METRIC RELATIONSHIPS IN ŞAHIN'S TRIANGLE

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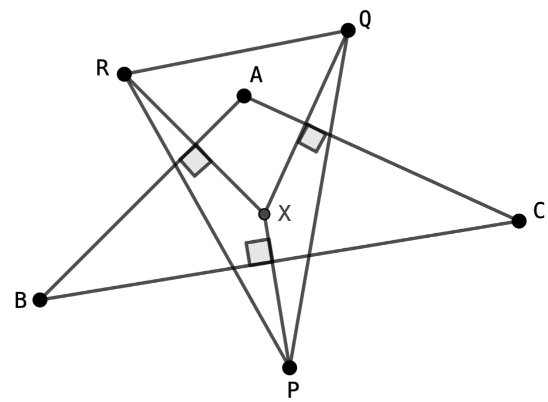
Abstract. In this article are proved a few metric relationships in a geometrical configuration created by the mathematician Mehmet Şahin from Ankara - Turkiye.



Theorem (Mehmet Şahin)
Let ΔABC be an acute triangle and $X \in Int(\Delta ABC)$ such that $XP \perp BC$; $XQ \perp AC; XR \perp AB; XP = BC; XQ = AC; XR = AB$ (such in above figure). In these conditions:

1. $QR = 2m_a, RP = 2m_b, PQ = 2m_c$
(m_a, m_b, m_c - medians in the original ΔABC)

2. $[PQR] = 3F$
($[PQR]$ - area; F - area of the original ΔABC)

3. $m_{a'} = \frac{3a}{2}; m_{b'} = \frac{3b}{2}; m_{c'} = \frac{3c}{2}$
($m_{a'}, m_{b'}, m_{c'}$ - medians in ΔPQR; a, b, c - sides of original ΔABC)

4. $R^* = \frac{8}{3} \cdot \frac{m_a m_b m_c R}{abc}$
(R^*, R - circumradii of $\Delta PQR, \Delta ABC$)

5. $R^* \le \frac{\sqrt{3}}{4} \cdot \frac{R^3}{r^2}$
(r - inradii of ΔABC)

6. $aR_a + bR_b + cR_c \le 9R^2$
(R_a, R_b, R_c - circumradii of $\Delta XQR, \Delta XRP, \Delta XPQ$)

7. $R^* = \frac{R_a R_b R_c}{3R^2}$

8. $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{m_a+m_b+m_c+a+b+c}{F}$
(r_1, r_2, r_3 - inradii of $\Delta XQR, \Delta XRP, \Delta XPQ$)

Proof (Daniel Sitaru).
1.
In ΔXQR by cosine law:

$$QR^2 = XQ^2 + XR^2 - 2XQ \cdot XR \cdot \cos(\pi - A)$$

($ARXQ$ is a cyclic quadrilateral; $\mu(\angle XRA) = \mu(\angle XQA) = \frac{\pi}{2}$)

$$QR^2 = b^2 + c^2 - 2bc\cos(\pi - A)$$
$$QR^2 = b^2 + c^2 + 2bc\cos A$$
$$QR^2 = b^2 + c^2 + 2bc \cdot \frac{b^2 + c^2 - a^2}{2bc}$$
$$QR^2 = 2(b^2 + c^2) - a^2$$
$$QR^2 = 4 \cdot \frac{2(b^2 + c^2) - a^2}{4}$$
$$QR^2 = 4m_a^2 \Rightarrow QR = 2m_a$$

Analogous: $RP = 2m_b, PQ = 2m_c$

2.
$$[PQR] = [XPQ] + [XQR] + [XRP] = \frac{1}{2}XP \cdot XQ \cdot \sin(\angle PXQ) +$$
$$+\frac{1}{2}XQ \cdot XR \cdot \sin(\angle QXR) + \frac{1}{2}XR \cdot XP \cdot \sin(\angle RXP) =$$
$$= \frac{1}{2}bc\sin(\pi - A) + \frac{1}{2}ca\sin(\pi - B) + \frac{1}{2}ab\sin(\pi - C) =$$
$$= \frac{1}{2}bc\sin A + \frac{1}{2}ca\sin B + \frac{1}{2}ab\sin C =$$
$$= F + F + F = 3F$$

3.
Denote:

$$a' = QR = 2m_a; b' = RP = 2m_b, c' = PQ = 2m_c$$
$$m_{a'}^2 = \frac{1}{2}(b'^2 + c'^2) - \frac{1}{4}a'^2 =$$
$$= \frac{1}{2}(4m_b^2 + 4m_c^2) - \frac{1}{4} \cdot 4m_a^2 =$$

$$= 2m_b^2 + 2m_c^2 - m_a^2 =$$
$$= 2\Big(\frac{1}{2}(a^2+c^2) - \frac{1}{4}b^2\Big) + 2\Big(\frac{1}{2}(a^2+b^2) - \frac{1}{4}c^2\Big) - \frac{1}{2}(b^2+c^2) + \frac{1}{4}a^2 =$$
$$= a^2 + c^2 - \frac{1}{2}b^2 + a^2 + b^2 - \frac{1}{2}c^2 - \frac{1}{2}b^2 - \frac{1}{2}c^2 + \frac{1}{4}a^2 =$$
$$= 2a^2 + \frac{1}{4}a^2 = \frac{9a^2}{4}$$
$$m_{a'}^2 = \frac{9a^2}{4} \Rightarrow m_{a'} = \frac{3a}{2}$$

Analogous: $m_{b'} = \frac{3b}{2}; m_{c'} = \frac{3c}{2}$

4.
$$R^* = \frac{a'b'c'}{4[PQR]} = \frac{2m_a \cdot 2m_b \cdot 2m_c}{4 \cdot 3F} = \frac{2m_a m_b m_c}{3 \cdot \frac{abc}{4R}} = \frac{8}{3} \cdot \frac{m_a m_b m_c R}{abc}$$

5.
We will use the known inequalities:
$$m_a \leq 2R\cos^2\frac{A}{2}; m_b \leq 2R\cos^2\frac{B}{2}; m_c \leq 2R\cos^2\frac{C}{2}$$
$$\frac{8}{3} \cdot \frac{m_a m_b m_c R}{abc} \leq \frac{8R}{3abc} \cdot 2R\cos^2\frac{A}{2} \cdot 2R\cos^2\frac{B}{2} \cdot 2R\cos^2\frac{C}{2} =$$
$$= \frac{64R^4}{3abc} \cdot \frac{s(s-a)}{bc} \cdot \frac{s(s-b)}{ca} \cdot \frac{s(s-c)}{ab} =$$
$$= \frac{64R^4 s^2 \cdot F^2}{3(abc)^3} = \frac{64R^4 s^2 \cdot F^2}{3 \cdot 16R^2F^2 \cdot 4RF}$$
$$= \frac{4R^2s^2}{3 \cdot 4RF} = \frac{R^2s^2}{3R \cdot rs} = \frac{Rs}{3r} \leq$$
$$\overset{\text{MITRINOVIC}}{\leq} \frac{R \cdot \frac{3\sqrt{3}}{2}R}{3r} = \frac{\sqrt{3}R^2}{2r} =$$
$$= \frac{\sqrt{3}R^3}{2rR} \overset{\text{EULER}}{\leq} \frac{\sqrt{3}R^3}{2r \cdot 2r} = \frac{\sqrt{3}}{4} \cdot \frac{R^3}{r^2}$$

6.
$$R_a = \frac{XQ \cdot XR \cdot RQ}{4[XQR]} = \frac{b \cdot c \cdot 2m_a}{4F} =$$
$$= \frac{\frac{2F}{\sin A} \cdot 2m_a}{4F} = \frac{m_a}{\sin A} = \frac{m_a}{\frac{a}{2R}} = \frac{2Rm_a}{a}$$
$$a \cdot R_a + b \cdot R_b + c \cdot R_c = 2R(m_a + m_b + m_c) \leq 2R \cdot \frac{9R}{2} = 9R$$

(The inequality: $m_a + m_b + m_c \leq \frac{9R}{2}$ is known)

7.
$$\frac{R_aR_bR_c}{3R^2} = \frac{\frac{2Rm_a}{a} \cdot \frac{2Rm_b}{b} \cdot \frac{2Rm_c}{c}}{3R^2} =$$
$$= \frac{8R^3}{3R^2} \cdot \frac{m_a m_b m_c}{abc} = \frac{8}{3} \cdot \frac{m_a m_b m_c R}{abc} = R^*$$

8.
$$\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=$$
$$=\frac{1}{\frac{2[XRQ]}{b+c+2m_a}}+\frac{1}{\frac{2[XPR]}{c+a+2m_b}}+\frac{1}{\frac{2[XQP]}{a+b+2m_c}}=$$
$$=\frac{b+c+2m_a}{2F}+\frac{c+a+2m_b}{2F}+\frac{a+b+2m_c}{2F}=$$
$$=\frac{2(b+c+a+m_a+m_b+m_c)}{2F}=$$
$$=\frac{a+b+c+m_a+m_b+m_c}{F}$$

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References

[1] Romanian Mathematical Magazine - Interactive Journal, www.ssmrmh.ro

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