THE CONTRAHARMONIC MEAN AND CONNECTIONS

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Abstract. In this paper is presented the contraharmonic mean with properties and a few connections with the others means.

Let be $a_1, a_2, \ldots, a_n > 0; n \in \mathbb{N}^*$. Define:

$$C(a_1, a_2, \ldots, a_n) = \frac{a_1^2 + a_2^2 + \ldots + a_n^2}{a_1 + a_2 + \ldots + a_n}$$

Property 1
In these conditions $C(a_1, a_2, \ldots, a_n)$ is a mean.

Proof. Let be $m = \min_{1 \le i \le n} a_i; M = \max_{1 \le i \le n} a_i$

$$a_1^2 + a_2^2 + \ldots + a_n^2 = a_1 \cdot a_1 + a_2 \cdot a_2 + \ldots + a_n \cdot a_n \ge$$

$$\ge ma_1 + ma_2 + \ldots + ma_n = m(a_1 + a_2 + \ldots + a_n)$$

(1) $$\frac{a_1^2 + a_2^2 + \ldots + a_n^2}{a_1 + a_2 + \ldots + a_n} \ge m$$

$$a_1^2 + a_2^2 + \ldots + a_n^2 = a_1 \cdot a_1 + a_2 \cdot a_2 + \ldots + a_n \cdot a_n \le$$

$$\le Ma_1 + Ma_2 + \ldots + Ma_n = M(a_1 + a_2 + \ldots + a_n)$$

(2) $$\frac{a_1^2 + a_2^2 + \ldots + a_n^2}{a_1 + a_2 + \ldots + a_n} \le M$$

By (1); (2):

(3) $$m \le C(a_1, a_2, \ldots, a_n) \le M$$

$$C(a_1, a_2, \ldots, a_n) = C(a, a, \ldots, a) = \frac{na^2}{na} = a$$

(4) $$C(a_1, a_2, \ldots, a_n) = a$$

By (3); (4) $\Rightarrow C(a_1, a_2, \ldots, a_n)$ is a mean.
Recall:
The harmonic mean:

$$H(a, b) = \frac{2ab}{a + b}$$

The geometric mean:

$$G(a, b) = \sqrt{ab}$$

The logarithmic mean:

$$L(a, b) = \begin{cases} a; a = b \\ \frac{b - a}{\log b - \log a}; a \ne b \end{cases}$$

The arithmetic mean:

$$A(a,b)=\frac{a+b}{2}$$

The generalized mean:

$$M(a,b)=\sqrt[3]{\frac{a^3+b^3}{2}}$$

The quadratic mean:

$$Q(a,b)=\sqrt{\frac{a^2+b^2}{2}}$$

It is known that:

$$m\le H(a,b)\le G(a,b)\le L(a,b)\le M(a,b)\le A(a,b)\le Q(a,b)\le M \tag{5}$$

□

Property 2

$$Q(a_1,a_2,\ldots,a_n)\le C(a_1,a_2,\ldots,a_n) \tag{6}$$

Proof.

$$(6)\Leftrightarrow\sqrt{\frac{a_1^2+a_2^2+\ldots+a_n^2}{n}}\le\frac{a_1^2+a_2^2+\ldots+a_n^2}{a_1+a_2+\ldots+a_n}$$

$$\frac{a_1^2+a_2^2+\ldots+a_n^2}{n}\le\frac{(a_1^2+a_2^2+\ldots+a_n^2)^2}{(a_1+a_2+\ldots+a_n)^2}$$

$$(a_1+a_2+\ldots+a_n)^2\le n(a_1^2+a_2^2+\ldots+a_n^2)$$

$$a_1^2+a_2^2+\ldots+a_n^2+2\sum_{1\le i<j\le n}a_ia_j\le n(a_1^2+a_2^2+\ldots+a_n^2)$$

$$(n-1)(a_1^2+a_2^2+\ldots+a_n^2)-2\sum_{1\le i<j\le n}a_ia_j\ge 0$$

$$\sum_{1\le i<j\le n}(a_i-a_j)^2\ge 0$$

In these conditions (5) can be written:

$$m\le H(a,b)\le G(a,b)\le L(a,b)\le M(a,b)\le A(a,b)\le Q(a,b)\le C(a,b)\le M$$

□

Property 3

If $x>0$ then:

$$C(xa_1,xa_2,\ldots,xa_n)=xC(a_1,a_2,\ldots,a_n)$$

Proof.

$$C(xa_1,xa_2,\ldots,xa_n)=\frac{(xa_1)^2+(xa_2)^2+\ldots+(xa_n)^2}{xa_1+xa_2+\ldots+xa_n}=$$

$$=\frac{x^2(a_1^2+a_2^2+\ldots+a_n^2)}{x(a_1+a_2+\ldots+a_n)}=x\cdot\frac{a_1^2+a_2^2+\ldots+a_n^2}{a_1+a_2+\ldots+a_n}=$$

$$=xC(a_1,a_2,\ldots,a_n)$$

□

Property 4

$$C(a,b)=2A(a,b)-H(a,b)$$

Proof.

$$2A(a,b) - H(a,b) = 2 \cdot \frac{a+b}{2} - \frac{2ab}{a+b} =$$
$$= \frac{(a+b)^2 - 2ab}{a+b} = \frac{a^2+b^2}{a+b} = C(a,b)$$

□

Property 5

$$A(H(a,b), C(a,b)) = A(a,b)$$

Proof.

$$A(H(a,b), C(a,b)) = \frac{\frac{2ab}{a+b} + \frac{a^2+b^2}{a+b}}{2} =$$
$$= \frac{2ab + a^2 + b^2}{2(a+b)} = \frac{(a+b)^2}{2(a+b)} = \frac{a+b}{2} = H(a,b)$$

□

Property 6

$$G(A(a,b), C(a,b)) = Q(a,b)$$

Proof.

$$G(A(a,b), C(a,b)) = \sqrt{A(a,b) \cdot C(a,b)} =$$
$$= \sqrt{\frac{a+b}{2} \cdot \frac{a^2+b^2}{a+b}} = \sqrt{\frac{a^2+b^2}{2}} = Q(a,b)$$

□

Property 7

$$G(A(a,b), H(a,b)) = G(a,b)$$

Proof.

$$G(A(a,b), H(a,b)) = \sqrt{A(a,b) \cdot H(a,b)} =$$
$$= \sqrt{\frac{a+b}{2} \cdot \frac{2ab}{a+b}} = \sqrt{ab} = G(a,b)$$

□

Property 8

$$\frac{L(a^2,b^2)}{L(a,b)} = A(a,b)$$

Proof.

$$\frac{L(a^2,b^2)}{L(a,b)} = \frac{\frac{b^2-a^2}{\log b^2 - \log a^2}}{\frac{b-a}{\log b - \log a}} =$$
$$= \frac{(b-a)(b+a)}{2(\log b - \log a)} \cdot \frac{\log b - \log a}{b-a} = \frac{a+b}{2} = A(a,b)$$

□

Property 9

$$\sqrt{\frac{L(a,b)}{L(\frac{1}{a},\frac{1}{b})}} = G(a,b)$$

Proof.

$$\sqrt{\frac{L(a,b)}{L(\frac{1}{a},\frac{1}{b})}}=\sqrt{\frac{b-a}{\log b-\log a}\cdot\frac{\log\frac{1}{b}-\log\frac{1}{a}}{\frac{1}{b}-\frac{1}{a}}}=$$

$$=\sqrt{\frac{b-a}{\log b-\log a}\cdot\frac{ab}{a-b}\cdot(-\log b+\log a)}=\sqrt{ab}=G(a,b)$$

□

Property 10

$$\frac{L(\frac{1}{a},\frac{1}{b})}{L(\frac{1}{a^2},\frac{1}{b^2})}=H(a,b)$$

Proof.

$$\frac{L(\frac{1}{a},\frac{1}{b})}{L(\frac{1}{a^2},\frac{1}{b^2})}=\frac{\frac{\frac{1}{b}-\frac{1}{a}}{\log\frac{1}{b}-\log\frac{1}{a}}}{\frac{1}{b^2}-\frac{1}{a^2}}=$$

$$=\left(\frac{1}{b}-\frac{1}{a}\right)\cdot\frac{1}{-\log b+\log a}\cdot\frac{a^2b^2}{a^2-b^2}\cdot(-2\log b+2\log a)=$$

$$=\frac{a-b}{ab}\cdot\frac{1}{\log b-\log a}\cdot\frac{2a^2b^2}{(a-b)(a+b)}(\log b-\log a)=\frac{2ab}{a+b}=H(a,b)$$

□

Property 11

$$C(a,b)+C(b,c)+C(c,a)\geq 3G(a,b,c)$$

Proof.

$$C(a,b)\geq G(a,b)\Rightarrow\sum_{cyc}C(a,b)\geq\sum_{cyc}G(a,b)=$$

$$=\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\overset{\text{AM-GM}}{\geq}$$

$$\geq 3\cdot\sqrt[3]{(\sqrt{ab})\cdot(\sqrt{bc})\cdot(\sqrt{ca})}=3\sqrt[3]{abc}=3G(a,b,c)$$

□

Property 12

$$\frac{C^2(a,b)}{C^2(b,c)}+\frac{C^2(b,c)}{C(c,a)}+\frac{C^2(c,a)}{C(a,b)}\geq\frac{C(a,b)}{C(b,c)}+\frac{C(b,c)}{C(c,a)}+\frac{C(c,a)}{C(a,b)} \tag{7}$$

Proof.

Denote: $u=C(a,b), v=C(b,c), w=C(c,a)$

$$(7)\Leftrightarrow\sum_{cyc}\frac{u^2}{v^2}\geq\sum_{cyc}\frac{u}{v}\Leftrightarrow$$

$$\Leftrightarrow\sum_{cyc}\frac{u^2}{v^2}(uvw)^2\geq\sum_{cyc}\frac{u}{v}(uvw)^2\Leftrightarrow$$

$$\Leftrightarrow\sum_{cyc}u^4w^2\geq(uvw)^2\sum_{cyc}\frac{u}{v}$$

$$\sum_{cyc}u^4w^2=\frac{1}{6}\sum_{cyc}(6u^4w^2)=$$

$$= \frac{1}{6}\sum_{cyc}(4u^4w^2 + u^4w^2 + u^4w^2) =$$

$$= \frac{1}{6}\sum_{cyc}(4u^4w^2 + v^4u^2 + w^4v^2) \geq$$

$$\overset{\text{AM-GM}}{\geq} \frac{1}{6}\sum_{cyc} 6 \cdot \sqrt[6]{(u^4w^2)^4 \cdot v^4 \cdot u^2 \cdot w^4v^2} =$$

$$= \sum_{cyc} \sqrt[6]{u^{18} \cdot v^6 \cdot w^{12}} =$$

$$= \sum_{cyc} u^3vw^2 = \sum_{cyc}(uvw)^2 \cdot \frac{u}{v} = (uvw)^2 \sum_{cyc} \frac{u}{v}$$

□

References

[1] Romanian Mathematical Magazine - Interactive Journal, www.ssmrmh.ro

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