THE PENTAGON JOURNAL CHALLENGES - (I)

DANIEL SITARU - ROMANIA

780. Prove that if $a, b, c \in [1, \infty)$, then $ab + bc + ca \geq 3 + 2\ln(a^b b^c c^a)$.

Proposed by Daniel Sitaru - Romania

Solution by Richdad Phuc, University of Sciences, Hanoi, Vietnam.
We have

$$LHS - RHS = b(a - 2\ln a) + c(b - 2\ln b) + a(c - 2\ln c) - 3$$

$$= \frac{b}{a}a(a - 2\ln a) + \frac{c}{b}b(b - 2\ln b) + \frac{a}{c}c(c - 2\ln c) - 3.$$

Let $f(x) = x(x - 2\ln x)$ for $x \geq 1$. The derivative is $f'(x) = 2x - 2\ln x - 2$ and $f''(x) = 2 - \frac{2}{x} \geq 0$ for all $x \geq 1$, so $f'(x) \geq f'(1) = 0$. This means that $f(x)$ is strictly increasing on $[0, 1)$. Then $f(x) \geq f(1)$ for all $x \geq 1$. Hence $a(a - 2\ln a) \geq 1$, $b(b - 2\ln b) \geq 1$, and $c(c - 2\ln c) \geq 1$. Then

$$LHS - RHS \geq \frac{b}{a} + \frac{c}{b} + \frac{a}{c} - 3$$

so $LHS - RHS \geq 0$ by the AM-GM inequality. Equality holds if $a = b = c = 1$. $\square$

781. Prove that if $a, b, c \in (0, \infty)$, then

$$\sum a\sqrt{\frac{(b^4 + c^4)}{2}} \geq a^2(b + c) + b^2(a + c) + c^2(a + b) - 3abc.$$

Proposed by Daniel Sitaru - Romania

Solution by the proposer.
We prove that if $x, y \in (0, \infty)$, then

$$x + y - \sqrt{xy} \geq \sqrt{\frac{x^2 + y^2}{2}} \tag{1}$$

We denote $u = \sqrt{\frac{x^2+y^2}{2}}$, which means $2u^2 = x^2 + y^2$, and let $v = \sqrt{xy}$ so $v^2 = xy$. With these notations, we have

$$2u^2 + 2v^2 = x^2 + 2xy + y^2 = (x + y)^2.$$

We can rewrite (1) as $x + y - v \geq u$ or

$$(x + y)^2 \geq (u + v)^2 \Leftrightarrow 2u^2 + 2v^2 - u^2 - v^2 - 2uv \geq 0 \Leftrightarrow (u - v)^2 \geq 0.$$

Now replace x with $\frac{x}{y}$ and y with $\frac{y}{x}$ in (1) to get

$$\frac{x}{y} + \frac{y}{x} \geq \sqrt{\frac{(\frac{x}{y})^2 + (\frac{y}{x})^2}{2}} + \sqrt{\frac{x}{y} \cdot \frac{y}{x}} \Leftrightarrow \frac{x^2 + y^2}{xy} \geq \frac{1}{xy}\sqrt{\frac{x^4 + y^4}{2}} + 1$$

$$\Leftrightarrow x^2+y^2 \geq \sqrt{\frac{x^4+y^4}{2}}+xy.$$

For $x=a$ and $y=b$ and multiplying by c we have

$$a^2c+b^2c \geq c\sqrt{\frac{a^4+b^4}{2}}+abc.$$

Analogously,

$$b^2a+c^2a \geq a\sqrt{\frac{b^4+c^4}{2}}+abc$$

and

$$c^2b+a^2b \geq b\sqrt{\frac{c^4+a^4}{2}}+abc.$$

Adding the last three inequalities gives the desired result. □

790. Prove that if $a,b \in \mathbb{R}$ with $a<b$, then

$$\ln\left|\left(\frac{2+\sin 2b}{2+\sin 2a}\right)\right| \leq \frac{2\sqrt{3}}{3}(b-a).$$

Proposed by Daniel Sitaru - Romania

Solution by Richdad Phuc, Hanoi, Vietnam.
Let $f(x)=\ln|2+2\sin x|$. Then $f'(x)=\frac{2\cos 2x}{2+\sin 2x}$. By the Mean Value Theorem, there is a $c\in(a,b)$ with $f(b)-f(a)=f'(c)(b-a)$ or

$$\left|\ln\left(\frac{2+\sin 2b}{2+\sin 2a}\right)\right| \leq \frac{2\cos 2c}{2+\sin 2c}(b-a).$$

But

$$\frac{2\cos 2c}{2+\sin 2c} \leq \frac{2\sqrt{3}}{3} \Leftrightarrow \sqrt{3}\cos 2c-\sin 2c \leq 2 \Leftrightarrow \cos\left(2c+\frac{\pi}{6}\right) \leq 1,$$

which is clearly true. □

799. Prove that if $a,b,c \in (0,2]$ then

$$3\sqrt{2} \leq \sum \frac{b(\sqrt{a}+\sqrt{2-a})}{c} \leq 2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right).$$

Proposed by Daniel Sitaru - Romania

Solution by Henry Ricardo, Westchester Area Math Circle, Purchase, NY.
Elementary calculus shows that on the interval $(0,2]$ the function
$f(x)=\sqrt{x}+\sqrt{2-x}$ attains a maximum of 2 at $x=1$ and a minimum of $\sqrt{2}$ at $x=2$. Therefore

$$\sum_{cyclic} \frac{b(\sqrt{a}-\sqrt{2-a})}{c} \leq 2\sum_{cyclic}\frac{b}{c} = 2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right).$$

Similarly, the AM-GM inequality yields

$$\sum_{cyclic} \frac{b(\sqrt{a}+\sqrt{2-a})}{c} \geq \sqrt{2}*3\sqrt[3]{\frac{b}{c}*\frac{c}{a}*\frac{a}{b}} = 3\sqrt{2}.$$

Equality holds on the left side of the original inequality if and only if $a=b=c=2$ on the right side if and only if $a=b=c=1$. □

800. Prove that if $a \in \mathbb{R}$, then

$$\int_{a+3}^{a+5} \ln(1+e^x)dx + \int_{a+6}^{a+8} \ln(1+e^x)dx \leq \int_{a}^{a+2} \ln(1+e^x)dx + \int_{a+9}^{a+11} \ln(1+e^x)dx$$

Proposed by Daniel Sitaru - Romania

Solution by Ángel Plaza. Department of Mathematics, Universidad de Las Palmas de Gran Canaria, Spain.
Notice that by a change of variables

$$\int_{a+3}^{a+5} \ln(1+e^x)dx = \int_{a}^{a+2} \ln(1+e^{x-3})dx,$$

$$\int_{a}^{a+2} \ln(1+e^{x-3})dx + \int_{a}^{a+2} \ln(1+e^{x-6})dx \leq$$

$$\leq \int_{a}^{a+2} \ln(1+e^{x})dx + \int_{a}^{a+2} \ln(1+e^{x-9})dx.$$

Using the properties of the natural logarithm, we get

$$\int_{a}^{a+2} \ln\Big(\frac{1+e^{x-3}+e^{x-6}+e^{2x-9}}{1+e^{x}+e^{x-9}+e^{2x-9}}\Big)dx \leq 0$$

The result will follows by showing that $e^{x-3}+e^{x-6} \leq e^x + e^{x-9}$ for all x. This is equivalent to $\frac{e^{x-6}-e^{x-9}}{3} \leq \frac{e^{x}-e^{x-3}}{3}$ and by Lagrange's Mean Value Theorem the left side is equal to e^{μ} for some μ in $(x-9, x-6)$ and the right side is equal to e^{η} for some η in $(x-3, x)$. Since the function e^x is increasing, the inequality is true. □

808. Prove that if $a, b, c \in [1, \infty)$ then

$$\frac{e^{a+b+c}}{e^{\frac{b}{a}+\frac{c}{b}+\frac{a}{c}}} \leq a^b b^c c^a \leq \frac{e^{ab+bc+ca}}{e^{a+b+c}}$$

Proposed by Daniel Sitaru - Romania

Solution by Brent Dozier, North Carolina Wesleyan College, Rocky Mount, NC.
Let $a \in (1, \infty)$. Using the fact that $f(x) = \ln x$ is concave on $[1, a]$, we have

$$f'(a) = \frac{1}{a} \leq \frac{\ln a - \ln 1}{a-1} \leq 1 = f'(1)$$

which yields

$$1 - \frac{1}{a} \leq \ln a \leq a - 1.$$

Raising e to all three parts of this inequality produces $\frac{e}{e^{\frac{1}{a}}} \leq a \leq \frac{e^a}{e}$. (Note that this is true for $a = 1$.) Raising this inequality to the power b, we get $\frac{e^b}{e^{\frac{b}{a}}} \leq a^b \leq \frac{e^{ab}}{e^b}$. Similarly, using the pairs b, c and c, a we have $\frac{e^c}{e^{\frac{c}{b}}} \leq b^c \leq \frac{e^{bc}}{e^c}$ and $\frac{e^a}{e^{\frac{a}{c}}} \leq c^a \leq \frac{e^{ca}}{e^a}$. Multiplying all corresponding sides of the three inequalities gives the desired inequality. □

809. Prove that if $a, b, c \in (2, \infty)$ then

$$\sqrt{2}\sum(\sqrt{a(b-a)}+\sqrt{b(a-2)}) \leq 3\sqrt{abc}$$

Proposed by Daniel Sitaru - Romania

Solution by Marian Ursărescu, National College, "Roman Voda", Roman, Romania.

The inequality is equivalent to

$$\sqrt{2}\sum(\sqrt{a}(\sqrt{b-2}+\sqrt{c-2})) < 3\sqrt{abc}$$

$$\Leftrightarrow \sum(\sqrt{a}(\sqrt{2(b-2)}+\sqrt{2(c-2)})) < 3\sqrt{abc}$$

Now we use Mahler's inequality:

$$\sqrt{x_1x_2}+\sqrt{y_1y_2} \leq \sqrt{(x_1+y_1)(x_2+y_2)}, \forall x_i, y_i > 0$$

which says that

$$\sqrt{2(b-2)}+\sqrt{(c-2)\cdot 2} < \sqrt{c\cdot b} \Rightarrow \sqrt{a}(\sqrt{2(b-2)}+\sqrt{2(c-2)}) < \sqrt{abc}.$$

Similarly,

$$\sqrt{b}(\sqrt{2(a-2)}+\sqrt{2(c-2)}) < \sqrt{abc}$$

and

$$\sqrt{c}(\sqrt{2(a-2)}+\sqrt{2(b-2)}) < \sqrt{abc}.$$

Summing the three inequalities gives the desired result. □

810. Compute

$$L = \lim_{n\to\infty}\frac{1}{n}\int_1^n \frac{x^4+4x^3+12x^2+9x}{(x+3)^5-x^5-243}dx$$

Proposed by Daniel Sitaru - Romania

Solution by Andrea Fanchini, Cantu, Italy.

$$I = \int_1^n \frac{x^4+4x^3+12x^2+9x}{(x+3)^5-x^5-243}dx = \int_1^n \frac{x(x+1)(x^2+3x+9)}{15x(x+3)(x^2+3x+9)}dx$$

$$= \frac{1}{15}\int_1^n \frac{x+1}{x+3}dx$$

$$= \frac{1}{15}\int_1^n 1-\frac{2}{x+3}dx = \frac{1}{15}(n-2\log(n+3)-1+\log 16).$$

Finally,

$$\frac{1}{15}\lim_{n\to\infty}\frac{n-2\log(n+3)-1+\log 16}{n} = \frac{1}{15}.$$

□

821. Prove that if $a, b, c \in \mathbb{R}$ then

$$4\sum_{cyclic} a|b(1-b^2)| \leq \sum_{cyclic} a(1+b^2)^2.$$

Proposed by Daniel Sitaru - Romania

Solution by Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania.
We have

$$4a|b(1-b^2)| \leq a(1+b^2)^2 \Leftrightarrow 4a\sqrt{(b(1-b^2))^2} \leq a(1+b^2)^2$$

$$\Leftrightarrow 16a^2b^2(1-b^2)^2 \leq a^2(1+b^2)^4$$

$$\Leftrightarrow a^2(b^8 - 12b^6 + 38b^4 - 12b^2 + 1) \geq 0$$

$$\Leftrightarrow a^2(b^2 + 2b - 1)^2(b^2 - 2b - 1)^2 \geq 0.$$

The last inequality is true so the first is true and then summing, we get the desired result. □

822. Prove that in any acute-angled ΔABC you have

$$2\sum_{cyclic} \tan^3 A \geq \sum_{cyclic} \sqrt{\frac{\tan^6 A + \tan^6 B}{2}} + 3(\tan A + \tan B + \tan C)$$

Proposed by Daniel Sitaru - Romania

Solution by proposer.
Lemma. If $a, b \in (0,1)$ then $a + b \geq \sqrt{\frac{a^2+b^2}{2}} + \sqrt{ab}$.

Proof.

$$\text{Denote } \begin{cases} x = \sqrt{\frac{a^2+b^2}{2}} \\ y = \sqrt{ab} \end{cases} \Rightarrow \begin{cases} a^2 + b^2 = 2x^2 \\ ab = y^2 \end{cases}. \text{ Then}$$

$$a + b \geq x = y \Leftrightarrow (a+b)^2 \geq (x+y)^2 \Leftrightarrow 2x^2 + 2y^2 \geq (x+y)^2$$

$$\Leftrightarrow 2x^2 + 2y^2 \geq x^2 + 2xy + y^2 \Leftrightarrow x^2 - 2xy + y^2 \geq 0 \Leftrightarrow (x-y)^2 \geq 0,$$

which is true. Now replace a and b in the Lemma with $a = \tan^3 A; B = \tan^3 B$ and get

$$\tan^3 A + \tan^3 B \geq \sqrt{\frac{\tan^6 A + \tan^6 B}{2}} + \sqrt{\tan^3 A \tan^3 B}$$

so that

$$\sum \tan^3 A + \tan^3 B \geq \sum \sqrt{\frac{\tan^6 A + \tan^6 B}{2}} + \sum \tan A \tan B \sqrt{\tan A \tan B}$$

and by the AM-GM

$$2\sum \tan^3 A \geq \sum \sqrt{\frac{\tan^6 A + \tan^6 B}{2}} + 3\sqrt[3]{\tan^3 A \tan^3 B \tan^3 C}$$

$$= \sum \sqrt{\frac{\tan^6 A + \tan^6 B}{2}} + 3(\tan A + \tan B + \tan C)$$

□

829. Let
$$\Omega_n = \binom{n}{7} + 2\binom{n-1}{7} + 3\binom{n-2}{7} + \dots (n-6)\binom{7}{7}$$
for all $n \geq 7$. Find
$$\Omega = \lim_{n\to\infty} \sqrt[n]{\Omega_n}$$
$\square$

Proposed by Daniel Sitaru - Romania

Solution by Brian Bradie, Christopher Newport University, Newport News, VA.
Let $n \geq 7$ and
$$\Omega_n = \binom{n}{7} + 2\binom{n-1}{7} + 3\binom{3-2}{7} + \dots (n-6)\binom{7}{7} = \sum_{j=7}^{n}\sum_{i=7}^{j}\binom{i}{7}$$
By the Hockey Stick Identity,
$$\sum_{i=7}^{j}\binom{i}{7} = \binom{j+1}{8} \text{ and } \sum_{j=7}^{n}\binom{j+1}{8} = \sum_{j=8}^{n+1}\binom{j}{8} = \binom{n+2}{9}.$$
Thus
$$\Omega_n = \frac{(n+2)(n+1)n(n-1)\dots(n-6)}{9!} = \frac{1}{9!}n^9\Big(1+\frac{2}{n}\Big)\Big(1+\frac{1}{n}\Big)\dots\Big(1-\frac{6}{n}\Big)$$
and
$$\Omega = \lim_{n\to\infty}\sqrt[n]{\Omega_n} = \lim_{n\to\infty}(\sqrt[n]{n})^9\sqrt[n]{\frac{1}{9!}\Big(1+\frac{2}{n}\Big)\Big(1+\frac{1}{n}\Big)\dots\Big(1-\frac{6}{n}\Big)} = 1.$$
$\square$

830. If $x \in (0, \frac{\pi}{2})$, prove that $2(\sin x)^{1-\sin x} \cdot (1-\sin x)^{\sin x} \leq 1$.

Proposed by Daniel Sitaru - Romania

Solution by Henry Ricardo, Westchester Area Math Circle, Purchase, NY.
Noting that $0 < \sin x < 1$ for $x \in (0, \frac{\pi}{2})$, we apply the weighted AGM inequality twice to see that
$$\begin{aligned}2(\sin x)^{1-\sin x} \cdot (1-\sin x)^{\sin x} &\leq 2[(1-\sin x)\sin x + \sin x(1-\sin x)]\\ &= 4\sin x(1-\sin x)\\ &\leq 4\Big(\frac{\sin x + (1-\sin x)}{2}\Big)^2 = 1,\end{aligned}$$
with equality when $x = \frac{\pi}{6}$. $\square$

831. If $\Delta ABC \sim \Delta A'B'C'$, prove that
$$\sum \frac{(a'+b')(a'+c')}{b'c'} + 3 \geq \frac{15(b+c)(c'+a')(a'+b')}{8ab'c'}$$

Proposed by Daniel Sitaru - Romania

Solution by Daniel Văcaru, "Maria Teiuleanu" National Economic College, Pitesti, Romania.

With $\Delta ABC \sim \Delta A'B'C'$, we have $a = ka', b = kb', c = kc'$ and we obtain

$$\frac{(a'+b')(a'+c')}{b'c'} = \frac{(a+b)(a+c)}{bc}$$

and

$$\frac{15(b+c)(c'+a')(a'+b')}{8ab'c'} = \frac{15(b+c)(c+a)(a+b)}{8abc}$$

That is

$$\sum \frac{(a+b)(a+c)}{bc} + 3 \geq \frac{15(b+c)(c+a)(a+b)}{8abc}$$

Multiplying by $\frac{abc}{(b+c)(c+a)(a+b)}$, we obtain

$$\sum \frac{a}{b+c} + \frac{3abc}{(a+b)(b+c)(c+a)} \geq \frac{15}{8}$$

and we write this as

$$\sum \frac{a}{b+c} - \frac{3}{2} \geq \frac{3}{8} - \frac{3abc}{(a+b)(b+c)(c+a)}.$$

By a calculation, we obtain the LHS is equal to $\frac{\sum[(a+b)(a-b)^2]}{2(a+b)(b+c)(c+a)}$ and the RHS is equal to $\frac{\sum a(b-c)^2}{8(a+b)(b+c)(c+a)}$. But

$$4(a+b) \geq c \Rightarrow 4(a+b)(a-b)^2 \geq c(a-b)^2$$

which implies the required inequality. □

843. Prove that in ΔABC you have

$$\sqrt{(2^{h_a} + 2^{h_b} + 2^{h_c})(2^{m_a} + 2^{m_b} + 2^{m_c})} < 2^a + 3^b + 4^c$$

Proposed by Daniel Sitaru - Romania

Solution by Ioannis Sfikas, Athens, Greece.
It is well-known in every triangle: $h_s \leq m_a < \frac{b+c}{2}$. If we assume that the function $f(x) = 2^x$, then $f'(x) = 2^x \ln 2 > 0$ and $f''(x) = 2^x(\ln 2)^2 > 0$. So, the function $f(x)$ is increasing and convex function. Also, we have:

$$2^{h_a} \leq 2^{m_a} < 2^{\frac{b+c}{2}} \leq \frac{2^b + 2^c}{2}$$

and $2^{m_a} + 2^{m_b} + 2^{m_c} < 2^a + 2^b + 2^c$ and

$$\begin{aligned}\sqrt{(2^{h_a} + 2^{h_b} + 2^{h_c})(2^{m_a} + 2^{m_b} + 2^{m_c})} &\leq 2^{m_a} + 2^{m_b} + 2^{m_c} \\ &< 2^a + 2^b + 2^c \\ &< 2^a + 3^b + 4^c.\end{aligned}$$

□

844. Prove that if $0 < a < b < c < 1$, then

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a \ln a & b \ln b & c \ln c \end{vmatrix} >$$

$$> \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ (a-1)\ln(a^2+1) & (b-1)\ln(b^2+1) & (c-1)\ln(c^2+1) \end{vmatrix}$$

Proposed by Daniel Sitaru - Romania

Solution by Michel Bataille, Rouen, France.
Let $f : (0,1) \to \mathbb{R}$. Then

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ f(a) & f(b) & f(c) \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-b \\ f(a) & f(b)-f(c) & f(c)-f(b) \end{vmatrix}$$

$$= (b-a)(c-b)\Big(\frac{f(c)-f(b)}{c-b} - \frac{f(b)-f(a)}{b-a}\Big).$$

Applying this result first with $f(x) = 2x \ln x$ and then with $f(x) = (x-1)\ln(x^2+1)$ and observing that $(b-a)(c-b) > 0$, we obtain that the proposed inequality is equivalent to

$$\frac{g(c)-g(b)}{c-b} > \frac{g(b)-g(a)}{b-a} \tag{1}$$

where g denotes the function defined by $g(x) = 2x \ln x - (x-1)\ln(x^2+1)$.
Now we calculate the first two derivatives of g:

$$g'(x) = 2 + 2\ln x - \ln(x^2+1) - \frac{2x^2-2x}{x^2+1}$$

$$g''(x) = \frac{2(1-x^2)(1+x)}{x(x^2+1)^2}$$

We deduce that $g''(x)$ is positive when x in $(0,1)$. Thus g is convex on the interval $(0,1)$ and (1) follows since $a < b < c$. □

845. If $a, b, c \in [0,1)$, then

$$8\int_0^a \left(\int_0^b \left(\int_0^c \frac{\sin^{-1} x \cdot \sin^{-1} y \cdot \sin^{-1} z}{(1+\sin^{-1} x)(1+\sin^{-1} y)(1+\sin^{-1} z)} dz\right) dy\right) dx$$

Proposed by Daniel Sitaru - Romania

Solution by the Missouri State University Problem Solving Group, Springfield, MO.

Let $t \in [0,1)$. Consider the function $f(t) = t(1 + \arcsin t) - \arcsin t$.
Since we have that

$$\frac{t-1}{\sqrt{1-t^2}} = -\frac{1-t}{\sqrt{(1-t)(1+t)}} = -\sqrt{\frac{1-t}{1+t}} \geq -1,$$

we see

$$f'(t) = 1 + \arcsin t + \frac{t}{\sqrt{1-t^2}} - \frac{1}{\sqrt{1-t^2}} \geq 0.$$

Since $f(0) = 0$, it follows that $t(1 + \arcsin t) \geq \arcsin t$ and so $\frac{\arcsin t}{1+\arcsin t} \leq t$. We therefore have

$$\begin{aligned} &8 \int_0^a \int_0^b \int_0^c \frac{\arcsin x \arcsin y \arcsin z}{(1+\arcsin x)(1+\arcsin y)(1+\arcsin z)} dz dy dx \\ &= \int_0^a 2\frac{\arcsin x}{1+\arcsin x} dx \int_0^b 2\frac{\arcsin y}{1+\arcsin y} dy \int_0^c 2\frac{\arcsin z}{1+\arcsin z} dz \\ &\leq \int_0^a 2t dt \int_0^b 2t dt \int_0^c 2t dt \\ &= a^2 b^2 c^2. \end{aligned}$$

□

Mathematics Department, National Economic College "Theodor Costescu", Drobeta Turnu - Severin, Romania

Email address: dansitaru63@yahoo.com