INEQUALITIES WITH FIBONACCI AND LUCAS NUMBERS

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Abstract. In this paper we present some IMO type inequalities with Fibonacci and Lucas numbers.

Theorem 1.

$$n^{m-1}\cdot\sum_{k=1}^{n}F_k^{2m}\geq F_n^m\cdot F_{n+1}^m \text{ for any positive integer } n \text{ and } m\geq 1.$$

Proof.
The function $f:(0,+\infty)\to(0,+\infty), f(x)=x^m$ is convex on $(0,+\infty)$. Hence,

$$f(x_1)+f(x_2)+\ldots+f(x_n)\geq n\cdot f\Big(\frac{x_1+x_2+\ldots+x_n}{n}\Big)$$

$$\Leftrightarrow n^{m-1}\cdot\sum_{k=1}^{n}f(x_k)\geq\left(\sum_{k=1}^{n}x_k\right)^m \tag{1}$$

If we take $x_k=F_k^2$, then by (1) we obtain:

$$n^{m-1}\cdot\sum_{k=1}^{n}F_k^{2m}\geq\left(\sum_{k=1}^{n}F_k^2\right)^m,$$

where we take into account that $\sum_{k=1}^{n}F_k^2=F_nF_{n+1}$ which yields

$$n^{m-1}\sum_{k=1}^{n}F_k^{2m}\geq F_n^mF_{n+1}^m, \forall n\in\mathbb{N}^*.$$

□

Theorem 2.

$$\frac{F_1}{(F_1^2+F_2^2)^{m+1}}+\frac{F_2}{(F_1^2+F_2^2+F_3^2)^{m+1}}+\ldots+\frac{F_n}{(F_1^2+F_2^2+\ldots+F_{n+1}^2)^{m+1}}\geq\frac{1}{F_{n+2}^m}-\frac{1}{F_{n+2}^{m+1}},$$

for any positive integer n and any positive real number m.

Proof.
Taking into account that

$F_1^2+F_2^2=F_2F_3, F_1^2+F_2^2+F_3^2=F_3F_4,\ldots,F_1^2+F_2^2+\ldots+F_{n+1}^2=F_{n+1}F_{n+2}$, then

$$LHS=\frac{F_1}{(F_2F_3)^{m+1}}+\frac{F_2}{(F_3F_4)^{m+1}}+\ldots+\frac{F_n}{(F_{n+1}F_{n+2})^{m+1}}=$$
$$=\frac{(\frac{F_1}{F_2F_3})^{m+1}}{F_1^m}+\frac{(\frac{F_2}{F_3F_3})^{m+1}}{F_2^m}+\ldots+\frac{(\frac{F_n}{F_{n+1}F_{n+2}})^{m+1}}{F_n^m}.$$

By J. Radon's inequality and well known formulas we have

$$LHS = \frac{(\frac{F_1}{F_2F_3})^{m+1}}{F_1^m} + \frac{(\frac{F_2}{F_3F_4})^{m+1}}{F_2^m} + \ldots + \frac{(\frac{F_n}{F_{n+1}F_{n+2}})^{m+1}}{F_n^m} \overset{\text{Radon}}{\geq}$$

$$\overset{\text{Radon}}{\geq} \frac{\left(\sum_{k=1}^n \frac{F_k}{F_{k+1}F_{k+2}}\right)^{m+1}}{\left(\sum_{k=1}^n F_k\right)^m} = \frac{\left(\sum_{k=1}^n \frac{F_{k+2}-F_{k+1}}{F_{k+1}F_{k+2}}\right)^{m+1}}{(F_{n+2}-1)^m} =$$

$$= \frac{\left(\sum_{k=1}^n \left(\frac{1}{F_{k+1}} - \frac{1}{F_{k+2}}\right)\right)^{m+1}}{(F_{n+2}-1)^m} = \frac{(1-\frac{1}{F_{n+2}})^{m+1}}{(F_{n+2}-1)^m} = \frac{(F_{n+2}-1)^{m+1}}{(F_{n+2}-1)^m F_{n+2}^{m+1}} =$$

$$= \frac{F_{n+2}-1}{F_{n+2}^{m+1}} = \frac{1}{F_{n+2}^m} - \frac{1}{F_{n+2}^{m+1}},$$

and we are done. □

Theorem 3.

$$\frac{L_1}{(L_1^2+L_2^2+2)^2} + \frac{L_2}{(L_1^2+L_2^2+L_3^2+2)^2} + \cdots \frac{L_n}{(L_1^2+L_2^2+\ldots+L_{n+1}^2+2)^2} \geq \frac{(L_{n+2}-1)^2}{L_{n+2}^2(L_{n+2}-3)}$$

for any positive integer n.

Proof.
We have

$$LHS = \frac{L_1}{(L_1^2+L_2^2+2)^2} + \frac{L_2}{(L_1^2+L_2^2+L_3^2+2)^2} + \ldots + \frac{L_n}{(L_1^2+L_2^2+\ldots+L_{n+1}^2+2)^2} =$$

$$= \frac{\left(\frac{L_1}{L_1^2+L_2^2+2}\right)^2}{L_1} + \frac{\left(\frac{L_2}{L_1^2+L_2^2+L_3^2+2}\right)^2}{L_2} + \ldots + \frac{\left(\frac{L_n}{L_1^2+L_2^2+\ldots+L_{n+1}^2+2}\right)^2}{L_n}.$$

By Bergström inequality we have

$$(1) \qquad LHS \overset{\text{Bergström}}{\geq} \frac{\left(\frac{L_1}{L_1^2+L_2^2+2} + \frac{L_2}{L_1^2+L_2^2+L_3^2+2} + \ldots + \frac{L_n}{L_1^2+L_2^2+\ldots+L_{n+1}^2+2}\right)^2}{\sum_{k=1}^n L_k}$$

Since, $\sum_{k=1}^{n} L_k = L_{n+2} - 3$, and $\sum_{i=1}^{k} L_k^2 + 2 = L_k L_{k+1}$ by (1) we deduce that

$$LHS \geq \frac{\left(\frac{L_3-L_2}{L_2L_3} + \frac{L_4-L_3}{L_3L_4} + \ldots + \frac{L_{n+2}-L_{n+1}}{L_{n+1}L_{n+2}}\right)^2}{L_{n+2}-3} =$$

$$= \frac{\left(\frac{1}{L_2} - \frac{1}{L_3} + \frac{1}{L_3} - \frac{1}{L_4} + \ldots + \frac{1}{L_{n+1}} - \frac{1}{L_{n+2}}\right)^2}{L_{n+2}-3} = \frac{\left(1-\frac{1}{L_{n+2}}\right)^2}{L_{n+2}-3} = \frac{(L_{n+2}-1)^2}{L_{n+2}^2(L_{n+2}-3)}$$

□

Theorem 4.
If $m \geq 0$ and $n \in \mathbb{N}^*$ then $(\sqrt{F_{2n+1} - F_{n+1}})^m + (\sqrt{F_{2n+1}} + F_{n+1})^m \geq 2F_n^m$.

Proof.
By AM-GM inequality we have

$$(\sqrt{F_{2n+1}}-F_{n+1})^m+(\sqrt{F_{2n+1}}+F_{n+1})^m \geq 2(\sqrt{(\sqrt{F_{2n+1}}-F_{n+1})(\sqrt{F_{2n+1}}+F_{n+1})^m} =$$

$$= 2\sqrt{(F_{2n+1}-F_{n+1}^2)^m} = 2\sqrt{F_n^{2m}} = 2F_n^m,$$

where we take into account $F_n^2+F_{n+1}^2 = F_{2n+1}$.
The equality holds if and only if $m = 0$. □

Theorem 5.

$$\sqrt{F_1^4 - F_1^2F_2^2 + F_2^4} + \sqrt{F_2^4 - F_2^2F_3^2 + F_3^4} + \ldots$$

$$\ldots + \sqrt{F_{n-1}^4 - F_{n-1}^2F_n^2 + F_n^4} + \sqrt{F_n^4 - F_n^2 + 1} > F_nF_{n+1},$$

for any $n \in \mathbb{N}^*$.

Proof.
We have $\sqrt{x^2-xy+y^2} \geq \frac{1}{2}(x+y)$, for any $x, y \in \mathbb{R}_+$.

$$\text{Indeed, } \sqrt{x^2-xy+y^2} \geq \frac{1}{2}(x+y) \Leftrightarrow 4(x^2-xy+y^2) \geq x^2+2xy+y^2 \Leftrightarrow$$

$$\Leftrightarrow 3(x^2-2xy+y^2) \geq 0 \Leftrightarrow (x-y)^2 \geq 0, \text{ true.}$$

The equality occurs if and only if $x = y$. Yields that

$$\sum_{k=1}^{n-1}\sqrt{F_k^4 - F_k^2F_{k+1}^2 + F_{k+1}^4} + \sqrt{F_n^4 - F_n^2F_1^2 + F_1^2} \geq$$

$$\geq \frac{1}{2}\sum_{k=1}^{n-1}(F_k^2 + F_{k+1}^2) + \frac{1}{2}(F_n^2 + F_1^2) =$$

$$= \frac{1}{2}\cdot 2\sum_{k=1}^{n} F_k^2 = \sum_{k=1}^{n} F_k^2 = F_nF_{n+1},$$

for any $n \in \mathbb{N}^*$.
Because $F_k \neq F_{k+1}, \forall k \geq 2$ yields that the inequality from the statement is strictly. □

Theorem 6.

$$\sqrt[3]{\frac{F_n}{5F_{n+2}}} + \sqrt[3]{\frac{F_{n+1}}{5F_{n+2}+3F_{n+1}}} + \sqrt[3]{\frac{F_{n+2}}{5F_{n+2}+3F_n}} < \sqrt[3]{4}, \text{ for any } n \in \mathbb{N}^*.$$

Proof.
For any $a, b \in \mathbb{R}$ we have the inequality

$$\sqrt[3]{a} + \sqrt[3]{b} \leq \sqrt[3]{4(a+b)} \tag{1}$$

with equality if and only if $a = b$.
Indeed, we have $4(x^3+y^3) - (x+y)^3 = 4(x+y)(x^2-xy+y^2) - (x+y)^3 =$

$$= (x+y)(4x^2-4xy+4y^2-x^2-y^2-2xy) = (x+y)(3x^2+3y^2-6xy) =$$

$$= 3(x+y)(x-y)^2 \geq 0, \forall x, y \in \mathbb{R}$$

with equality if and only if $x = y$. So we have $\sqrt[3]{4(x^3+y^3)} \geq x + y$, where we put $x = \sqrt[3]{a}, y = \sqrt[3]{b}$ and we obtain (1). By (1) yields that

$$\sqrt[3]{\frac{a}{16(a+b)+4c}} = \sqrt[3]{\frac{a}{4(4(a+b)+c)}} = \frac{\sqrt[3]{a}}{\sqrt[3]{4((\sqrt[3]{4(a+b)})^3 + (\sqrt[3]{c})^3)}} \leq$$

$$\leq \frac{\sqrt[3]{a}}{\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}} \tag{2}$$

Hence,

$$\sum_{cyclic} \sqrt[3]{\frac{a}{16(a+b)+4c}} \leq \sum_{cyclic} \frac{\sqrt[3]{a}}{\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}} = 1 \tag{3}$$

The inequality from the statement is equivalent with

$$\sqrt[3]{\frac{F_n}{20F_{n+2}}} + \sqrt[3]{\frac{F_{n+1}}{20F_{n+2}+12F_{n+1}}} + \sqrt[3]{\frac{F_{n+2}}{20F_{n+2}+12F_n}} < 1$$

(4)

$$\Leftrightarrow \sqrt[3]{\frac{F_n}{16(F_n+F_{n+1})+4F_{n+2}}} + \sqrt[3]{\frac{F_{n+1}}{16(F_{n+1}+F_{n+2})+4F_n}} + \sqrt[3]{\frac{F_{n+2}}{16(F_{n+2}+F_n)+4F_{n+1}}} < 1$$

If we take $a = F_n, b = F_{n+1}, c = F_{n+2}$, then by (3) we obtain (4) and we are done. □

Theorem 7.
If $(x_n)_{n\geq 1}, x_n \in \mathbb{R}$,

$$2 \cdot \left(\sum_{k=1}^{n} F_k \cdot \sin x_k\right) \cdot \left(\sum_{k=1}^{n} F_k \cdot \cos x_k\right) \leq n \cdot F_n \cdot F_{n+1}, \forall n \in \mathbb{N}^*.$$

Proof.
By Cauchy-Buniakowski-Schwarz' inequality, we have:

$$\left|\sum_{k=1}^{n} F_k \sin x_k\right|^2 \leq \left(\sum_{k=1}^{n} F_k^2\right)\left(\sum_{k=1}^{n} \sin^2 x_k\right) = F_n F_{n+1} \cdot \sum_{k=1}^{n} \sin^2 x_k, \tag{1}$$

and

$$\left|\sum_{k=1}^{n} F_k \cos x_k\right|^2 \leq \left(\sum_{k=1}^{n} F_k^2\right)\left(\sum_{k=1}^{n} \cos^2 x_k\right) = F_n F_{n+1} \cdot \sum_{k=1}^{n} \cos^2 x_k, \tag{2}$$

By (1), (2) and AM-GM inequality we deduce that

$$\left|\sum_{k=1}^{n} F_k \sin x_k\right| \cdot \left|\sum_{k=1}^{n} F_k \cos x_k\right| \leq F_n F_{n+1} \cdot \sqrt{\left(\sum_{k=1}^{n} \sin^2 x_k\right)\left(\sum_{k=1}^{n} \cos^2 x_k\right)} \overset{\text{AM-GM}}{\leq}$$

$$\overset{\text{AM-GM}}{\leq} \frac{F_n F_{n+1}}{2} \cdot \sum_{k=1}^{n} (\sin^2 x_k + \cos^2 x_k) = \frac{n \cdot F_n \cdot F_{n+1}}{2}, \forall n \in \mathbb{N}^*,$$

hence

$$\left(\sum_{k=1}^{n} F_k \sin x_k\right)\left(\sum_{k=1}^{n} F_k \cos x_k\right) \leq \left|\sum_{k=1}^{n} F_k \sin x_k\right| \cdot \left|\sum_{k=1}^{n} F_k \cos x_k\right| \leq \frac{n \cdot F_n \cdot F_{n+1}}{2}, \forall n \in \mathbb{N}^*.$$

and we are done. □

Theorem 8.

$$\frac{F_1}{(F_1^2+F_2^2)^2} + \frac{F_2}{(F_1^2+F_2^2+F_3^2)^2} + \dots$$
$$\dots + \frac{F_n}{(F_1^2+F_2^2+\dots+F_{n+1}^2)^2} \geq \frac{1}{F_{n+2}} - \frac{1}{F_{n+2}^2}$$

for any positive integer n.

Proof.
By Bergström's inequality we have

$$LHS = \frac{F_1}{(F_1^2+F_2^2)^2} + \frac{F_2}{(F_1^2+F_2^2+F_3^2)^2} + \dots + \frac{F_n}{(F_1^2+F_2^2+\dots+F_{n+1}^2)^2} =$$
$$= \frac{\left(\frac{F_1}{F_1^2+F_2^2}\right)^2}{F_1} + \frac{\left(\frac{F_2}{F_1^2+F_2^2+F_3^2}\right)^2}{F_2} + \dots + \frac{\left(\frac{F_n}{F_1^2+F_2^2+\dots+F_{n+1}^2}\right)^2}{F_n} \overset{\text{Bergström}}{\geq}$$

(1) $$\overset{\text{Bergström}}{\geq} \frac{\left(\frac{F_1}{F_1^2+F_2^2} + \frac{F_2}{F_1^2+F_2^2+F_3^2} + \dots + \frac{F_n}{F_1^2+F_2^2+F_3^2+\dots+F_{n+1}^2}\right)^2}{\sum_{k=1}^{n} F_k}$$

It is well-known that

(2) $$F_1^2 + F_2^2 + \dots + F_{m+1}^2 = F_{m+1}F_{m+2}, \text{ for any positive integer } n$$

By (1) and (2) we obtain that

$$LHS \geq \frac{\left(\frac{F_1}{F_2F_3} + \frac{F_2}{F_3F_4} + \dots + \frac{F_n}{F_{n+1}F_{n+2}}\right)^2}{F_{n+2}-1} = \frac{\left(\frac{F_3-F_2}{F_2F_3} + \frac{F_4-F_3}{F_3F_4} + \dots + \frac{F_{n+2}-F_{n+1}}{F_{n+1}F_{n+2}}\right)^2}{F_{n+2}-1} =$$
$$= \frac{\left(\frac{1}{F_2} - \frac{1}{F_3} + \frac{1}{F_3} - \frac{1}{F_4} + \dots + \frac{1}{F_{n+1}} - \frac{1}{F_{n+2}}\right)^2}{F_{n+2}-1} = \frac{\left(\frac{1}{F_2} - \frac{1}{F_{n+2}}\right)^2}{F_{n+2}-1} = \frac{\left(1 - \frac{1}{F_{n+2}}\right)^2}{F_{n+2}-1} =$$
$$= \frac{(F_{n+2}-1)^2}{F_{n+2}^2(F_{n+2}-1)} = \frac{1}{F_{n+2}} - \frac{1}{F_{n+2}^2}$$

□

Theorem 9.

$$\frac{L_1}{(L_1^2+L_2^2+2)^{m+1}} + \frac{L_2}{(L_1^2+L_2^2+L_3^2+2)^{m+1}} + \dots$$
$$\dots + \frac{L_n}{(L_1^2+L_2^2+\dots+L_{n+1}^2+2)^{m+1}} \geq \frac{(L_{n+2}-1)^{m+1}}{L_{n+2}^{m+1}(L_{n+2}-3)^m}$$

for any positive integer n and m.

Proof.
It is well-known that

$L_1^2+L_2^2=L_2L_3-2, L_1^2+L_2^2+L_3^2=L_3L_4-2, L_1^2+L_2^2+\ldots+L_{n+1}^2=L_{n+1}L_{n+2}-2,$

and that $L_1+L_2+\ldots+L_n=L_{n+2}-3$.
Then by above and by Radon's inequality we obtain that

$$LHS=\frac{L_1}{(L_1^2+L_2^2+2)^{m+1}}+\frac{L_2}{(L_1^2+L_2^2+L_3^2+2)^{m+1}}+\ldots+\frac{L_n}{(L_1^2+L_2^2+\ldots+L_{n+1}^2+2)^{m+1}}=$$

$$=\frac{L_1}{(L_2L_3)^{m+1}}+\frac{L_2}{(L_3L_4)^{m+1}}+\ldots+\frac{L_n}{(L_{n+1}L_{n+2})^{m+1}}=$$

$$=\frac{\left(\frac{L_1}{L_2L_3}\right)^{m+1}}{L_1^m}+\frac{\left(\frac{L_2}{L_3L_4}\right)^{m+1}}{L_2^m}+\ldots+\frac{\left(\frac{L_n}{L_{n+1}L_{n+2}}\right)^{m+1}}{L_n^m}\overset{\text{Radon}}{\geq}$$

$$\overset{\text{Radon}}{\geq}\frac{\left(\frac{L_1}{L_2L_3}+\frac{L_2}{L_3L_4}+\ldots+\frac{L_n}{L_{n+1}L_{n+2}}\right)^{m+1}}{(L_1+L_2+\ldots+L_n)^m}=\frac{\left(\frac{L_3-L_2}{L_2L_3}+\frac{L_4-L_3}{L_3L_4}+\ldots+\frac{L_{n+2}-L_{n+1}}{L_{n+1}L_{n+2}}\right)^{m+1}}{(L_{n+2}-3)^m}=$$

$$=\frac{\left(\frac{1}{L_2}-\frac{1}{L_3}+\frac{1}{L_3}-\frac{1}{L_4}+\ldots+\frac{1}{L_{n+1}}-\frac{1}{L_{n+2}}\right)^{m+1}}{(L_{n+2}-3)^m}=\frac{\left(1-\frac{1}{L_{n+2}}\right)^{m+1}}{(L_{n+2}-3)^m}=\frac{(L_{n+2}-1)^{m+1}}{(L_{n+2}-3)^mL_{n+2}^{m+1}}$$

□

Theorem 10.
If $a,b>0$, then

$$\sum_{k=1}^n\frac{F_k^4}{aL_k+bF_k^2}>\frac{F_n^2F_{n+1}^2}{a(L_{n+2}-3)+bF_nF_{n+1}}$$

for any positive integer n.

Proof.
By Bergström's inequality we have

$$\sum_{k=1}^n\frac{F_k^4}{aL_k+bF_k^2}=\sum_{k=1}^n\frac{(F_k^2)^2}{aL_k+bF_k^2}\overset{\text{Bergström}}{\geq}\frac{\left(\sum_{k=1}^nF_k^2\right)^2}{\sum_{k=1}^n(aL_k+bF_k^2)}=$$

$$=\frac{\left(\sum_{k=1}^nF_k^2\right)^2}{a\sum_{k=1}^nL_k+b\sum_{k=1}^nF_k^2}\tag{1}$$

It is well-known that

$$\sum_{k=1}^nF_k^2=F_nF_{n+1}\text{ and }\sum_{k=1}^nL_k=L_{n+2}-3\tag{2}$$

By (1) and (2) yields the conclusion. □

Theorem 11.
If $a,b>0$, then

$$\sum_{k=1}^n\frac{F_k^4}{aF_{n+2}+bF_k-a}>\frac{F_n^2F_{n+1}^2}{(an+b)(F_{n+2}-1)},$$

for any positive integer n.

Proof.
By Bergström's inequality we have

$$\sum_{k=1}^{n} \frac{F_k^4}{aF_{n+2} + bF_k - a} = \sum_{k=1}^{n} \frac{(F_k^2)^2}{aF_{n+2} + bF_k - a} \overset{\text{Bergström}}{\geq} \frac{\left(\sum_{k=1}^{n} F_k^2\right)^2}{\sum_{k=1}^{n}(aF_{n+2} + bF_k - a)} =$$

$$= \frac{F_n^2 F_{n+1}^2}{an(F_{n+2} - 1) + b\sum_{k=1}^{n} F_k} = \frac{F_n^2 F_{n+1}^2}{(an + b)(F_{n+2} - 1)}$$

We use well-known identities $\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}$ respectively $\sum_{k=1}^{n} F_k = F_{n+2} - 1$.

□

Theorem 12.

$$\sum_{k=1}^{n+1} \frac{\binom{n}{k-1}^{m+1}}{F_k^{2m}} > \frac{2^{n(m+1)}}{F_{n+1}^m F_{n+2}^m}$$

for any $m \geq 0$ and any positive integer $n > 1$.

Proof.
By Radon's inequality we have

$$\sum_{k=1}^{n+1} \frac{\binom{n}{k-1}^{m+1}}{F_k^{2m}} = \sum_{k=1}^{n+1} \frac{\binom{n}{k-1}^{m+1}}{(F_k^2)^m} \overset{\text{Radon}}{\geq} \frac{\left(\sum_{k=1}^{n+1} \binom{n}{k-1}\right)^{m+1}}{\left(\sum_{k=1}^{n+1} F_k^2\right)^m} =$$

$$= \frac{\left(\sum_{k=1}^{n} \binom{n}{k}\right)^{m+1}}{(F_n F_{n+1} + F_{n+1}^2)^m} = \frac{2^{n(m+1)}}{F_{n+1}^m F_{n+2}^m}$$

We use the well-known identities $\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}$ respectively $\sum_{k=1}^{n} \binom{n}{k} = 2^n$.

The inequality from above is strictly because the conditions from equality in Radon's inequality are not satisfied. □

Theorem 13.

$$\sum_{k=1}^{n+1} \frac{F_k^{m+1}}{\binom{n}{k-1}^m} > \frac{(F_{n+3} - 1)^{m+1}}{2^{mn}}$$

for any $m \geq 0$ and any positive integer $n > 1$.

Proof.
By Radon's inequality we have

$$\sum_{k=1}^{n+1} \frac{F_k^{m+1}}{\binom{n}{k-1})^m} \overset{\text{Radon}}{\geq} \frac{\left(\sum_{k=1}^{n+1} F_k\right)^{m+1}}{\left(\sum_{k=1}^{n+1} \binom{n}{k-1}\right)^m} = \frac{\left(\sum_{k=1}^{n} F_k + F_{n+1}\right)^{m+1}}{\left(\sum_{k=1}^{n} \binom{n}{k}\right)^m} =$$

$$= \frac{(F_{n+2} - 1 + F_{n+1})^{m+1}}{(2^n)^m} = \frac{(F_{n+3} - 1)^{m+1}}{2^{mn}}$$

We use the well-known identites $\sum_{k=1}^{n} F_k = F_{n+2} - 1$ respectively $\sum_{k=1}^{n} \binom{n}{k} = 2^n$

The inequality from above is strictly because the conditions from equality in Radon's inequality are not satisfied. □

Theorem 14.

$$\frac{F_n^4 + F_1^4}{F_n^2 + F_1^2} + \sum_{k=1}^{n-1} \frac{F_k^4 + F_{k+1}^4}{F_{2k+1}} > F_n F_{n+1}$$

for any positive integer $n > 1$.

Proof.
We have that

(1) $$F_k^2 + F_{k+1}^2 = F_{2k+1}, \forall k \in \mathbb{N},$$

and

(2) $$\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}, \forall n \in \mathbb{N}^*$$

We have

(3) $$\frac{x^4 + y^4}{x^2 + y^2} \geq \frac{1}{2}(x^2 + y^2), \forall x, y \in \mathbb{R}_+^*$$

Indeed, $\frac{x^4+y^4}{x^2+y^2} \geq \frac{1}{2}(x^2 + y^2) \Leftrightarrow (x^2 - y^2)^2 \geq 0, \forall x, y \in \mathbb{R}_+^*$, with equality iff $x = y$.
From $(1), (2)$ and (3) we obtain

$$\frac{F_n^4 + F_1^4}{F_n^2 + F_1^2} + \sum_{k=1}^{n-1} \frac{F_k^4 + F_{k+1}^4}{F_{2k+1}} \overset{(1)}{=}$$

$$= \frac{F_n^4 + F_1^4}{F_n^2 + F_1^2} + \sum_{k=1}^{n-1} \frac{F_k^4 + F_{k+1}^4}{F_k^2 + F_{k+1}^2} \overset{(3)}{>} \frac{1}{2}(F_n^2 + F_1^2) + \frac{1}{2}\sum_{k=1}^{n-1}(F_k^2 + F_{k+1}^2) =$$

$$= \frac{1}{2} \cdot 2 \cdot \sum_{k=1}^{n} F_k^2 \overset{(2)}{=} F_n F_{n+1}$$

and we are done! □

Theorem 15.

$$\sqrt{2} \cdot \sqrt{1 + F_n^4} + \sum_{k=1}^{n-1} \sqrt{(F_k^4 + 1)(F_{k+1}^4 + 1)} > 2F_n F_{n+1},$$

for any positive integer $n > 1$.

Proof.
We have that

(1) $$\sqrt{(x^4 + 1)(y^4 + 1)} \geq x^2 + y^2, \forall x, y > 0$$

Indeed $\sqrt{(x^4+1)(y^4+1)} \geq x^2+y^2$ is equivalent with $(x^2y^2-1)^2 > 0, \forall x, y > 0$, which is true; equality occurs if and only if $xy = 1$.
The equality is written as

$$\sqrt{(F_n^4+1)(F_1^4+1)} + \sum_{k=1}^{n-1} \sqrt{(F_k^4+1)(F_{k+1}^4+1)} > 2F_nF_{n+1},$$

then by (1) we deduce that
(2)

$$\sqrt{(F_n^4+1)(F_1^4+1)}+\sum_{k=1}^{n-1} \sqrt{(F_k^4+1)(F_{k+1}^4+1)} \geq F_n^2+F_1^2+\sum_{k=1}^{n-1}(F_k^2+F_{k+1}^2) = 2\sum_{k=1}^{n} F_k^2$$

By (2) and the well-known

(3) $$\sum_{k=1}^{n} F_k^2 = F_nF_{n+1}, \forall n \in \mathbb{N}^*$$

we obtain

$$\sqrt{2} \cdot \sqrt{1+F_n^4} + \sum_{k=1}^{n-1} \sqrt{(F_k^4+1)(F_{k+1}^4+1)} > 2F_nF_{n+1}$$

and we are done!
The inequality is strict because $F_kF_{k+1} \neq 1, \forall k \in \mathbb{N}^*$. □

Theorem 16.

$$\frac{F_n^4+F_n^2+1}{F_n} + \sum_{k=1}^{n-1} \frac{F_k^4+F_k^2F_{k+1}^2+F_{k+1}^4}{F_kF_{k+1}} > 3F_nF_{n+1}$$

for any positive integer $n > 1$.

Proof.
We have the inequality

$$\frac{x^4+x^2y^2+y^4}{xy} \geq \frac{2(x^4+x^2y^2+y^4)}{x^2+y^2} \geq \frac{3}{2}(x^2+y^2), \forall x, y > 0.$$

Indeed, $\frac{x^4+x^2y^2+y^4}{xy} \geq \frac{2(x^4+x^2y^2+y^4)}{x^2+y^2} \Leftrightarrow (x-y)^2 \geq 0$, true! and

$$\frac{2(x^4+x^2y^2+y^4)}{x^2+y^2} \geq \frac{3}{2}(x^2+y^2) \Leftrightarrow (x^2-y^2)^2 \geq 0, \text{ true!}$$

So,

(1) $$\frac{x^4+x^2y^2+y^4}{xy} \geq \frac{3}{2}(x^2+y^2)$$

We have equalities from above iff $x = y$.
Therefore,

$$\frac{F_n^4+F_n^4+1}{F_n} + \sum_{k=1}^{n-1} \frac{F_k^4+F_k^2F_{k+1}^2+F_{k+1}^4}{F_kF_{k+1}} =$$

$$= \frac{F_n^4+F_n^2F_1^2+F_1^4}{F_nF_1} + \sum_{k=1}^{n-1} \frac{F_k^4+F_k^2F_{k+1}^2+F_{k+1}^4}{F_kF_{k+1}} \geq$$

$$(2) \qquad \overset{(1)}{\geq} \frac{3}{2}(F_n^2+F_1^2)+\frac{3}{2}\sum_{k=1}^{n-1}(F_k^2+F_{k+1}^2)=3\sum_{k=1}^{n}F_k^2$$

$$(3) \qquad \text{It is well-known that } \sum_{k=1}^{n}F_k^2=F_nF_{n+1}$$

From (2) and (3) we obtain the desired inequality!
The inequality is strict because $F_k \neq F_{k+1}, \forall k \in \mathbb{N}^* - \{1\}$. □

Theorem 17.

$$\frac{F_n^4+1}{F_n^2-F_n+1}+\sum_{k=1}^{n-1}\frac{F_k^4+F_{k+1}^4}{F_k^2-F_kF_{k+1}+F_{k+1}^2} > 2F_nF_{n+1},$$

for any positive integer $n > 1$.

Proof.
We have the inequality

$$(1) \qquad \frac{x^4+y^4}{x^2-xy+y^2} \geq x^2+y^2, \forall x,y>0$$

Indeed (1) is equivalent with $xy(x-y)^2 \geq 0$, which is true.
In (1) we have equality iff $x = y$.
(2)

$$\text{So, } \frac{F_n^4+1}{F_n^2-F_n+1}+\sum_{k=1}^{n-1}\frac{F_k^4+F_{k+1}^4}{F_k^2-F_kF_{k+1}+F_{k+1}^2} \overset{(1)}{\geq} F_n^2+F_1^2+\sum_{k=1}^{n-1}(F_k^2+F_{k+1}^2)=2\sum_{k=1}^{n}F_k^2$$

$$(3) \qquad \text{It is well-known that } \sum_{k=1}^{n}F_k^2=F_nF_{n+1}$$

From (2) and (3) we obtain the desired inequality!
The inequality is strict because $F_k \neq F_{k+1}, \forall k \in \mathbb{N}^* - \{1\}$. □

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