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SP.404 In ΔABC the following relationship holds:

$$3(a^2+b^2+c^2)+4\left(h_a^2+h_b^2+h_c^2\right)\geq 24\sqrt{3}F$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer, Solution 2 by Daniel Văcaru-Romania

Solution 1 by proposer

$$2h_a^2-2m_a^2=2h_a^2-2\cdot\frac{2(b^2+c^2)-a^2}{4}=2h_a^2-b^2-c^2+\frac{a^2}{2}$$

Hence,

$$a^2+b^2+c^2+2h_a^2-2m_a^2=2h_a^2+a^2+\frac{a^2}{2}=2h_a^2+\frac{3}{2}a^2\overset{AM-GM}{\geq}$$

$$\geq 2\sqrt{2h_a^2\cdot\frac{3a^2}{2}}=2\sqrt{3a^2h_a^2}=2\sqrt{2(2F)^2}=4\sqrt{3}F$$

$$3(a^2+b^2+c^2)+2\left(h_a^2+h_b^2+h_c^2\right)-2\left(m_a^2+m_b^2+m_c^2\right)\geq 12\sqrt{3}F$$

Thus,

$$3(a^2+b^2+c^2)+2\left(h_a^2+h_b^2+h_c^2\right)-2\cdot\frac{3}{4}(a^2+b^2+c^2)\geq 12\sqrt{3}F$$

$$\frac{3}{2}(a^2+b^2+c^2)+2\left(h_a^2+h_b^2+h_c^2\right)\geq 12\sqrt{3}F$$

Therefore,

$$3(a^2+b^2+c^2)+4\left(h_a^2+h_b^2+h_c^2\right)\geq 24\sqrt{3}F$$

Equality holds for $a=b=c$.

Solution 2 by Daniel Văcaru-Romania

We have $3a^2+4h_a^2=3a^2+\frac{16F^2}{a^2}\overset{AGM}{\geq} 8F\sqrt{3}$ (and analogs for b and c)

Summing these relationships, we find:

$$3(a^2+b^2+c^2)+4\left(h_a^2+h_b^2+h_c^2\right)\geq 24\sqrt{3}F.$$