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In ΔABC the following relationship holds:

$$\frac{6r}{R}\sqrt[3]{abc} \le \sum_{cyc} \frac{bc}{b+c-a} \le \frac{s}{2}\left(\frac{R}{r}+2\right)$$

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Solution by George Florin Șerban-Romania

$$\sum_{cyc} \frac{bc}{b+c-a} = \frac{1}{2}\sum_{cyc} \frac{bc}{s-a} = \frac{s}{2}\sum_{cyc} \frac{1}{\frac{s(s-a)}{bc}} = \frac{s}{2}\sum_{cyc} \frac{1}{\cos^2\frac{A}{2}} = \frac{s}{2}\cdot\frac{s^2+(4R+r)^2}{s^2}$$

$$\sum_{cyc} \frac{bc}{b+c-a} \overset{(1)}{\le} \frac{s}{2}\left(\frac{R}{r}+2\right) \Leftrightarrow \frac{s}{2}\cdot\frac{s^2+(4R+r)^2}{s^2} \le \frac{s}{2}\left(\frac{R}{r}+2\right)$$

$$1+\frac{(4R+r)^2}{s^2} \le \frac{R}{r}+2 \Leftrightarrow \frac{(4R+r)^2}{s^2} \le \frac{R}{r}+1 \Leftrightarrow$$

$$s^2 \ge \frac{r(4R+r)^2}{R+r}(Blundon-Gerretsen)$$

$$\frac{6r}{R}\sqrt[3]{abc} \overset{(2)}{\le} \sum_{cyc} \frac{bc}{b+c-a} \Leftrightarrow \frac{s}{2}\cdot\frac{s^2+(4R+r)^2}{s^2} \ge \frac{6r}{R}\sqrt[3]{abc}$$

$$\frac{6r}{R}\sqrt[3]{abc} \overset{AGM}{\le} \frac{6r}{R}\cdot\frac{a+b+c}{3} = \frac{4sr}{R} \overset{(2)}{\le} \frac{s}{2}\cdot\frac{s^2+(4R+r)^2}{s^2}$$

$$(2) \Leftrightarrow 9 = \frac{8r}{R} \le 1 + \frac{(4R+r)^2}{s^2} \Leftrightarrow \frac{(4R+r)^2}{s^2} \overset{?}{\ge} \frac{8r}{R} - 1$$

$$\frac{(4R+r)^2}{s^2} \overset{Blundon-G}{\ge} \frac{(4R+r)^2}{\frac{R(4R+r)^2}{2(2R-r)}} = \frac{4R-2r}{R} \overset{?}{\ge} \frac{8r}{R} - 1 \Leftrightarrow R \ge 2r\ (Euler).$$

Therefore,

$$\frac{6r}{R}\sqrt[3]{abc} \le \sum_{cyc} \frac{bc}{b+c-a} \le \frac{s}{2}\left(\frac{R}{r} + 2\right)$$

Equality holds if and only if triangle is equilateral.