b}{e^{\frac{b}{a}}} \le a^b \le \frac{e^{ab}}{e^b} \tag{2}$$

Analogous:

$$\frac{e^c}{e^{\frac{c}{b}}} \le b^c \le \frac{e^{bc}}{e^c} \tag{3}$$

$$\frac{e^a}{e^{\frac{a}{c}}} \le c^a \le \frac{e^{ca}}{e^a} \tag{4}$$

By multiplying (2); (3); (4)⇒

$$\Rightarrow \frac{e^{a+b+c}}{e^{\frac{b}{a}+\frac{c}{b}+\frac{a}{c}}} \le a^b b^c c^a \le \frac{e^{ab+bc+ca}}{e^{a+b+c}}$$

Equality holds if $a = b = c$.

809. Prove that if $a, b, c \in (2, \infty)$ then:

$$\sqrt{2}\sum(\sqrt{a(b-2)} + \sqrt{b(a-2)}) < 3\sqrt{abc}$$

Proposed by Daniel Sitaru - Romania

Proof.

We prove that if $a, b \in (2, \infty)$ then:

$$\sqrt{2(a-2)} + \sqrt{2(b-2)} < \sqrt{ab} \tag{1}$$

Let be $a = (1 + x) \cdot 2; b = (1 + y) \cdot 2; x, y > 1$ □

The inequality can be written:

$$\sqrt{2 \cdot 2x} + \sqrt{2 \cdot 2y} \le \sqrt{(1+x) \cdot 2(1+y) \cdot 2}$$

$$2\sqrt{x} + 2\sqrt{y} \le 2\sqrt{(1+x)(1+y)}$$

$$x + y + 2\sqrt{xy} \le 1 + x + y + xy$$

$$xy - 2\sqrt{xy} + 1 \ge 0 \Leftrightarrow (\sqrt{xy} - 1)^2 \ge 0$$

We multiply (1) with $\sqrt{c}$:

$$\sqrt{2c(a-2)} + \sqrt{2c(b-2)} < \sqrt{abc}$$

Analogous:

$$\sqrt{2a(b-2)} + \sqrt{2a(c-a)} < \sqrt{abc}$$

$$\sqrt{2b(c-2)} + \sqrt{2b(a-2)} < \sqrt{abc}$$

$$\sum(\sqrt{2a(b-2)} + \sqrt{2b(a-2)}) < 3\sqrt{abc}$$

$$\sqrt{2}\sum(\sqrt{a(b-2)} + \sqrt{b(a-2)}) < 3\sqrt{abc}$$

810. Compute

$$L = \lim_{n\to\infty} \frac{1}{n}\int_1^n \frac{x^4 + 4x^3 + 12x^2 + 9x}{(x+3)^5 - x^5 - 243}dx$$

Solution 1 by Amit Dutta - Jamshedpur - India.

$$(a+b)^5 = {}^5C_0a^5 + {}^5C_1a^4b^1 + {}^5C_2a^3b^2 + {}^5C_3a^2b^3 + {}^5C_4ab^4 + {}^5C_5b^5$$

$$(a+b)^5 = a^5 + b^5 + 5a^4b + 5ab^4 + 10a^3b^2 + 10a^2b^3$$

$$(a+b)^5 - a^5 - b^5 = 5ab(a^3+b^3) + 10a^2b^2(a+b)$$

$$put\ a = x, b = 3$$

$$(x+3)^5 - x^5 - 3^5 = 5 \times x \times 3(x^3+3^3) + 10 \times 9 \times x^2(x+3)$$

$$= 15x(x^3+6x^2+18x+27)$$

$$\because L = \lim_{n\to\infty}\int_1^n \frac{x(x^3+4x^2+12x+9)}{15x(x^3+6x^2+18x+27)}dx$$

$$L = \frac{1}{15}\lim_{n\to\infty}\frac{1}{n}\int_1^n 1 - \frac{2(x^2+3x+9)}{x^3+6x^2+18x+27}dx$$

$$L = \frac{1}{15}\lim_{n\to\infty}\frac{1}{n}\int_1^n 1 - \frac{2(x^2+3x+9)}{(x+3)(x^2+3x+9)}dx$$

$$L = \frac{1}{15}\lim_{n\to\infty}\frac{1}{n}\int_1^n 1 - \frac{2}{n+3}dx$$

$$L = \frac{1}{15}\left\{\lim_{n\to\infty}\left(\frac{n-1}{n}\right) - \frac{2}{n}\ln\left(\frac{n+3}{n}\right)\right\} = \frac{1}{15}\times 1$$

$$L = \frac{1}{15}$$

□

Solution 2 by Lazaros Zachariadis - Thessaloniki - Greece.

$$x^4+4x^3+12x^2+9x = x(x^3+4x^2+12x+9) = x(x+1)(x^2+3x+9)$$

$$\begin{array}{cccc|l} 1 & 4 & 12 & 9 & -1 \\ \downarrow & -1 & -3 & -9 & \\ 1 & 3 & 9 & 0 & \end{array}$$

$$(x+5)^3 - x^5 - 243 = x^5 + 15x^4 + 90x^3 + 270x^2 + 405x + 243 - x^5 - 243 =$$

$$= 15(x^4+6x^3+18x^2+27x) = 15x(x+3)(x^2+3x+9)$$

$$\begin{array}{cccc|l} 1 & 6 & 18 & 27 & -3 \\ \downarrow & -3 & -9 & -27 & \\ 1 & 3 & 9 & 0 & \end{array}$$

$$\frac{x^4+4x^3+12x^2+9x}{(x+3)^5-x^5-243} = \frac{x(x+1)(x^2+3x+9)}{15x(x+3)(x^2+3x+9)} = \frac{x+1}{15(x+3)}$$

$$= \frac{x+3}{15(x+3)} - \frac{2}{15(x+3)} = \frac{1}{15} - \frac{2}{15(x+3)} = \left(\frac{x}{15} - \frac{2}{15}\ln(x+3)\right)'$$

$$\int_1^n \left(\frac{x}{15} - \frac{2\ln(x+3)}{15}\right)'dx = \frac{n-2\ln(n+3)}{15} - \left(\frac{1}{15} - \frac{2\ln 4}{15}\right)$$

$$= \frac{n-2\ln(n+3)-1+4\ln 2}{15}$$

$$\frac{n-2\ln(n+3)-1+4\ln 2}{15n} = \frac{1-2\frac{\ln(n+3)}{n} - \frac{1}{n} + \frac{4\ln 2}{n}}{15}$$

$$\lim_{n\to+\infty}\frac{1-2\cdot\frac{\ln(n+3)}{n} - \frac{1}{n} + \frac{4\ln 2}{n}}{15} = \frac{1}{15}$$

because $\lim\limits_{n\to+\infty}\dfrac{\ln(n+3)}{n}\overbrace{=}^{dlh}\lim\limits_{n\to+\infty}\dfrac{1}{n+3}=0$ *and* $\lim\limits_{n\to+\infty}\dfrac{1}{n}=\lim\limits_{n\to+\infty}\dfrac{4\ln 2}{n}=0$

□

Solution 3 by Nawar Alasadi - Babylon - Iraq.

$$\lim_{n\to\infty}\frac{1}{n}\int_1^n\frac{x^4+x^3+9x^2+9x+3x^3+3x^2}{15x^4+90x^3+270x^2+405x}dx$$
$$=\lim_{n\to\infty}\frac{1}{n}\int_1^n\frac{x^3(x+1)+9x(x+1)+3x^2(x+1)}{15[x^3(x+3)+3x^2(x+3)+9x(x+3)]}dx$$
$$=\lim_{n\to\infty}\frac{1}{n}\int_1^n\frac{x(x+1)(x^2+9+3x)}{15x(x+3)(x^2+3x+9)}dx$$
$$=\lim_{n\to\infty}\frac{1}{n}\int_1^n\frac{x+1}{15(x+3)}dx$$
$$=\lim_{n\to\infty}\frac{1}{15n}\int_1^n\frac{x+3-2}{x+3}dx$$
$$=\lim_{n\to\infty}\frac{1}{15n}\int_1^n\Big(\frac{(x+3)}{(x+3)}-\frac{2}{x+3}\Big)dx$$
$$=\lim_{n\to\infty}\frac{1}{15n}\int_1^n[x-2\ln|x+3|]_1^n$$
$$=\lim_{n\to\infty}\frac{1}{15n}\int_1^n[n-2\ln|n+3|-1+2\ln(4)]$$

By LiR

$$=\lim_{n\to\infty}\Big(\frac{1}{15}\Big)=\frac{1}{15}$$

□

Solution 4 by Remus Florin Stanca - Romania.

$$x^4+4x^3+12x^2+9x=x^4+x^3+3x^3+3x^2+9x^2+9x=$$
$$=x^3(x+1)+3x^2(x+1)+9x(x+1)=$$
$$(x+1)(x^3+3x^2+9x)=x(x+1)(x^2+3x+9)$$
$$(x+3)^3-x^5-243=C_5^0x^5+C_5^1x^4\cdot 3+C_5^3x^3\cdot 9+C_5^3x^2\cdot 27+C_5^4x\cdot 81+C_5^5 243-x^5-243$$
$$=x^5+15x^4+90x^3+270x^2+405x+243-x^5-243=$$
$$=15x^4+90x^3+270x^2+405x=15x(x^3+6x^2+18x+27)=$$
$$=15x^4+90x^3+270x^2+405x=15x(x^3+6x^2+18x+27)=$$
$$15x[(x+3)^3-3x^2-9x]=15x[(x+3)^3-3x(x+3)]$$
$$=15x(x+3)(x^2+3x+9)$$
$$\Leftrightarrow\int_1^n\frac{x^4+4x^3+12x^2+9x}{(x+3)^5-x^5-243}dx=\int_1^n\frac{x(x+1)(x^2+3x+9)}{15x(x+3)(x^2+3x+9)}dx=$$
$$=\int_1^n\frac{x+1}{15(x+3)}dx=\frac{1}{15}\int_1^n\frac{x+1}{x+3}dx=$$
$$\int\frac{x+1}{x+3}dx=\int(x+1)\cdot\frac{1}{x+3}dx=(x+1)\ln(|x+3|)-\int\ln(|x+3|)dx$$
$$\int\ln(x+3)dx=\frac{1}{2}\int(2x+6)\cdot\frac{\ln(|x+3|)}{x+3}dx=$$

$$= \frac{1}{2}\left\{(x+3)^2 \cdot \frac{\ln(|x+3|)}{x+3} - \int (x+3)^2 \cdot \left[\frac{\ln(|x+3|)}{x+3}\right]' dx\right\}$$

$$\Rightarrow \int \ln(|x+3|)dx = \frac{1}{2}\left\{(x+3)\ln(|x+3|) - x + \int \ln(|x+3|)dx\right\}$$

$$\int \ln(|x+3|)dx \overbrace{=}^{not} a \Rightarrow a = \frac{1}{2}[(x+3)\ln(|x+3|) - x + a]$$

$$\Rightarrow 2a = (x+3)\ln(|x+3|) - x + a \Rightarrow a = (x+3)\ln(|x+3|) - x$$

$$\Rightarrow \int \ln(|x+3|)dx = (x+3)\ln(|x+3|) - x + C$$

$$\Rightarrow \int \frac{x+1}{x+3}dx = (x+1)\ln(|x+3|) - (x+3)\ln(|x+3|) + x - C$$

$$\Rightarrow \int_1^n \frac{x+1}{x+3}dx = (n+1)\ln(n+3) - (n+3)\ln(n+3) + n - 2\ln 4 + 4\ln 4 - 1$$

$$= -2\ln(n+3) + n + 2\ln 4 - 1$$

$$\Rightarrow \lim_{n\to\infty} \frac{1}{n}\int_1^n E(x)dx = \lim_{n\to\infty} \frac{1}{15} \cdot \frac{-2\ln(n+3) + n + 2\ln 4 - 1}{n} =$$

$$\lim_{n\to\infty} \frac{1}{15}\left(\frac{-2\ln(n+3)}{n} + 1\right) = \frac{1}{15}\left(\lim_{n\to\infty} \frac{-2\ln(n+3)}{n} + 1\right) = \lim \tag{1}$$

$$\lim_{n\to\infty} \frac{-2\ln(n+3)}{n} = \lim_{n\to\infty} \frac{-2\ln\frac{n+4}{n+3}}{n+1-n} = 0$$

$$\overbrace{\Rightarrow}^{(1)} \lim = \frac{1}{15} \textit{(Q.E.D.)}$$

□

Solution 5 by Sagar Kumar - Kolkata - India.

$$L = \lim_{n\to\infty} \frac{1}{n}\int_1^n \frac{(x^4 + 4x^3 + 12x^2 + 9x)}{(x+3)^5 - (x^5 + 3^5)}dx$$

$$(x+3)^5 - (x^5 + 3^5) = x^5 + 15x^4 + 90x^3 + 270x^2 + 405x + 243 - x^5 - 243$$

$$= 15x^4 + 90x^3 + 270x^2 + 405x$$

$$= 15(x^4 + 6x^3 + 18x^2 + 27x)$$

$$L = \lim_{n\to\infty} \frac{1}{n} \cdot \frac{1}{15}\int_1^n \frac{x^3 + 4x^2 + 12x + 9}{x^3 + 6x^2 + 18x + 27}dx$$

$$L = \lim_{n\to\infty} \frac{1}{n} \cdot \frac{1}{15}\int_1^n \frac{(x+1)(x^2 + 3x + 9)}{(x+3)(x^2 + 3x + 9)}dx$$

$$L = \lim_{n\to\infty} \frac{1}{15n}\int_1^n \frac{x+1}{x+3}dx$$

$$L = \frac{1}{15}\lim_{n\to\infty}\left(\frac{n - 1 - 2 \quad 1 \ \textit{of} \ \frac{(n+3)}{4}}{n}\right)$$

$$L = \frac{1}{15}\lim_{n\to\infty}\left(1 - \frac{1}{n} - \frac{2 \quad 1 \ \textit{of} \ \frac{(n+3)}{4}}{n}\right)$$

$$= \frac{1}{15} \ AM$$

□

Solution 6 by Shivam Sharma - New Delhi - India.

$$(x+3)^5 - x^5 - 243 = 15(x^4 + 6x^3 + 18x^2 + 27x)$$

(OR)

$$(x+3)^5 - x^5 - 243 = 15(x+3)(x^2 + 3x + 9) \tag{1}$$

Using (1)*, we get,*

$$\Rightarrow \lim_{n\to\infty} \frac{1}{15n} \int_1^n \frac{x^4 + 4x^3 + nx^2 + 9x}{(x+3)(x^2+3x+9)} dx$$

$$\Rightarrow \lim_{n\to\infty} \frac{1}{15n} \int_1^n \frac{(x+1)(x^2+3x+9)}{(x+3)(x^2+3x+9)} dx$$

$$\Rightarrow \lim_{n\to\infty} \frac{1}{15n} \int_1^n \frac{x+3}{x+3} dx + \lim_{n\to\infty} \frac{1}{15n} \int_1^n \frac{-3+1}{x+3} dx$$

$$\Rightarrow \lim_{n\to\infty} \frac{1}{15} [n - 1 + (-2)\ln(n+3) + 2\ln(4)]$$

$$\Rightarrow \lim_{n\to\infty} \left[\frac{1}{15} - \frac{1}{15n} - \frac{2\ln(n+3)}{15n} + \frac{2\ln(4)}{15n}\right]$$

(OR)

$$L = \frac{1}{15} \text{ (Answer)}$$

□

Solution 7 by proposer.

$$x^4 + 4x^3 + 12x^2 + 9x = x(x^3 + 4x^2 + 12x + 9) =$$
$$= x(x^3 + x^2 + 3x^2 + 3x + 9x + 9) =$$
$$= x\Big(x^2(x+1) + 3x(x+1) + 9(x+1)\Big) =$$
$$= x(x+1)(x^2 + 3x + 9)$$

For the denominator we use the identity:

$$(a+b+c)^5 - a^5 - b^5 - c^5 = 5(a+b)(b+c)(c+a)(a^2 + b^2 + c^2 + ab + ac + bc)$$
$$(x+3+0)^5 - x^5 - 3^5 - 0^5 = 5(x+3)(x+0)(3+0)(x^2 + 3^2 + 3x) =$$
$$= 15x(x+3)(x^2 + 3x + 9)$$

$$L = \lim_{n\to\infty} \frac{1}{n} \int_1^n \frac{x(x+1)(x^2+3x+9)}{15x(x+3)(x^2+3x+9)} dx =$$
$$= \lim_{n\to\infty} \frac{1}{n} \cdot \frac{1}{15} \int_1^n \frac{x+1}{x+3} dx =$$
$$= \lim_{n\to\infty} \left(\frac{1}{15n}(n-1) + \frac{1}{15n} \cdot (-2)\ln\left(\frac{n+3}{4}\right)\right) =$$
$$= \frac{1}{15}$$

□

821. If $a, b, c \in \mathbb{R}$ then:

$$4\sum a\left|b(1-b^2)\right| \leq \sum a(1+b^2)^2$$

Proposed by Daniel Sitaru - Romania

Proof.

Lemma 0.1. *If $a \in \mathbb{R}$ then:*

$$\left|\frac{a(1-a^2)}{(1+a^2)}\right| \leq \frac{1}{4} \tag{1}$$

Proof. Denote $a = \tan\frac{x}{2}; x \in (-\pi, \pi)$:
(1) can be written:

$$\left|\frac{\tan\frac{x}{2}(1-\tan^2\frac{x}{2})}{(1+\tan^2\frac{x}{2})^2}\right| \leq \frac{1}{4} \Leftrightarrow \left|\frac{\tan\frac{x}{2}\cos x}{1+\tan^2\frac{x}{2}}\right| \leq \frac{1}{4}$$

$$\left|\frac{\sin x\cos x}{2}\right| \leq \frac{1}{4} \Leftrightarrow |\sin 2x| \leq 1$$

which is true.

$$4|a(1-a^2)| \leq (1+a^2)$$

Multiplying with b:

$$4b|a(1-a^2)| \leq b(1+a^2)^2$$

By adding analogous relationships:

$$4\sum a|b(1-b^2)| \leq \sum a(1+b^2)^2$$

□

□

822. Prove that in any acute-angled ΔABC you have:

$$2\sum_{cyclic}\tan^3 A \geq \sum_{cyclic}\sqrt{\frac{\tan^6 A+\tan^6 B}{2}} + 3(\tan A+\tan B+\tan C)$$

Solution 1 by Soumava Chakraborty - SoftWebTechnologies - Kolkata - India.

$$\sqrt{\frac{x^2+y^2}{2}} + \sqrt{xy} \overset{\text{CBS}}{\leq} \sqrt{2}\sqrt{\frac{x^2+y^2}{2}+xy}$$

$$= \sqrt{2}\frac{(x+y)}{\sqrt{2}} = x+y \quad (x, y > 0)$$

$$\Rightarrow \forall x, y > 0, \sqrt{\frac{x^2+y^2}{2}} \overset{(1)}{\leq} x+y-\sqrt{xy}$$

Setting $x = \tan^3 A, y = \tan^3 B$, (1) $\Rightarrow$

$$\sqrt{\frac{\tan^6 A+\tan^6 B}{2}} \overset{(i)}{\leq} \tan^3 A+\tan^3 B-\sqrt{\tan^3 A\tan^3 B}$$

Similarly, $\sqrt{\frac{\tan^6 B+\tan^6 C}{2}} \overset{(ii)}{\leq} \tan^3 B + \tan^3 C - \sqrt{\tan^3 B + \tan^3 C}$ and

$$\sqrt{\frac{\tan^6 C + \tan^6 A}{2}} \overset{(iii)}{\leq} \tan^3 C + \tan^3 A - \sqrt{\tan^3 C \tan^3 A}$$

$$(ii) + (ii) + (iii) \Rightarrow RHS \leq 2\sum \tan^3 A - \sum \sqrt{\tan^3 A \tan^3 B} + 3\tan A \tan B \tan C$$

$$(\because \sum \tan A = \tan A \tan B \tan C)$$

$$\overset{?}{\leq} 2\sum \tan^3 A$$

$$\Leftrightarrow \sum \sqrt{\tan^3 A \tan^3 B} \overset{?}{\geq} 3\tan A \tan B \tan C$$

$$\rightarrow \text{ true } \because \sum \sqrt{\tan^3 A \tan^3 B} \overset{\text{A-G}}{\geq} 3\sqrt[3]{\tan^3 A \tan^3 B \tan^3 C} = 3\tan A \tan B \tan C$$

(Hence proved) □

Solution 2 by Tran Hong-Dong Thap - Vietnam (student).

$$(1) \qquad \sqrt{\frac{a^6 + b^6}{2}} + \sqrt{a^3 b^3} \leq a^3 + b^3 \quad (\forall a, b > 0)$$

$$\Leftrightarrow \left(\sqrt{\frac{a^6 + b^6}{2}} - \sqrt{a^3 b^3}\right)^2 \geq 0. \text{ true}$$

Using (1) with $a = \tan A, b = \tan B, c = \tan C$
$(a, b, c >$ and $a + b + c = abc)$

$$\because 3\sum \tan A + \sum \sqrt{\frac{\tan^6 A + \tan^6 B}{2}} \leq 3\sum \tan A + \sum(\tan^3 A + \tan^3 B)$$

$$-\sum \sqrt{\tan^3 A \tan^3 B} = 3\prod \tan A + 2\sum \tan^3 A - \sum \sqrt{(\tan A \tan B)^3}$$

It must be shown that:

$$3\prod \tan A \leq \sum \sqrt{(\tan A \tan B)^3}$$

It is true, because:

$$3\prod \tan A \overset{\text{Cauchy}}{\leq} \sqrt{\tan^3 A \tan^3 B} + \sqrt{\tan^3 B \tan^3 C} + \sqrt{\tan^3 C \tan^3 A}$$

Proved. Equality $\Leftrightarrow A = B = C$ □

Solution 3 by proposer.

Lemma 0.2. *: If $a, b \in (0, \infty)$ then:*

$$(1) \qquad a + b \geq \sqrt{\frac{a^2 + b^2}{2}} + \sqrt{ab}$$

Proof. Denote $\begin{cases} x = \sqrt{\frac{a^2+b^2}{2}} \\ y = \sqrt{ab} \end{cases} \Rightarrow \begin{cases} a^2 + b^2 = 2x^2 \\ ab = y^2 \end{cases}$ □

$$(a+b)^2 = a^2 + 2ab + b^2 = 2x^2 + 2y^2 = 2(x^2+y^2)$$
$$1 \Leftrightarrow a+b \geq x+y \Leftrightarrow (a+b)^2 \geq (x+y)^2$$
$$\Leftrightarrow 2(x^2+y^2) \geq (x+y)^2 \Leftrightarrow 2x^2+2y^2 \geq x^2+2xy+y^2$$
$$\Leftrightarrow x^2 - 2xy + y^2 \geq 0 \Leftrightarrow (x-y)^2 \geq 0$$

In (1) *for* $a = \tan^3 A; b = \tan^3 B$

$$\tan^3 A + \tan^3 B \geq \sqrt{\frac{\tan^6 A + \tan^6 B}{2}} + \sqrt{\tan^3 A + \tan^3 B}$$
$$\sum(\tan^3 A + \tan^3 B) \geq \sum \sqrt{\frac{\tan^6 A + \tan^6 B}{2}} + \sum \tan A \tan B \sqrt{\tan A \tan B}$$
$$\overbrace{\geq}^{AM\text{-}GM} \sum \sqrt{\frac{\tan^6 A + \tan^6 B}{2}} + 3\sqrt[3]{\tan^3 A \tan^3 B \tan^3 C}$$
$$2\sum \tan^3 A \geq \sum \sqrt{\frac{\tan^6 A + \tan^6 B}{2}} + 3 \tan A \tan B \tan C =$$
$$= \sum \sqrt{\frac{\tan^6 A + \tan^6 B}{2}} + 3\sum \tan A$$

□

829. $\Omega_n = \binom{n}{7} + 2\binom{n-1}{7} + 3\binom{n-2}{7} + \cdots + (n-6)\binom{7}{7}; n \geq 7$

$$Find: \ \Omega = \lim_{n\to\infty} \sqrt[n]{\Omega_n}$$

Proposed by Daniel Sitaru - Romania

Proof.

$$\Omega = \lim_{n\to\infty} \sqrt[n]{\Omega_n} = \lim_{n\to\infty} \frac{\Omega_{n+1}}{\Omega_n} =$$
$$= \lim_{n\to\infty} \frac{\binom{n+1}{7} + 2\binom{n}{7} + \cdots + (n-5)\binom{7}{7}}{\binom{n}{7} + 2\binom{n-1}{7} + \cdot + (n-6)\binom{7}{7}} =$$
$$\overset{STOLZ-CESARO}{=} \lim_{n\to\infty} \frac{\Omega_{n+2} - \Omega_{n+1}}{\Omega_{n+1} - \Omega_n} =$$
$$= \lim_{n\to\infty} \frac{\binom{n+3}{7}}{\binom{n+2}{7}} =$$
$$\lim_{n\to\infty} \frac{(n+3)!}{7!(n+3-7)!} \cdot \frac{7!(n+2-7)!}{(n+2)!}$$
$$\lim_{n\to\infty} \frac{(n+2)! \cdot (n+3) \cdot (n-5)!}{(n-5)! \cdot (n-4) \cdot (n+2)!} = \lim_{n\to\infty} \frac{n+3}{n-4} = 1$$

□

830. If $x \in (0.\frac{\pi}{2})$ then:

$$2(\sin x)^{1-\sin x} \cdot (1-\sin x)^{\sin x} \le 1$$

Proposed by Daniel Sitaru - Romania

Solution:

$$\begin{gathered} Denote\ a = \sin x;\ b = 1 - \sin x \\ x \in (0, \frac{\pi}{2}) \Rightarrow a, b \in (0,1) \\ 2(\sin x)^{1-\sin x} \cdot (1-\sin x)^{\sin x} = 2a^b \cdot b^a \le \\ \overset{AM-GM}{\ge} 2 \cdot \left(\frac{a \cdot b + b \cdot a}{a+b}\right)^{a+b} \overset{AM-GM}{\ge} 2 \cdot \left(\frac{a+b}{2}\right)^{a+b} = \\ = 2 \cdot \left(\frac{\sin x + 1 - \sin x}{2}\right)^{a+b} = 2 \cdot \left(\frac{1}{2}\right)^{\sin x + 1 - \sin x} = 2 \cdot \left(\frac{1}{2}\right)^1 = 2 \cdot \frac{1}{2} = 1 \end{gathered}$$

831. If $\Delta ABC \sim \Delta A'B'C'$ then:

$$\sum \frac{(a'+b')(a'+c')}{b'c'} + 3 \ge \frac{15(b+c)(c'+a')(a'+b')}{8ab'c'}$$

Proposed by Daniel Sitaru - Romania

Proof.

$$\Delta ABC \sim \Delta A'B'C' \Rightarrow \frac{a'}{a} = \frac{b'}{b} = \frac{c'}{c} = k \Rightarrow a' = ka;\ b' = kb;\ c' = ck$$

Inequality can be written:

$$\begin{gathered} \sum \frac{(ka+kb)(ka+kc)}{kb \cdot kc} + 3 \ge \frac{15(b+c)(kc+ka)(ka+kb)}{8a \cdot kb \cdot kc} \\ \sum \frac{(a+b)(a+c)}{bc} + 3 \ge \frac{15(b+c)(c+a)(a+b)}{8abc} \\ \frac{s^2 - r^2 - Rr}{Rr} + 3 \ge \frac{15}{8} \cdot \frac{2s(s^2 + r^2 + 2Rr)}{4Rr} \\ \frac{s^2 - r^2 - Rr + 3Rr}{Rr} \ge \frac{15(s^2 + r^2 + 2Rr)}{16Rr} \\ 16(s^2 - r^2 + 2Rr) \ge 15(s^2 + r^2 + 2Rr) \\ 16s^2 - 16r^2 + 32Rr \ge 15s^2 + 15r^2 + 30Rr \\ s^2 \ge 31r^2 - 2Rr\ (to\ prove) \end{gathered}$$

By Gerretsen's inequality:

$$s^2 \geq 16Rr - 5r^2 \geq 31r^2 - 2Rr \Leftrightarrow 18Rr \geq 36r^2$$
$$R \geq 2r \ (Euler's\ inequality)$$

□

843. Prove that in ΔABC:

$$\sqrt{(2^{h_a} + 2^{h_b} + 2^{h_c})(2^{m_a} + 2^{m_b} + 2^{m_c})} < 2^a + 3^b + 4^c$$

Proof.

$$2^{h_a} < 2^{m_a} < 2^{\frac{b+c}{2}} = \sqrt{2^b \cdot 2^c} \leq \frac{2^b + 2^c}{2}$$
$$2^{h_b} < 2^{m_b} < 2^{\frac{a+c}{2}} = \sqrt{2^a \cdot 2^c} \leq \frac{2^a + 2^c}{2}$$
$$2^{h_c} < 2^{m_c} < 2^{\frac{b+a}{2}} = \sqrt{2^b \cdot 2^a} \leq \frac{2^b + 2^a}{2}$$
$$2^{h_a} + 2^{h_b} + 2^{h_c} < 2^{m_a} + 2^{m_b} + 2^{m_c} < 2^a + 2^b + 2^c$$
$$(2^{h_a} + 2^{h_b} + 2^{h_c})(2^{m_a} + 2^{m_b} + 2^{m_c}) < (2^a + 2^b + 2^c)^2$$
$$\sqrt{(2^{h_a} + 2^{h_b} + 2^{h_c})(2^{m_a} + 2^{m_b} + 2^{m_c})} < 2^a + 2^b + 2^c < 2^a + 3^b + 4^c$$
$$\sqrt{(2^{h_a} + 2^{h_b} + 2^{h_c})(2^{m_a} + 2^{m_b} + 2^{m_c})} < 2^a + 3^b + 4^c$$

□

844. Prove that if $0 < a < b < c < 1$ then:

$$2\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a\ln a & b\ln b & c\ln c \end{vmatrix} > \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ (a-1)\ln(a^2+1) & (b-1)\ln(b^2+1) & (c-1)\ln(c^2+1) \end{vmatrix}$$

Proof. Let be $f : (0,1) \to \mathbb{R}; f(x) = 2x\ln x - (x-1)\ln(1+x^2)$

$$f'(x) = 2\ln x + 2 - \ln(1+x^2) - \frac{2x(x-1)}{1+x^2}$$
$$f'(x) = \ln\frac{x^2}{x^2+1} + \frac{2(x+1)}{x^2+1}$$
$$f''(x) = \frac{\left(\frac{x^2}{x^2+1}\right)'}{\frac{x^2}{x^2+1}} + \frac{2(x^2+1) - 4x(x+1)}{(x^2+1)^2}$$
$$f''(x) = \frac{2}{x(x^2+1)} + \frac{-2x^2 - 4x + 2}{(x^2+1)^2}$$
$$f''(x) = \frac{-2x^3 - 2x^2 + 2x + 2}{x(x^2+1)^2} = \frac{2(x+1)(1-x^2)}{x(x^2+1)^2} > 0$$

It follows f strictly convex on $(0,1) \Rightarrow$

$$\Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ f(a) & f(b) & f(c) \end{vmatrix} > 0 \Rightarrow$$

$$\Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ 2a\ln a-(a-1)\ln(a^2+1) & 2b\ln b-(b-1)\ln(b^2+1) & 2c\ln c-(c-1)\ln(c^2+1) \end{vmatrix} > 0$$

$$\Rightarrow 2\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a\ln a & b\ln b & c\ln c \end{vmatrix} > \begin{vmatrix} 1 & 1 & 1 \\ (a-1)\ln(a^2+1) & (b-1)\ln(b^2+1) & (c-1)\ln(c^2+1) \end{vmatrix}$$

$\square$

845. If $a,b,c\in[0,1)$ then:

$$8\int_0^a\left(\int_0^b\left(\int_0^c \frac{\sin^{-1}x\cdot\sin^{-1}y\cdot\sin^{-1}z}{(1+\sin^{-1}x)(1+\sin^{-1}y)(1+\sin^{-1}z)}dz\right)dy\right)dx\le a^2b^2c^2$$

Proof. We will prove that:

$$x\ge\frac{\sin^{-1}x}{1+\sin^{-1}x};(\forall)x\in[0,1)$$

This is equivalent:

$$x(1+\sin^{-1}x)\ge\sin^{-1}x$$
$$\sin^{-1}x\cdot(1-x)\le x$$
$$\sin^{-1}x\le\frac{x}{1-x}$$

Let be $f:[0,1)\to\mathbb{R}; f(x)=\frac{x}{1-x}-\sin^{-1}x$

$$f'(x)=\frac{1}{(1-x)^2}-\frac{1}{\sqrt{1-x^2}}=$$

$$=\frac{1}{\sqrt{1-x}}\left(\frac{1}{(1-x)\sqrt{1-x}}-\frac{1}{\sqrt{1+x}}\right)\ge 0$$

because $x\in[0,1)\Rightarrow(1-x)\sqrt{1-x}\le 1$ and $\sqrt{x+1}\ge 1$
So, f is increasing on $[0,1)\Rightarrow$

$$\Rightarrow f(x)\ge f(0)=0,(\forall)x\in[0,1)$$

Hence:

$$x\ge\frac{\sin^{-1}x}{1+\sin^{-1}x};y\ge\frac{\sin^{-1}y}{1+\sin^{-1}y};z\ge\frac{\sin^{-1}z}{1+\sin^{-1}z};(\forall)x,y,z\in[0,1)$$

By multiplying:

$$\frac{\sin^{-1}x}{1+\sin^{-1}x}\cdot\frac{\sin^{-1}y}{1+\sin^{-1}y}\cdot\frac{\sin^{-1}z}{1+\sin^{-1}z}\le xyz$$

$$\int_0^a\left(\int_0^b\left(\int_0^c \frac{\sin^{-1}x\cdot\sin^{-1}y\cdot\sin^{-1}z}{(1+\sin^{-1}x)(1+\sin^{-1}y)(1+\sin^{-1}z)}dz\right)dy\right)dx\le$$

$$\le\int_0^a\left(\int_0^b\left(\int_0^c xyzdz\right)dy\right)dx=$$

$$= \int_0^a \left(\int_0^b \frac{xyc^2}{2} dy \right) dx = \int_0^a \left(\frac{xb^2c^2}{4} \right) dx = \frac{a^2b^2c^2}{8}$$

$$8 \int_0^a \left(\int_0^b \left(\int_0^c \frac{\sin^{-1} x \sin^{-1} y \sin^{-1} z}{(1 + \sin^{-1} x)(1 + \sin^{-1} y)(1 + \sin^{-1} z)} dz \right) dy \right) dx \leq a^2b^2c^2$$

$\square$

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