FEW AMAZING PROPERTIES OF MEANS

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Abstract. In this paper are presented a few amazing properties of means.

Notations:

$$A_n = \frac{a_1 + a_2 + \ldots + a_n}{n}; G_n = \sqrt[n]{a_1 a_2 \cdot \ldots \cdot a_n}$$

Property 1:
If $0 < a_1 \leq a_2 \leq \ldots \leq a_n$ then:

$$A_{k-1} \leq A_k \leq a_n; \forall k \in \overline{2, n}, n \in \mathbb{N}, n \geq 2 \tag{1}$$

Proof.

$$A_{k-1} \leq A_k \Leftrightarrow \frac{a_1 + a_2 + \ldots + a_{k-1}}{k-1} \leq \frac{a_1 + a_2 + \ldots + a_k}{k} \Leftrightarrow$$

$$k(a_1 + a_2 + \ldots + a_{k-1}) \leq (k-1)(a_1 + a_2 + \ldots + a_{k-1} + a_k)$$

$$k(a_1 + a_2 + \ldots + a_{k-1}) \leq k(a_1 + a_2 + \ldots + a_{k-1}) + ka_k - (a_1 + a_2 + \ldots + a_{k-1} + a_k)$$

$$a_1 + a_2 + \ldots + a_{k-1} + a_k \leq ka_k$$

which result by adding all relationships $a_i \leq a_k; \forall i \in \overline{1, k}$

$$A_k \leq a_n \Leftrightarrow \frac{a_1 + a_2 + \ldots + a_k}{k} \leq a_n \Leftrightarrow a_1 + a_2 + \ldots + a_k \leq ka_n$$

which result by adding all relationships $a_i \leq a_n; \forall i \in \overline{1, k}$.

□

Corollary 1:
If $0 < a \leq b \leq c$ then:

$$\frac{a+b}{2} \leq \frac{a+b+c}{3} \leq c \tag{2}$$

Proof.

We take in (1) : $n = 3, a_1 = a, a_2 = b, a_3 = c$.

□

Corollary 2:
If $0 < a \leq b \leq c \leq d$ then:

$$\frac{a+b+c}{3} \leq \frac{a+b+c+d}{4} \leq d \tag{3}$$

Proof.

We take in (1) : $n = 3, a_1 = a, a_2 = b, a_3 = c, a_4 = d$.

□

Corollary 3:
If $x, y > 0$ then:

$$\frac{\sqrt{xy} + \frac{x+y}{2}}{2} \leq \frac{\sqrt{xy} + \frac{x+y}{2} + \sqrt{\frac{x^2+y^2}{2}}}{3} \leq \sqrt{\frac{x^2+y^2}{2}}$$

Proof.

$$\text{We take in (2)} : a = \sqrt{xy}; b = \frac{x+y}{2}; c = \sqrt{\frac{x^2+y^2}{2}}$$

□

Corollary 4:
If $x, y > 0$ then:

$$\frac{\frac{2xy}{x+y} + \sqrt{xy} + \frac{x+y}{2}}{3} \leq \frac{\frac{2xy}{x+y} + \sqrt{xy} + \frac{x+y}{2} + \sqrt{\frac{x^2+y^2}{2}}}{4} \leq \sqrt{\frac{x^2+y^2}{2}}$$

Proof.

$$\text{We take in (3)} : a = \frac{2xy}{x+y}; b = \sqrt{xy}; c = \frac{x+y}{2}; d = \sqrt{\frac{x^2+y^2}{2}}$$

□

Corollary 5:
If $x, y, z > 0$ then:

$$\frac{\sqrt[3]{xyz} + \frac{x+y+z}{3}}{2} \leq \frac{\sqrt[3]{xyz} + \frac{x+y+z}{3} + \sqrt{\frac{x^2+y^2+z^2}{3}}}{3} \leq \sqrt{\frac{x^2+y^2+z^2}{3}}$$

Proof.

$$\text{We take in (2)} : a = \sqrt[3]{xyz}; b = \frac{x+y+z}{3}; c = \sqrt{\frac{x^2+y^2+z^2}{3}}$$

□

Property 2:
If $0 < a_1 \leq a_2 \leq \ldots \leq a_n$ then:

$$a_k \cdot A_{k-1}^{k-1} \leq A_k^k; k \in \overline{2, n}, n \in \mathbb{N}, n \geq 2 \tag{4}$$

Proof.

$$\text{By (1)} : A_{k-1} \leq A_k \Rightarrow \frac{A_k}{A_{k-1}} \geq 2; \forall k \in \overline{2, n}$$

$$\frac{A_k^k}{A_{k-1}^{k-1}} = A_{k-1} \cdot \left(\frac{A_k}{A_{k-1}}\right)^k \overset{\text{Bernoulli}}{\geq} A_{k-1}\left(1 + k\left(\frac{A_k}{A_{k-1}} - 1\right)\right) =$$

$$= A_{k-1} + kA_k - kA_{k-1} = kA_k - (k-1)A_{k-1} =$$

$$= k \cdot \frac{a_1 + a_2 + \ldots a_k}{k} - (k-1) \cdot \frac{a_1 + a_2 + \ldots + a_{k-1}}{k-1}$$

$$\frac{A_k^k}{A_{k-1}^{k-1}} \geq a_k \Rightarrow a_k \cdot A_{k-1}^{k-1} \leq A_k^k$$

□

Corollary 6:
If $0 < a \leq b \leq c$ then:

$$c\Big(\frac{a+b}{2}\Big)^2 \leq \Big(\frac{a+b+c}{3}\Big)^3 \tag{5}$$

Proof.

$$\text{We take in (4)} : n = 3, a_1 = a, a_2 = b, a_3 = c.$$

□

Corollary 7:
If $0 < a \leq b \leq c \leq d$ then:

$$d\Big(\frac{a+b+c}{3}\Big)^3 \leq \Big(\frac{a+b+c+d}{4}\Big)^4 \tag{6}$$

Proof.

$$\text{We take in (4)} : n = 4, a_1 = a, a_2 = b, a_3 = c, a_4 = d$$

□

Corollary 8
If $x, y > 0$ then:

$$\frac{x+y}{2}\left(\frac{\frac{2xy}{x+y} + \sqrt{xy}}{2}\right)^2 \leq \left(\frac{\frac{2xy}{x+y} + \sqrt{xy} + \frac{x+y}{2}}{3}\right)^3$$

Proof.

$$\text{We take in (5)} : a = \frac{2xy}{x+y}, b = \sqrt{xy}, x = \frac{x+y}{2}.$$

□

Corollary 9:
If $x, y > 0$ then:

$$\sqrt{\frac{x^2+y^2+z^2}{3}}\left(\frac{\frac{3xyz}{xy+yz+zx} + \sqrt[3]{xyz} + \frac{x+y+z}{3\S}}{3}\right)^3 \leq$$

$$\leq \left(\frac{\frac{3xyz}{xy+yz+zx} + \sqrt[3]{xyz} + \frac{x+y+z}{3} + \sqrt{\frac{x^2+y^2+z^2}{3}}}{4}\right)^4$$

Proof.

$$\text{We take in (6)} : a = \frac{3xyz}{xy+yz+zx}, b = \sqrt[3]{xyz}, c = \frac{x+y+z}{3}, d = \sqrt{\frac{x^2+y^2+z^2}{3}}$$

□

Property 3:
If $0 < a_1 \leq a_2 \leq \ldots \leq a_n, n \in \mathbb{N}, n \geq 2$ then:

$$G_{k-1} \leq G_k \leq a_n; \forall k \in \overline{2, n} \tag{7}$$

Proof.

$$\left(\frac{G_k}{G_{k-1}}\right)^{k(k-1)}=\left(\frac{\sqrt[k]{a_1a_2\cdot\ldots\cdot a_k}}{\sqrt[k-1]{a_1a_2\cdot\ldots\cdot a_{k-1}}}\right)^{k(k-1)}=\frac{(a_1a_2\cdot\ldots\cdot a_k)^{k-1}}{(a_1a_2\cdot\ldots\cdot a_{k-1})^k}=$$

$$=\frac{a_k^{k-1}}{a_1a_2\cdot\ldots\cdot a_{k-1}}=\frac{a_k}{a_1}\cdot\frac{a_k}{a_2}\cdot\ldots\cdot\frac{a_{k-1}}{a_k}\geq 1 \text{ because } a_k\geq a_k\geq a_i;\forall i\in\overline{1,k}.$$

$$\left(\frac{G_k}{G_{k-1}}\right)^{k(k-1)}\geq 1\Rightarrow\frac{G_k}{G_{k-1}}\geq 1\Rightarrow G_k\geq G_{k-1}$$

$$\left(\frac{a_n}{G_n}\right)^n=\left(\frac{a_n}{\sqrt[n]{a_1a_2\cdot\ldots\cdot a_n}}\right)^n=\frac{a_n^n}{a_1a_2\cdot\ldots\cdot a_n}=\frac{a_n}{a_1}\cdot\frac{a_n}{a_2}\cdot\ldots\cdot\frac{a_n}{a_{n-1}}\geq 1$$

$$\text{because } a_n\geq a_i,\forall i\in\overline{1,n-1}$$

$$\left(\frac{a_n}{G_n}\right)^n\geq 1\Rightarrow\frac{a_n}{G_n}\geq 1\Rightarrow G_n\leq a_n$$

$$G_2\leq G_3\leq G_4\leq\ldots\leq G_n\leq a_n.$$

□

Property 4:
If $0<a_1\leq a_2\leq\ldots\leq a_n, n\in\mathbb{N}, n\geq 2$ then:

(8) $$k\cdot G_k-(k-1)G_{k-1}\leq a_n;\forall k\in\overline{2,n}$$

Proof.

$$\text{By (7)}: G_k\geq G_{k-1}\Rightarrow\frac{G_k}{G_{k-1}}\geq 1;\forall k\in\overline{2,n}$$

$$a_k=\frac{a_1a_2\cdot\ldots\cdot a_{k-1}a_k}{a_1a_2\cdot\ldots\cdot a_{k-1}}=\frac{(\sqrt[k]{a_1a_2\cdot\ldots\cdot a_k})^k}{(\sqrt[k-1]{a_1a_2\cdot\ldots\cdot a_{k-1}})^{k-1}}=\frac{G_k^k}{G_{k-1}^{k-1}}=$$

$$=G_{k-1}\cdot\left(\frac{G_k}{G_{k-1}}\right)^k\overset{\text{Bernoulli}}{\geq}G_{k-1}\cdot\left(1+k\left(\frac{G_k}{G_{k-1}}-1\right)\right)=$$

$$=G_{k-1}+kG_k-kG_{k-1}=(1-k)G_{k-1}+kG_k=kG_k-(k-1)G_{k-1}$$

$$a_n\geq a_k\geq kG_k-(k-1)G_{k-1};\forall k\in\overline{2,n}$$

□

Corollary 10:
If $0<a\leq b\leq c$ then:

(9) $$3\cdot\sqrt[3]{abc}-2\sqrt{ab}\leq c$$

Proof.

$$\text{We take in (8)}: k=3, n=3, a_1=a, a_2=b, a_3=c.$$

□

Corollary 11:
If $0<a\leq b\leq c\leq d$ then:

$$4\sqrt[4]{abc}-3\sqrt[3]{abc}\leq d$$

Proof.

$$\text{We take in (8)}: k=n=3, a_1=a, a_2=b, a_3=c, a_4=d.$$

□

Corollary 12:
If $x, y > 0$ then:

$$3\sqrt{xy} - 2\sqrt{\frac{2xy\sqrt{xy}}{x+y}} \leq \frac{x+y}{2}$$

Proof.

$$a = \frac{2xy}{x+y}, b = \sqrt{xy}, c = \frac{x+y}{2}$$

□

References

[1] Romanian Mathematical Magazine - Interactive Journal, www.ssmrmh.ro

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