A SIMPLE PROOF FOR REDHEFFER-WILLIAMS' INEQUALITY

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Abstract. In this paper is presented a simple proof for the famous Redheffer-Williams' inequality.

Lemma: If $x \in [0,1)$ then:

$$(1+x^2)\prod_{k=2}^{n}\left(1-\frac{x^2}{k^2}\right) \geq 1+\frac{x^2}{n}; n \geq 2$$

Proof.

$$\text{For } n=2:$$
$$(1+x^2)\left(1-\frac{x^2}{4}\right) \geq 1+\frac{x^2}{2}$$
$$1-\frac{x^2}{4}+x^2-\frac{x^4}{4} \geq 1+\frac{x^2}{2}$$
$$\frac{x^2}{2}-\frac{x^2}{4}+x^2-\frac{x^4}{4} \geq 0$$
$$2x^2-x^2+4x^2-x^4 \geq 0$$
$$x^2(5-x^2) \geq 0, \text{ true for all } x \in [0,1).$$

Equality holds for $x=0$.
By induction, suppose that it is true.

$$P(n) : (1+x^2)\prod_{k=2}^{n}\left(1-\frac{x^2}{k^2}\right) \geq 1+\frac{x^2}{n}$$

$$P(n+1) : (1+x^2)\prod_{k=2}^{n+1}\left(1-\frac{x^2}{k^2}\right) \geq 1+\frac{x^2}{n+1} \text{ to prove.}$$

$$(1+x^2)\prod_{k=2}^{n+1}\left(1-\frac{x^2}{k^2}\right) = (1+x^2)\left(1-\frac{x^2}{n+1}\right)\prod_{k=2}^{n}\left(1-\frac{x^2}{k^2}\right) \overset{P(n)}{\geq}$$
$$\geq \left(1+\frac{x^2}{n}\right)\left(1-\frac{x^2}{(n+1)^2}\right)$$

Remains to prove:

$$\left(1+\frac{x^2}{n}\right)\left(1-\frac{x^2}{(n+1)^2}\right) \geq 1+\frac{x^2}{n+1}$$
$$1-\frac{x^2}{(n+1)^2}+\frac{x^2}{n}-\frac{x^4}{n(n+1)^2} \geq 1+\frac{x^2}{n+1}$$
$$\frac{x^2}{n}-\frac{x^2}{(n+1)^2}-\frac{x^2}{n+1}-\frac{x^4}{n(n+1)^2} \geq 0$$

$$x^2\Big(\frac{1}{n}-\frac{1}{(n+1)^2}-\frac{1}{n+1}-\frac{x^2}{n(n+1)^2}\Big)\geq 0$$

$$\frac{x^2}{n(n+1)^2}((n+1)^2-n-n(n+1)-x^2)\geq 0$$

$$\frac{x^2}{n(n+1)}(n^2+2n+1-n-n^2-n-x^2)\geq 0$$

$$\frac{x^2}{n(n+1)^2}(1-x^2)\geq 0 \text{ true } \forall x\in[0,1)$$

Equality holds for $x=0$. □

Theorem (REDHEFFER-WILLIAMS' INEQUALITY)
If $x>0$ then:

$$\frac{\sin x}{x}\geq\frac{\pi^2-x^2}{\pi^2+x^2}$$

Proof.

If $x\in(0,1)$ by Lemma:

$$(1+x^2)\prod_{k=2}^{n}\Big(1-\frac{x^2}{k^2}\Big)\geq 1+\frac{x^2}{n}\geq 1\geq 1-x^2; n\geq 2$$

$$\prod_{k=2}^{n}\Big(1-\frac{x^2}{k^2}\Big)\geq\frac{1-x^2}{1+x^2}$$

$$\prod_{k=2}^{\infty}\Big(1-\frac{x^2}{k^2}\Big)\geq\frac{1-x^2}{1+x^2}$$

$$\frac{\sin(\pi x)}{\pi x}\geq\frac{1-x^2}{1+x^2}\Leftrightarrow\frac{\sin x}{x}\geq\frac{1-(\frac{x}{\pi})^2}{1+(\frac{x}{\pi})^2}$$

$$\frac{\sin x}{x}\geq\frac{\pi^2-x^2}{\pi^2+x^2}$$

If $x>1$,

$$\frac{1-x^2}{1+x^2}-\frac{\sin(\pi x)}{\pi x}=\frac{1-x^2}{1+x^2}+\frac{\sin(\pi x-\pi)}{\pi x}=\frac{1-x^2}{1+x^2}+\frac{\pi(x-1)}{\pi x}\cdot\frac{\sin\pi(x-1)}{\pi(x-1)}\leq$$

$$\leq\frac{1-x^2}{1+x^2}+\frac{x-1}{x}=\frac{x-x^3+x-1+x^3-x^2}{x(1+x^2)}=$$

$$=\frac{-x^2+2x-1}{x(1+x^2)}=\frac{-(x-1)^2}{x(1+x^2)}<0$$

$$\frac{1-x^2}{1+x^2}-\frac{\sin(\pi x)}{\pi x}\leq 0\Leftrightarrow\frac{\sin(\pi x)}{\pi x}\geq\frac{1-x^2}{1+x^2}\Leftrightarrow$$

$$\frac{\sin x}{x}\geq\frac{1-(\frac{x}{\pi})^2}{1+(\frac{x}{\pi})^2}\Leftrightarrow\frac{\sin x}{x}\geq\frac{\pi^2-x^2}{\pi^2+x^2}$$

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References

[1] Romanian Mathematical Magazine - Interactive Journal, www.ssmrmh.ro

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