METRIC RELATIONSHIPS IN ŞAHIN'S TRIANGLE (II)

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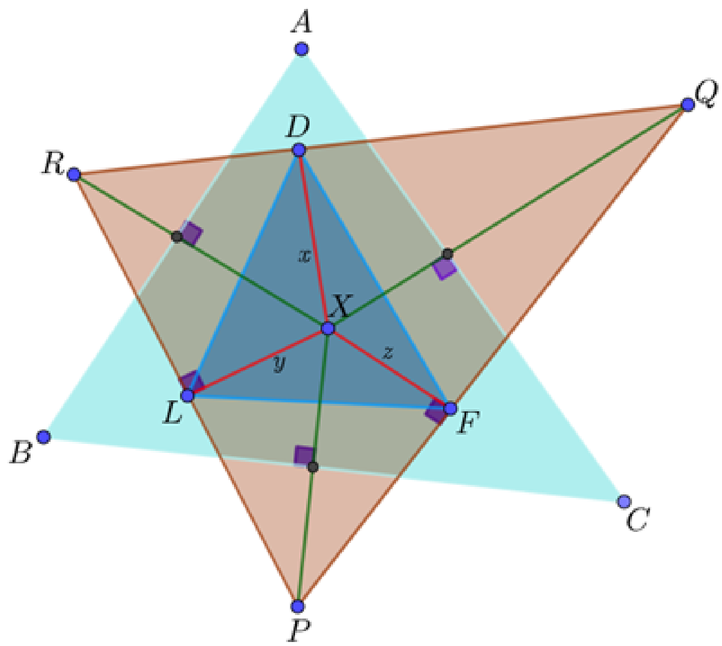
ABSTRACT. This article follows to [1] and prove more metric relationships in a geometrical configuration created by the mathematician Mehmet Şahin from Ankara - Turkiye.



Theorem (Mehmet Şahin)
Let ΔABC be an acute triangle and $X \in Int(\Delta ABC)$ such that $XP \perp BC$; $XQ \perp AC; XR \perp AB; XQ = AC; XP = BC; XR = AB$ (such in above figure) and let ΔDEF be the pedal triangle of X according to ΔPQR.
In these conditions:

1. If r^* is inradii of ΔPQR then:

$$r^* = \frac{3F}{m_a + m_b + m_c}$$

2. If $XD = x; XE = y; XF = z; XD \perp RQ, XE \perp PR, XF \perp QP$ then:

$$x = \frac{F}{m_a}; y = \frac{F}{m_b}; z = \frac{F}{m_c}$$

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{m_a+m_b+m_c}{F}$$

3.

$$\cos(\angle RPQ)=\cos(\angle P)=\frac{5a^2-b^2-c^2}{8m_bm_c}$$
$$\cos(\angle PQR)=\cos(\angle Q)=\frac{5b^2-c^2-a^2}{8m_cm_a}$$
$$\cos(\angle QRP)=\cos(\angle R)=\frac{5c^2-a^2-b^2}{8m_am_b}$$

4.

$$\sin(\angle RPQ)=\sin(\angle P)=\frac{3F}{2m_bm_c}$$
$$\sin(\angle PQR)=\sin(\angle Q)=\frac{3F}{2m_cm_a}$$
$$\sin(\angle QRP)=\sin(\angle R)=\frac{3F}{2m_am_b}$$

5.

$$[DEF]=\frac{9F^3(a^2+b^2+c^2)}{16m_a^2m_b^2m_c^2}$$

6.

$$DE+EF+FD=\frac{3(am_a+bm_b+cm_c)}{2m_am_bm_c}$$

7.
If R_* is circumradii of ΔDEF then:

$$R_*=\frac{3abc}{2(a^2+b^2+c^2)}$$

Proof (Daniel Sitaru).
1.
According to [1]:

$$QR=2m_a, RP=2m_b, PQ=2m_c \text{ and } [PQR]=3F$$
$$r^*=\frac{[PQR]}{\frac{QR+RP+PQ}{2}}=\frac{3F}{\frac{2m_a+2m_b+2m_c}{2}}=\frac{3F}{m_a+m_b+m_c}$$

2.

$$[XQR]=\frac{x\cdot QR}{2}$$

According to [1]: $[XQR]=F; QR=2m_a$

$$F=\frac{x\cdot 2m_a}{2}\Rightarrow x=\frac{F}{m_a}$$

Analogous:

$$y=\frac{F}{m_b}; z=\frac{F}{m_c}$$

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{m_a}{F}+\frac{m_b}{F}+\frac{m_c}{F}=\frac{m_a+m_b+m_c}{F}$$

3.

$$\cos P=\frac{PR^2+PQ^2-QR^2}{2PR\cdot PQ}=\frac{4m_c^2+4m_b^2-4m_a^2}{2\cdot 2m_c\cdot 2m_b}=\frac{m_b^2+m_c^2-m_a^2}{2m_bm_c}$$
$$=\frac{\frac{1}{2}(a^2+c^2)-\frac{1}{4}b^2+\frac{1}{2}(a^2+b^2)-\frac{1}{4}a^2-\frac{1}{2}(b^2+c^2)+\frac{1}{4}a^2}{2m_bm_c}=\frac{5a^2-b^2-c^2}{8m_bm_c}$$

Analogous:

$$\cos Q=\frac{5b^2-c^2-a^2}{8m_cm_a};\cos R=\frac{5c^2-a^2-b^2}{8m_am_b}$$

4.

$$\sin P=\frac{QR}{2R^*}$$

R^* - circumradii of ΔPQR.
According to [1]:

$$R^*=\frac{8}{3}\cdot\frac{m_am_bm_cR}{abc};QR=2m_a$$
$$\sin P=\frac{2m_a}{2\cdot\frac{8}{3}\cdot\frac{m_am_bm_cR}{abc}}=\frac{3abc}{8m_bm_cR}=\frac{3\cdot 4RF}{8m_am_bm_cR}=\frac{3F}{2m_bm_c}$$

Analogous:

$$\sin Q=\frac{3F}{2m_cm_a};\sin R=\frac{3F}{2m_am_b}$$

5.

$$[DEF]=[DXE]+[EXF]+[FXD]=$$
$$=\frac{1}{2}xy\sin R+\frac{1}{2}yz\sin P+\frac{1}{2}zx\sin Q=$$
$$=\frac{1}{2}\sum_{cyc}xy\sin R=\frac{1}{2}\sum_{cyc}\frac{F}{m_a}\cdot\frac{F}{m_b}\cdot\frac{3F}{2m_am_b}=$$
$$=\frac{3F^3}{4}\sum_{cyc}\frac{1}{m_a^2m_b^2}=$$
$$=\frac{3F^3}{4m_a^2m_b^2m_c^2}(m_a^2+m_b^2+m_c^2)=$$
$$=\frac{3}{4}\cdot\frac{3}{4}\cdot\frac{(a^2+b^2+c^2)F^3}{m_a^2m_b^2m_c^2}=$$
$$=\frac{9(a^2+b^2+c^2)F^3}{16m_a^2m_b^2m_c^2}$$

6.

$$DE^2=x^2+y^2-2xy\cos(\angle EXD)=$$
$$=\frac{F^2}{m_a^2}+\frac{F^2}{m_b^2}-2\cdot\frac{F}{m_a}\cdot\frac{F}{m_b}\cdot\cos(\pi-R)=$$
$$=F^2\Big(\frac{1}{m_a^2}+\frac{1}{m_b^2}+2\cdot\frac{F}{m_a}\cdot\frac{F}{m_b}\cdot\cos R)=$$

$$= F^2\left(\frac{1}{m_a^2} + \frac{1}{m_b^2} + \frac{2}{m_a m_b} \cdot \frac{5c^2 - a^2 - b^2}{8m_a m_b}\right) =$$
$$= \frac{F^2}{4} \cdot \frac{4m_b^2 + 4m_a^2 + 5c^2 - a^2 - b^2}{m_a^2 m_b^2} =$$
$$= \frac{F^2}{4m_a^2 m_b^2}(2a^2 + 2c^2 - b^2 + 2b^2 + 2c^2 - a^2 + 5c^2 - a^2 - b^2) =$$
$$= \frac{9c^2 F^2}{4m_a^2 m_b^2} \Rightarrow DE = \frac{3cF}{2m_a m_b}$$

Analogous:

$$EF = \frac{3aF}{2m_b m_c}; FD = \frac{3bF}{2m_c m_a}$$
$$DE + EF + FD = \frac{3F}{2}\left(\frac{c}{m_a m_b} + \frac{a}{m_b m_c} + \frac{b}{m_c m_a}\right) =$$
$$= \frac{3F(am_a + bm_b + cm_c)}{2m_a m_b m_c}$$

7.

$$R_* = \frac{DE \cdot EF \cdot FD}{4[DEF]} = \frac{\frac{3cF}{2m_a m_b} \cdot \frac{3aF}{2m_b m_c} \cdot \frac{3bF}{2m_c m_a}}{4 \cdot \frac{9(a^2+b^2+c^2)F^3}{16m_a^2 m_b^2 m_c^2}} =$$
$$= \frac{27abcF^3}{8m_a^2 m_b^2 m_c^2} \cdot \frac{4m_a^2 m_b^2 m_c^2}{9(a^2 + b^2 + c^2)F^3} =$$
$$= \frac{27abc}{18(a^2 + b^2 + c^2)} = \frac{3abc}{2(a^2 + b^2 + c^2)}$$

References

[1] Daniel Sitaru, *Metric relationships in Şahin's triangle.* www.ssmrmh.ro
[2] Romanian Mathematical Magazine - Interactive Journal, www.ssmrmh.ro

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