NAPOLEON OUTER TRIANGLE REVISITED

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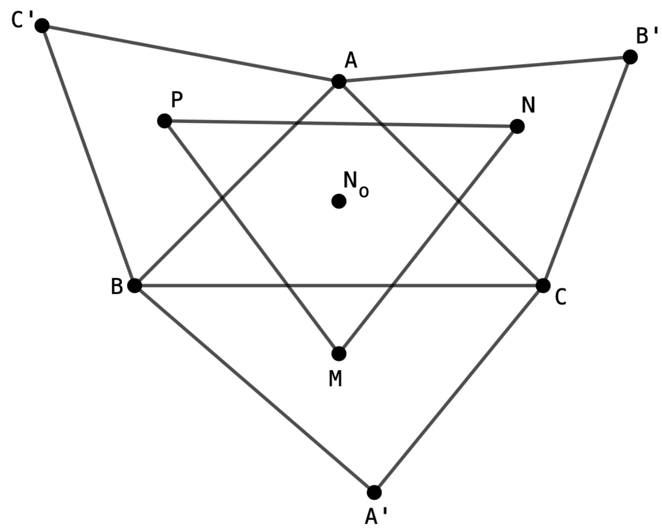
Abstract. In this paper is proved Napoleon's theorem and are made connections with famous inequalities as Ionescu-Weitzenbock's.

Napoleon's theorem for outer triangle



In the figure above, ABC is any fixed triangle, BCA', CAB', ABC' are equilateral triangles constructed on sides of ABC in exterior. The line connecting the centroids M, N, P of triangles BCA', CAB', ABC' form an equilateral triangle named Napoleon's outer triangle of ΔABC.

Proof.

$$BC = a, CA = b, AB = c, \mu(\angle PAN) = \mu(A) + \frac{\pi}{3},$$

$$AP = \frac{2}{3} \cdot \frac{\sqrt{3}}{2} \cdot c = \frac{\sqrt{3}c}{3}; AN = \frac{2}{3} \cdot \frac{\sqrt{3}}{2} \cdot b = \frac{\sqrt{3}b}{3}$$

By cosine law in ΔPAN:

$$MN^2 = AP^2 + AN^2 - 2AP \cdot AN \cdot \cos(\angle PAN)$$

$$MN^2 = \frac{1}{3}c^2 + \frac{1}{3}b^2 - 2bc \cdot \frac{1}{3} \cdot \cos\Big(A + \frac{\pi}{3}\Big)$$

$$MN^2 = \frac{b^2 + c^2}{3} - \frac{2bc}{3}\Big(\cos A \cos\frac{\pi}{3} - \sin A \sin\frac{\pi}{3}\Big)$$

$$MN^2 = \frac{b^2 + c^2}{3} - \frac{2bc}{3}\Big(\cos A \cdot \frac{1}{2} - \sin A \cdot \frac{\sqrt{3}}{2}\Big)$$

$$MN^2 = \frac{b^2+c^2}{3} - \frac{bc}{3} \cdot \frac{b^2+c^2-a^2}{2bc} + bc\sin A \cdot \frac{\sqrt{3}}{3}$$

$$MN^2 = \frac{b^2+c^2}{3} - \frac{b^2+c^2}{6} + \frac{a^2}{6} + 2F \cdot \frac{\sqrt{3}}{3}$$

$$MN^2 = \frac{b^2+c^2+a^2}{6} + \frac{2F\sqrt{3}}{3} \tag{1}$$

Expression (1) is symmetrical in terms of a, b, c hence $MN = NP = PM \Rightarrow \Delta MNP$ is an equilateral one.

Sides of Napoleon's outer triangle are given by:

$$MN = \sqrt{\frac{a^2+b^2+c^2}{6} + \frac{2F\sqrt{3}}{3}}; F = [ABC]$$

Area of Napoleon's outer triangle:

$$[MNP] = \frac{\sqrt{3}}{4} \cdot MN^2 = \frac{\sqrt{3}}{4} \cdot \left(\frac{b^2+c^2+a^2}{6} + \frac{2F\sqrt{3}}{3}\right)$$

$$[MNP] = \frac{(a^2+b^2+c^2)\sqrt{3}}{4} + \frac{F}{2}$$

Observation 1

If the original triangle ABC is an equilateral one ($a = b = c$) then:

$$[MNP] = \frac{3a^2\sqrt{3}}{24} + \frac{F}{2} = \frac{a^2\sqrt{3}}{8} + \frac{a^2\sqrt{3}}{8} = \frac{a^2\sqrt{3}}{4} = [ABC]$$

Observation 2

Using Ionescu-Weitzenbock's inequality $a^2+b^2+c^2 \geq 4\sqrt{3}F$ the following inequality can be obtained:

$$[MNP] = \frac{(a^2+b^2+c^2)\sqrt{3}}{24} + \frac{F}{2} \geq \frac{4\sqrt{3}F \cdot \sqrt{3}}{24} + \frac{F}{2} = \frac{F}{2} + \frac{F}{2} = F$$

$$[MNP] \geq F$$

Observation 3

Denote $k = [MNP]; K = [MNP]; s_k$ - semiperimeter of $\Delta MNP; r_k, R_k, r_a^k$ - inradii, circumradii, respectively exradii of $\Delta MNP, N_0$ - the center of ΔMNP.

$$k = \sqrt{\frac{a^2+b^2+c^2}{6} + \frac{2F\sqrt{3}}{3}}$$

$$K = \frac{(a^2+b^2+c^2)\sqrt{3}}{24} + \frac{F}{2}$$

$$s_k = \frac{3k}{2}$$

$$r_k = \frac{\sqrt{3}}{6} \cdot k = \frac{1}{6}\sqrt{\frac{a^2+b^2+c^2}{2} + 2F\sqrt{3}}$$

$$R_k = \frac{\sqrt{3}}{3} \cdot k = \frac{1}{3}\sqrt{\frac{a^2+b^2+c^2}{2} + 2F\sqrt{3}}$$

$$r_a^k = \frac{\sqrt{3}}{8} \cdot k = \frac{1}{8}\sqrt{\frac{a^2+b^2+c^2}{2} + 2F\sqrt{3}}$$

Observation 4
The trilinear coordinates of N_0 are:

$$\left(\sec\left(A-\frac{\pi}{3}\right);\sec\left(B-\frac{\pi}{3}\right),\sec\left(C-\frac{\pi}{3}\right)\right)$$

Observation 5
The barycentric coordinates of N_0 are:

$$\left(a\csc\left(A+\frac{\pi}{6}\right),b\csc\left(B+\frac{\pi}{6}\right),c\csc\left(C+\frac{\pi}{6}\right)\right)$$

Observation 6
Using Ionescu-Weitzenbock's inequality:

$$r_k=\frac{1}{6}\sqrt{\frac{a^2+b^2+c^2}{2}+2F\sqrt{3}}\geq\frac{1}{6}\sqrt{\frac{4\sqrt{3}F}{2}+2F\sqrt{3}}=\frac{1}{6}\sqrt{4F\sqrt{3}}=\frac{1}{3}\sqrt{F\sqrt{3}}$$

$$R_k=\frac{1}{3}\sqrt{\frac{a^2+b^2+c^2}{2}+2F\sqrt{3}}\geq\frac{1}{3}\sqrt{\frac{4\sqrt{3}F}{2}+2F\sqrt{3}}=\frac{1}{3}\sqrt{4F\sqrt{3}}=\frac{2}{3}\sqrt{F\sqrt{3}}$$

$$r_k+R_k\geq\frac{1}{3}\sqrt{F\sqrt{3}}+\frac{2}{3}\sqrt{F\sqrt{3}}=\sqrt{F\sqrt{3}}$$

$$r_a^k=\frac{1}{8}\sqrt{\frac{a^2+b^2+c^2}{2}+2F\sqrt{3}}\geq\frac{1}{8}\sqrt{\frac{4\sqrt{3}F}{2}+2F\sqrt{3}}=\frac{1}{8}\sqrt{4F\sqrt{3}}=\frac{1}{4}\sqrt{F\sqrt{3}}$$

References

[1] Romanian Mathematical Magazine - Interactive Journal, www.ssmrmh.ro

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