The Hyder Series

by Syed Shahabudeen

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Abstract

The Hyder Series is a generelized version of a special type of multiple infinite series.In this paper, we will be looking at some main aspect of this series in detail .

1 Introduction

Hyder Series is basically a generelized form of a special type of an infinite series. It is defined as

$$\mathcal{H}^q(\alpha_1, \alpha_2, \alpha_3 ... \alpha_k; p_1, p_2, p_3 ... p_k; \beta) = \sum_{m_1, m_2, m_3, ... m_k = 0}^{\infty} \frac{\prod\limits_{1 \le i \le k} \alpha_i^{m_i}}{\left(\sum\limits_{n=1}^{k} p_n m_n + \beta\right)^q}$$

where $\sum\limits_{m_1, m_2, m_3, ... m_k = 0} = \sum\limits_{m_1=0}^{\infty} \sum\limits_{m_2=0}^{\infty} \sum\limits_{m_3=0}^{\infty} .... \sum\limits_{m_k=0}^{\infty}$

In this paper we'll be looking at some special values, its respective proofs and relation of hyder series to hypergeometric series.

1.1 Notations

The q in the Hyder Notation Stands for the power order of the series. $p_1, p_2 ...., p_k$ are the coefficients of $m_1, m_2, ...m_k$.
If a number is being repeated for n number of times in the first two slots

of the Hyder Series, then it can be denoted as $\mathcal{H}^q(\alpha_{r(n)}; p_{r(n)}; \beta)$ where $r(n)$ stands for the n number of times repeated . Such that as per the definition it equals

$$\mathcal{H}^q(\alpha_{r(n)}; p_{r(n)}; \beta) = \sum_{m_1, m_2, m_3, \ldots m_n = 0}^{\infty} \frac{\alpha^{m_1+m_2+m_3+\ldots+m_n}}{\left(p \sum_{k=1}^{n} m_k + \beta\right)^q}$$

.It has to be noted that the number of sums repeated is equal to the number of terms written on any one of the first two slots of hyder notation.

1.2 Some Examples

These are some examples for the case when $q = 1$

$$\mathcal{H}\left(\frac{2}{3}, \frac{2}{3}; 2, 2; 1\right) = \frac{\sqrt{3}}{2\sqrt{2}} \tanh^{-1}\left(\sqrt{\frac{2}{3}}\right) + \frac{3}{2} \tag{1}$$

Proof.

$$\begin{aligned}
\mathcal{H}\left(\frac{2}{3}, \frac{2}{3}; 2, 2; 1\right) &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{\left(\frac{3}{2}\right)^{m+n}(2m + 2n + 1)} \\
&= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \int_0^1 \frac{x^{2m+2n}}{\left(\frac{3}{2}\right)^{m+n}} dx \\
&= \int_0^1 \left(\sum_{m=0}^{\infty} \left(\frac{2x^2}{3}\right)^m \sum_{n=0}^{\infty} \left(\frac{2x^2}{3}\right)^n \right) dx \\
&= \frac{3^2}{2^2} \underbrace{\int_0^1 \frac{1}{\left(\frac{3}{2} - x^2\right)^2} dx}_{I\left(\frac{3}{2}\right)}
\end{aligned}$$

Here

$$I(a) = \int_0^1 \frac{dx}{\left(a - x^2\right)^2}$$
$$= -\frac{\partial}{\partial a}\left(\frac{\tanh^{-1}\left(\frac{x}{\sqrt{a}}\right)}{\sqrt{a}}\right)\Bigg|_{x=0}^{1}$$
$$= \frac{\tanh^{-1}\left(\frac{1}{\sqrt{a}}\right)}{2a\sqrt{a}} + \frac{1}{2a^2\left(1 - \frac{1}{a}\right)}$$

$$\therefore I\left(\frac{3}{2}\right) = \frac{\sqrt{2}\tanh^{-1}\left(\frac{\sqrt{2}}{\sqrt{3}}\right)}{3\sqrt{3}} + \frac{2}{3}$$

So

$$\mathcal{H}\left(\frac{2}{3}, \frac{2}{3}; 2, 2; 1\right) = \frac{3^2}{2^2}\left(\frac{\sqrt{2}\tanh^{-1}\left(\frac{\sqrt{2}}{\sqrt{3}}\right)}{3\sqrt{3}} + \frac{2}{3}\right)$$
$$= \frac{\sqrt{3}}{2\sqrt{2}}\tanh^{-1}\left(\sqrt{\frac{2}{3}}\right) + \frac{3}{2}$$

□

$$\mathcal{H}\left(\left(\frac{1}{2}\right)_{r(3)}; 2_{r(3)}; 1\right) = \frac{3\tanh^{-1}\left(\frac{1}{\sqrt{2}}\right)}{4\sqrt{2}} + \frac{7}{4} \tag{2}$$

Proof.

$$\mathcal{H}\left(\left(\frac{1}{2}\right)_{r(3)}; 2_{r(3)}; 1\right) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \frac{1}{2^{m+n+p}\left(2m + 2n + 2p + 1\right)}$$
$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \int_0^1 \left(\frac{x^2}{2}\right)^{m+n+p} dx$$
$$= 2^3 \int_0^1 \frac{dx}{\left(2 - x^2\right)^3}$$
$$= \frac{3\tanh^{-1}\left(\frac{1}{\sqrt{2}}\right)}{4\sqrt{2}} + \frac{7}{4}$$

$\square$

Theorem 1. *This Theorem is a generalised version of the above two mentioned examples, It holds true for* $\alpha = \dfrac{1}{a}, p = 2$ *and* $q = 1$

$$\mathcal{H}\left(\left(\frac{1}{a}\right)_{r(m+1)}; 2_{r(m+1)}; 1\right) = \frac{1}{2}\left(\sum_{i=1}^{m}\binom{2m-2i}{m-i}\Omega\left(a,i\right)\right) + \frac{(2m)!}{2^{2m}(m!)^2 a^{m+\frac{1}{2}}}\tanh^{-1}\left(\frac{1}{\sqrt{a}}\right)$$

where

$$\Omega\left(a,i\right) = \frac{1}{i4^{m-i}(1-\frac{1}{a})^i}\sum_{k=0}^{i-1}\frac{\left(1-\frac{1}{a}\right)^k}{k!}\left(\frac{1}{2}\right)_k \quad a > 1,\ m \in \mathbb{N}$$

Before proceeding to the proof of theorem 1 we'll have to prove an important lemma

Lemma 1.

$$\frac{\partial^i}{\partial a^i}\tanh^{-1}\left(\frac{x}{\sqrt{a}}\right) = \frac{x\left(-1\right)^i\left(i-1\right)!}{2\sqrt{a}(a-x^2)^i}\sum_{k=0}^{i-1}\frac{\left(1-\frac{x^2}{a}\right)^k}{k!}\left(\frac{1}{2}\right)_k;\ \ i \in \mathbb{N} \qquad (3)$$

Proof. To find the 'i' th derivative of the given expression we'll make use of the Leibniz differentiation[9] formula which states that

$$\left(fg\right)^i = \sum_{k=0}^{i}\binom{i}{k} f^{(i-k)}g^{(k)} \qquad (4)$$

where f and g are n times differentiable functions. On differentiating $\tanh^{-1}\left(\dfrac{x}{\sqrt{a}}\right)$ with respect to a we get

$$\frac{\partial}{\partial a}\tanh^{-1}\left(\frac{x}{\sqrt{a}}\right) = \frac{-x}{2\sqrt{a}\left(a-x^2\right)}$$

let us take $f = \dfrac{-x}{2\left(a-x^2\right)}$ and $g = \dfrac{1}{\sqrt{a}}$
therefore 'n'th differentiation of both functions with respect to a is

$$f^{(n)} = \frac{-x(-1)^n n!}{2(a-x^2)^{n+1}} \text{ and } g^{(n)} = \frac{(-1)^n\,(2n-1)!!}{2^n a^{\frac{2n+1}{2}}}$$

on substituting the above values in eq 4 we'll get

$$(fg)^i = \frac{-x}{2}\sum_{k=0}^{i}\binom{i}{k}\frac{(-1)^i(i-k)!\,(2k-1)!!}{(a-x^2)^{i-k+1}2^k a^{\frac{2k+1}{2}}}$$

since $i \geq 1$ we have

$$(fg)^i = \frac{-x}{2}\sum_{k=0}^{i-1}\binom{i-1}{k}\frac{(-1)^{i-1}(i-k-1)!\,(2k-1)!!}{(a-x^2)^{i-k}2^k a^{\frac{2k+1}{2}}}$$

on rearranging and writing the double factorial in terms of Pochammer symbol[8]. i.e $\frac{(2k-1)!!}{2^k} = \left(\frac{1}{2}\right)_k$

we'll get

$$(fg)^i = \frac{x\,(-1)^i\,(i-1)!}{2\sqrt{a}(a-x^2)^i}\sum_{k=0}^{i-1}\frac{\left(1-\frac{x^2}{a}\right)^k}{k!}\left(\frac{1}{2}\right)_k$$

$\because (fg) = \frac{\partial}{\partial a}\tanh^{-1}\left(\frac{x}{\sqrt{a}}\right)$

$\therefore$

$$\frac{\partial^i}{\partial a^i}\tanh^{-1}\left(\frac{x}{\sqrt{a}}\right) = \frac{x\,(-1)^i\,(i-1)!}{2\sqrt{a}(a-x^2)^i}\sum_{k=0}^{i-1}\frac{\left(1-\frac{x^2}{a}\right)^k}{k!}\left(\frac{1}{2}\right)_k$$

□

Proof of Theorem 1

Proof.

$$\mathcal{H}\left(\left(\frac{1}{a}\right)_{r(m+1)};2_{r(m+1)};1\right) = \sum_{n_1,n_2,n_3,....,n_{m+1}\geq 0}\frac{1}{a^{n_1+n_2...+.n_{m+1}}\,(2n_1+2n_2+....+2n_{m+1}+1)}$$

$$= \int_0^1 \sum_{n_1,n_2,n_3,....,n_{m+1}\geq 0}\left(\frac{x^2}{a}\right)^{n_1+n_2+.....+n_{m+1}} dx$$

$$= a^{m+1}\underbrace{\int_0^1\frac{dx}{(a-x^2)^{m+1}}}_{I}$$

here

$$I = \int_0^1 \frac{dx}{(a-x^2)^{m+1}}$$
$$= \frac{(-1)^m}{m!} \frac{\partial^m}{\partial a^m} \left(\frac{\tanh^{-1}\left(\frac{1}{\sqrt{a}}\right)}{\sqrt{a}} \right)$$

Now to differentiate the above expression we'll make use of Leibniz differentiation formula

$$(fg)^m = \sum_{i=0}^{m} \binom{m}{i} f^{(m-i)} g^{(i)} \tag{5}$$

let us consider $f = \dfrac{1}{\sqrt{a}}$ and $g = \tanh^{-1} \dfrac{1}{\sqrt{a}}$

therefore its respective 'n'th Derivatives are $f^{(n)} = \dfrac{(-1)^n (2n)!}{2^{2n}(n!)a^{\frac{2n+1}{2}}}$ and $g^{(n)} = \dfrac{(-1)^n (n-1)!}{2\sqrt{a}(a-x^2)^n} \sum_{k=0}^{n-1} \dfrac{\left(1-\frac{1}{a}\right)^k}{k!} \left(\dfrac{1}{2}\right)_k$,(from **Lemma 1**)

we can rewrite Eq (5) as

$$(fg)^m = \sum_{i=1}^{m} \binom{m}{i} f^{(m-i)} g^{(i)} + f^{(m)} g \tag{6}$$

$\therefore$

$$\frac{\partial^m}{\partial a^m} \left(\frac{\tanh^{-1}\left(\frac{1}{\sqrt{a}}\right)}{\sqrt{a}} \right) = \frac{(-1)^m}{2a^{m+1}} \sum_{i=1}^{m} \binom{m}{i} \frac{(i-1)!(2m-2i)!}{4^{m-i}(m-i)!(1-\frac{1}{a})^i} \sum_{k=0}^{i-1} \frac{\left(1-\frac{1}{a}\right)^k}{k!} \left(\frac{1}{2}\right)_k + \frac{(-1)^m (2m)!}{2^{2m}(m!)a^{m+\frac{1}{2}}} \tanh^{-1}\left(\frac{1}{\sqrt{a}}\right)$$

the above expression can be rewritten as

$$\frac{\partial^m}{\partial a^m} \left(\frac{\tanh^{-1}\left(\frac{1}{\sqrt{a}}\right)}{\sqrt{a}} \right) = \frac{(-1)^m m!}{2a^{m+1}} \sum_{i=1}^{m} \frac{\binom{2m-2i}{m-i}}{i4^{m-i}(1-\frac{1}{a})^i} \sum_{k=0}^{i-1} \frac{\left(1-\frac{1}{a}\right)^k}{k!} \left(\frac{1}{2}\right)_k + \frac{(-1)^m (2m)!}{2^{2m}(m!)a^{m+\frac{1}{2}}} \tanh^{-1}\left(\frac{1}{\sqrt{a}}\right)$$

where

$$\Omega\left(a,i\right)=\frac{1}{i4^{m-i}(1-\frac{1}{a})^{i}}\sum_{k=0}^{i-1}\frac{\left(1-\frac{1}{a}\right)^{k}}{k!}\left(\frac{1}{2}\right)_{k}$$

we came to know that

$$I=\frac{(-1)^{m}}{m!}\frac{\partial^{m}}{\partial a^{m}}\left(\frac{\tanh^{-1}\left(\frac{1}{\sqrt{a}}\right)}{\sqrt{a}}\right)$$

$\therefore$

$$\begin{aligned}I&=\frac{(-1)^{m}}{m!}\left(\frac{(-1)^{m}m!}{2a^{m+1}}\left(\sum_{i=1}^{m}\binom{2m-2i}{m-i}\Omega\left(a,i\right)\right)+\frac{(-1)^{m}\left(2m\right)!}{2^{2m}(m!)a^{m+\frac{1}{2}}}\tanh^{-1}\left(\frac{1}{\sqrt{a}}\right)\right)\\&=\frac{1}{2a^{m+1}}\left(\sum_{i=1}^{m}\binom{2m-2i}{m-i}\Omega\left(a,i\right)\right)+\frac{(2m)!}{2^{2m}(m!)^{2}a^{m+\frac{1}{2}}}\tanh^{-1}\left(\frac{1}{\sqrt{a}}\right)\end{aligned}$$

$\because$

$$\mathcal{H}\left(\left(\frac{1}{a}\right)_{r(m+1)};2_{r(m+1)};1\right)=a^{m+1}I$$

and finally we'll get the result

$$\mathcal{H}\left(\left(\frac{1}{a}\right)_{r(m+1)};2_{r(m+1)};1\right)=\frac{1}{2}\left(\sum_{i=1}^{m}\binom{2m-2i}{m-i}\Omega\left(a,i\right)\right)+\frac{(2m)!}{2^{2m}(m!)^{2}a^{m+\frac{1}{2}}}\tanh^{-1}\left(\frac{1}{\sqrt{a}}\right)\tag{7}$$

□

Example 1. let us plug the value's $a=3$ and $m=4$ in Eq(7) i.e $\mathcal{H}\left(\left(\frac{1}{3}\right)_{r(5)};2_{r(5)};1\right)$ where 3 and 2 are repeated 5 times. Therefore from theorem 1 we'll get

$$\begin{aligned}\mathcal{H}\left(\left(\frac{1}{3}\right)_{r(5)};2_{r(5)};1\right)&=\frac{1}{2}\left(\sum_{i=1}^{4}\binom{8-2i}{4-i}\Omega\left(3,i\right)\right)+\frac{(8)!\sqrt{3}}{2^{8}\left(4!\right)^{2}}\tanh^{-1}\left(\frac{1}{\sqrt{3}}\right)\\&=\frac{249}{128}+\frac{35\sqrt{3}}{128}\tanh^{-1}\left(\frac{1}{\sqrt{3}}\right)\end{aligned}$$

Theorem 2. *This Equation holds true for $p = 2$,$\beta = 1$ and for the case when each values of a_i is greater than 1.*

$$\mathcal{H}\left(\left\{\frac{1}{a_i}\right\}_{i=1}^{n};2_{r(n)};1\right) = (-1)^{n+1}\sum_{cyc}\frac{\tanh^{-1}\left(\frac{1}{\sqrt{a_1}}\right)}{\sqrt{a_1}\prod_{1<j\leq n}(a_1-a_j)}\left(\prod_{i=1}^{n}a_i\right); \quad a_i>1$$

Proof. As per the definition of Hyder series we can write the expression

$$\begin{aligned}\mathcal{H}\left(\left\{\frac{1}{a_i}\right\}_{i=1}^{n};2_{r(n)};1\right) &= \sum_{k_1,k_2,k_3,....,k_n\geq 0}\frac{1}{{a_1}^{k_1}{a_2}^{k_2}....{a_n}^{k_n}\left(2k_1+2k_2+....+2k_n+1\right)}\\ &= \int_0^1\sum_{k_1,k_2,k_3,....,k_n\geq 0}\left(\frac{x^2}{a_1}\right)^{k_1}\left(\frac{x^2}{a_2}\right)^{k_1}.....\left(\frac{x^2}{a_n}\right)^{k_n}dx\\ &= (-1)^n\left(\prod_{i=1}^{n}a_i\right)\int_0^1\frac{1}{(x^2-a_1)(x^2-a_2)....(x^2-a_n)}dx\end{aligned}$$

The following integral can be solved by using partial fraction. It is quite intresting to note that while using partial fraction for this product,we can see a cyclic pattern in it. i.e

$$\frac{1}{(x^2-a_1)(x^2-a_2)....(x^2-a_n)} = \sum_{cyc}\frac{1}{(x^2-a_1)\prod_{1<j\leq n}(a_1-a_j)}$$

therefore

$$\begin{aligned}\int_0^1\frac{1}{(x^2-a_1)(x^2-a_2)....(x^2-a_n)}dx &= \int_0^1\sum_{cyc}\frac{1}{(x^2-a_1)\prod_{1<j\leq n}(a_1-a_j)}dx\\ &= \sum_{cyc}\frac{1}{\prod_{1<j\leq n}(a_1-a_j)}\int_0^1\frac{1}{(x^2-a_1)}dx\end{aligned}$$

since $\displaystyle\int_0^1\frac{1}{(x^2-a_1)}dx = \frac{-\tanh^{-1}\left(\frac{1}{\sqrt{a_1}}\right)}{\sqrt{a_1}}$

therefore

$$\int_0^1 \frac{1}{\left(x^2-a_1\right)\left(x^2-a_2\right)....\left(x^2-a_n\right)}dx = -\sum_{cyc} \frac{\tanh^{-1}\left(\frac{1}{\sqrt{a_1}}\right)}{\sqrt{a_1}\prod_{1<j\leq n}\left(a_1-a_j\right)}$$

and finally we'll get the result as

$$\mathcal{H}\left(\left\{\frac{1}{a_i}\right\}_{i=1}^{n}; 2_{r(n)}; 1\right) = (-1)^{n+1}\sum_{cyc} \frac{\tanh^{-1}\left(\frac{1}{\sqrt{a_1}}\right)}{\sqrt{a_1}\prod_{1<j\leq n}\left(a_1-a_j\right)}\left(\prod_{i=1}^{n} a_i\right) \tag{8}$$

□

Example 2. For $n = 3$ in Eq(8) we have

$$\begin{aligned}\mathcal{H}\left(\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}; 2_{r(3)}; 1\right) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \frac{1}{a_1^m a_2^n a_3^p\left(2m+2n+2p+1\right)} \\ &= a_1a_2a_3 \sum_{cyc} \frac{\tanh^{-1}\left(\frac{1}{\sqrt{a_1}}\right)}{\sqrt{a_1}\prod_{1<j\leq 3}\left(a_1-a_j\right)}\end{aligned}$$

on expanding the Cyclic sum we get

$$a_1a_2a_3\left(\frac{\tanh^{-1}\left(\frac{1}{\sqrt{a_1}}\right)}{\sqrt{a_1}(a_1-a_2)(a_1-a_3)} + \frac{\tanh^{-1}\left(\frac{1}{\sqrt{a_2}}\right)}{\sqrt{a_2}(a_2-a_1)(a_2-a_3)} + \frac{\tanh^{-1}\left(\frac{1}{\sqrt{a_3}}\right)}{\sqrt{a_3}(a_3-a_1)(a_3-a_2)}\right)$$

let us plug the values $a_1 = \pi$, $a_2 = \phi$, and $a_3 = e$, (where ϕ, e are golden ratio and exponential constant) therefore

$$\mathcal{H}\left(\frac{1}{\pi},\frac{1}{\phi},\frac{1}{e};2_{r(3)};1\right)=\pi\phi e\left(\frac{\tanh^{-1}\left(\frac{1}{\sqrt{\pi}}\right)}{\sqrt{\pi}(\pi-\phi)(\pi-e)}+\frac{\tanh^{-1}\left(\frac{1}{\sqrt{\phi}}\right)}{\sqrt{\phi}(\phi-\pi)(\phi-e)}+\frac{\tanh^{-1}\left(\frac{1}{\sqrt{e}}\right)}{\sqrt{e}(e-\pi)(e-\phi)}\right.$$

Theorem 3.

$$\mathcal{H}\left(a_{r(m)};p_{r(m)};\eta\right)=\frac{1}{\eta}\,{}_2F_1\left(m,\frac{\eta}{p};\frac{\eta}{p}+1;a\right);\quad a\in(0,1),m\in\mathbb{N}$$

where ${}_2F_1$ is a Hypergeometric Series[3]

Proof.

$$\begin{aligned}\mathcal{H}\left(a_{r(m)};p_{r(m)};\eta\right)&=\sum_{n_1,n_2,n_3,\ldots\ldots n_m\geq 0}\frac{a^{n_1+n_2+\ldots.n_m}}{(pn_1+pn_2+\ldots\ldots pn_m+\eta)}\\&=\frac{1}{a^{\frac{\eta-1}{p}}}\int_0^1\sum_{n_1,n_2,n_3,\ldots\ldots n_m\geq 0}(ax^p)^{n_1+n_2+n_3\ldots+n_m+\frac{\eta-1}{p}}dx\\&=\int_0^1\frac{x^{(\eta-1)}}{(1-ax^p)^m}dx&&(\text{Let }t=x^p)\\&=\frac{1}{p}\int_0^1\frac{t^{\frac{\eta}{p}-1}}{(1-at)^m}dt\end{aligned}$$

Since

$$\frac{\Gamma(\beta)\Gamma(\gamma-\beta)}{\Gamma(\gamma)}{}_2F_1(\alpha,\beta;\gamma;z)=\int_0^1t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha}dt\qquad(\text{Euler Integral })$$

Therefore

$$\frac{1}{p}\int_0^1\frac{t^{\frac{\eta}{p}-1}}{(1-at)^m}dt=\frac{1}{p}\frac{\Gamma\left(\frac{\eta}{p}\right)\Gamma(1)}{\Gamma\left(\frac{\eta}{p}+1\right)}{}_2F_1\left(m,\frac{\eta}{p};\frac{\eta}{p}+1;a\right)$$

$\Rightarrow$

$$\mathcal{H}\left(a_{r(m)};p_{r(m)};\eta\right)=\frac{1}{\eta}\,{}_2F_1\left(m,\frac{\eta}{p};\frac{\eta}{p}+1;a\right)\tag{9}$$

□

From the above theorem it is clear how the hyder series is related to the ${}_2F_1$ hypergeometric series . Now we will look forward to some special cases.

Corollary 3.1.

$$\mathcal{H}\left(\left(\frac{1}{2}\right)_{r(m)};1_{r(m)};m\right)=2^{m-1}\left(\psi\left(\frac{1+m}{2}\right)-\psi\left(\frac{m}{2}\right)\right);\quad m\in\mathbb{N}$$

Proof.

$$\begin{aligned}\mathcal{H}\left(\left(\frac{1}{2}\right)_{r(m)};1_{r(m)};m\right)&=\frac{1}{m}\,{}_2F_1\left(m,m;m+1;\frac{1}{2}\right) && \text{(from theorem 2)}\\ &=\frac{2^m}{m}\,{}_2F_1\left(m,1;m+1;-1\right) && \text{(Apply Pfaff Transformation)}\\ &=2^m\int_0^1\frac{t^{m-1}}{1+t}dt\end{aligned}$$

the following integral can be easily proved in terms of digamma function[5] ,A beautiful proof of it's can also be seen in the book "Almost Impossible Integral and Sums"[1] at page 67. Therefore

$$\int_0^1\frac{t^{m-1}}{1+t}dt=\frac{1}{2}\left(\psi\left(\frac{1+m}{2}\right)-\psi\left(\frac{m}{2}\right)\right)$$

Hence

$$\mathcal{H}\left(\left(\frac{1}{2}\right)_{r(m)};1_{r(m)};m\right)=2^{m-1}\left(\psi\left(\frac{1+m}{2}\right)-\psi\left(\frac{m}{2}\right)\right)\tag{10}$$

□

By making use of the Series definition of the Digamma function, we can write the digamma relation in Eq (9) in terms of the Lerch Transcendent[4] Notation i.e

$$\mathcal{H}\left(\left(\frac{1}{2}\right)_{r(m)};1_{r(m)};m\right)=\Phi(-1,1,m)$$

where $\Phi(-1,1,m)=\sum_{k=0}^{\infty}\frac{(-1)^k}{m+k}$

Example 3. let us plug the values $m = 4$ in Eq(9) Therefore as per the definition it equals to

$$\mathcal{H}\left(\left(\frac{1}{2}\right)_{r(4)};1_{r(4)};4\right) = \sum_{x=0}^{\infty}\sum_{y=0}^{\infty}\sum_{z=0}^{\infty}\sum_{l=0}^{\infty}\frac{1}{2^{x+y+z+l}(x+y+z+l+4)} \quad \text{(Apply result from Eq(9))}$$

$$= 2^3\left(\psi\left(\frac{5}{2}\right) - \psi(2)\right)$$

Since $\psi\left(\frac{5}{2}\right) = \frac{8}{3} - \gamma - \ln(4)$ and $\psi(2) = 1 - \gamma$. (where γ is Euler-Mascheroni[2] constant).
therefore

$$\mathcal{H}\left(\left(\frac{1}{2}\right)_{r(4)};1_{r(4)};4\right) = \frac{40}{3} - 16\ln(2)$$

1.3 Hyder Series of Higher order

The Hyder Series that we just encountered at the beginning of this paper were all just the order of 1 i.e $q = 1$. In this section we will be looking at some special cases of higher order of Hyder Series.

Lemma 2. *For* $q, \beta \in \mathbb{N}$

$$\int_0^1 \frac{x^{\beta+1}}{\left(1-\frac{x}{a}\right)^2}\log^q(x)\,dx = a^{\beta+2}(-1)^q\,(q\,!)\left(\operatorname{Li}_q\left(\frac{1}{a}\right) - (\beta+1)\operatorname{Li}_{q+1}\left(\frac{1}{a}\right) + (\beta+1)\sum_{k=1}^{\beta}\frac{1}{a^k k^{q+1}} - \sum_{k=1}^{\beta}\frac{1}{a^k k^q}\right)$$

where $\operatorname{Li}_s(z)$ *is the polylogarithm function*[6]

Proof.

$$\int_0^1 \frac{x^{\beta+1}}{\left(1-\frac{x}{a}\right)^2}\log^q(x)\,dx = a\int_0^1 \frac{\frac{x}{a}}{\left(1-\frac{x}{a}\right)^2}\,x^\beta\log^q(x)\,dx$$

□

The above mentioned Integral can be evaluated by using the series

$$\sum_{k=0}^{\infty} kx^k = \frac{x}{(1-x)^2}$$

In our's case x equals $\frac{x}{a}$. Therefore we can write the Integral as

$$a\int_0^1 \frac{\frac{x}{a}}{\left(1-\frac{x}{a}\right)^2}\, x^{\beta} \log^{q}(x)\, dx = a\sum_{k=0}^{\infty} \frac{k}{a^k} \int_0^1 x^{\beta+k} \log^{q}(x) dx$$

To evaluate the integral we'll make use of a well known result

$$\int_0^1 x^m \log^{q}(x) dx = \frac{(-1)^q\, q!}{(m+1)^{q+1}} \tag{11}$$

therefore

$$\begin{aligned} a\sum_{k=0}^{\infty} \frac{k}{a^k} \int_0^1 x^{\beta+k} \log^{q}(x) dx &= a(-1)^{q}\ (q\,!) \sum_{k=0}^{\infty} \frac{k}{a^k\,(\beta+k+1)^{q+1}} \\ &= a(-1)^{q}\ (q\,!) \left(\sum_{k=0}^{\infty} \frac{1}{a^k\,(\beta+k+1)^{q}} - (\beta+1) \sum_{k=0}^{\infty} \frac{1}{a^k\,(\beta+k+1)^{q+1}} \right) \\ &= a(-1)^{q}\ (q\,!) \left(\Phi\left(\frac{1}{a}, q, \beta+1\right) - (\beta+1)\Phi\left(\frac{1}{a}, q+1, \beta+1\right) \right) \end{aligned}$$

On writing the Lerch Transdescent in terms of Polylogarithm function we get

$$\Phi\left(\frac{1}{a}, q, \beta+1\right) = a^{\beta+1} \left(\text{ Li }_{q}\left(\frac{1}{a}\right) - \sum_{k=1}^{\beta} \frac{1}{a^k k^q} \right)$$

and

$$\Phi\left(\frac{1}{a}, q+1, \beta+1\right) = a^{\beta+1} \left(\text{ Li }_{q+1}\left(\frac{1}{a}\right) - \sum_{k=1}^{\beta} \frac{1}{a^k k^{q+1}} \right)$$

finally upon substitution, we will get the result as

$$\begin{aligned} \int_0^1 \frac{x^{\beta+1}}{\left(1-\frac{x}{a}\right)^2} \log^{q}(x)\, dx = a^{\beta+2}(-1)^{q}\ (q\,!) \Bigg(\text{ Li }_{q}\left(\frac{1}{a}\right) - (\beta+1) \text{ Li }_{q+1}\left(\frac{1}{a}\right) + (\beta+1) \sum_{k=1}^{\beta} \frac{1}{a^k k^{q+1}} \\ - \sum_{k=1}^{\beta} \frac{1}{a^k k^q} \Bigg) \end{aligned}$$

Theorem 4.

$$\mathcal{H}^{q+1}\left(\left(\frac{1}{a}\right)_{r(2)};1_{r(2)};\beta+2\right)=a^{\beta+2}\Bigg(\operatorname{Li}_q\left(\frac{1}{a}\right)-(\beta+1)\operatorname{Li}_{q+1}\left(\frac{1}{a}\right)+(\beta+1)\sum_{k=1}^{\beta}\frac{1}{a^k k^{q+1}}-\sum_{k=1}^{\beta}\frac{1}{a^k k^q}\Bigg)$$

Proof. As per the definition of Hyder Series

$$\mathcal{H}^{q+1}\left(\left(\frac{1}{a}\right)_{r(2)};1_{r(2)};\beta+2\right)=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{1}{a^{k_1+k_2}(k_1+k_2+\beta+2)^{q+1}}$$

To evaluate the series, we will make use of the Eq(11), In this case $m = k_1+k_2+\beta+1$
therefore

$$\begin{aligned}\mathcal{H}^{q+1}\left(\left(\frac{1}{a}\right)_{r(2)};1_{r(2)};\beta+2\right)&=\frac{(-1)^q}{q!}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{1}{a^{k_1+k_2}}\int_0^1 x^{k_1+k_2+\beta+1}\log^q(x)\,dx\\&=\frac{(-1)^q}{q!}\int_0^1 x^{\beta+1}\log^q(x)\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{x^{k_1+k_2}}{a^{k_1+k_2}}\,dx\\&=\frac{(-1)^q}{q!}\underbrace{\int_0^1\frac{x^{\beta+1}}{\left(1-\frac{x}{a}\right)^2}\log^q(x)\,dx}_{A}\end{aligned}$$

The above mentioned Integral A is the same Integral that we just came to prove in **Lemma 2**. Therefore the final result becomes

$$\mathcal{H}^{q+1}\left(\left(\frac{1}{a}\right)_{r(2)};1_{r(2)};\beta+2\right)=a^{\beta+2}\Bigg(\operatorname{Li}_q\left(\frac{1}{a}\right)-(\beta+1)\operatorname{Li}_{q+1}\left(\frac{1}{a}\right)+(\beta+1)\sum_{k=1}^{\beta}\frac{1}{a^k k^{q+1}}-\sum_{k=1}^{\beta}\frac{1}{a^k k^q}\Bigg)$$

□

Example 4. For $q = 2, \beta = 2$ and $a = 2$ in **Theorem 4** , from definition we have

$$\mathcal{H}^3\left(\left(\frac{1}{2}\right)_{r(2)}; 1_{r(2)}; 4\right) = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} \frac{1}{2^{k_1+k_2}(k_1+k_2+4)^3}$$
$$= 2^4\left(\text{ Li }_2\left(\frac{1}{2}\right) - 3\text{ Li }_3\left(\frac{1}{2}\right) + 3\sum_{k=1}^{2}\frac{1}{2^k k^3} - \sum_{k=1}^{2}\frac{1}{2^k k^2}\right)$$

here

$$\text{Li }_2\left(\frac{1}{2}\right) = \frac{\pi^2}{12} - \frac{\log^2(2)}{2}$$

and

$$\text{Li }_3\left(\frac{1}{2}\right) = \frac{\log^3(2)}{6} + \frac{7\zeta(3)}{8} - \frac{\pi^2\log(2)}{12}$$

therefore

$$\mathcal{H}^3\left(\left(\frac{1}{2}\right)_{r(2)}; 1_{r(2)}; 4\right) = \frac{4\pi^2}{3} + 4\pi^2\log(2) + \frac{33}{2} - 8\log^3(2) - 8\log^2(2) - 42\zeta(3)$$

Corollary 4.1. *The following equation hold's true for the case when $q \geq 2$*

$$\mathcal{H}^{q+1}\left(1_{r(2)}; 1_{r(2)}; 2\right) = \zeta(q) - \zeta(q+1) \tag{12}$$

where $\zeta(q)$ is the zeta function[7]

Proof.

$$\mathcal{H}^{q+1}\left(1_{r(2)}; 1_{r(2)}; 2\right) = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{1}{(k_1+k_2+2)^{q+1}} \qquad \text{(Apply Theorem 4)}$$
$$= \text{ Li }_q(1) - \text{ Li }_{q+1}(1)$$

Since $\text{Li }_q(1) = \zeta(q)$. therefore

$$\mathcal{H}^{q+1}\left(1_{r(2)}; 1_{r(2)}; 2\right) = \zeta(q) - \zeta(q+1)$$

□

Example 5. Here are some examples for the above case

$$\begin{aligned}\mathcal{H}^3\left(1_{r(2)};1_{r(2)};2\right) &= \zeta(2)-\zeta(3)\\ &= \frac{\pi^2}{6}-\zeta(3)\end{aligned}$$

$$\begin{aligned}\mathcal{H}^4\left(1_{r(2)};1_{r(2)};3\right) &= \zeta(3)-\zeta(4)\\ &= \zeta(3)-\frac{\pi^4}{90}\end{aligned}$$

2 Conclusion

This paper was just an introduction to Hyder Series. We came to see some important results and some special cases of Hyder series of higher order. This series is named in honour of my late Grandfather Syed Hyder , who was a chief executive engineer in the water department of state Tamil Nadu. He enjoyed solving mathematical problems during his leisure times and was a man of wit and humour. I hope this Series could have many more other intresting result to be discovered and could also have unique relation to some special functions.

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Web:http://www.ssmrmh.ro

Warriorshahab@gmail.com
Kmea Engineering College
Ernakulam,Kerala-India