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ROMANIAN MATHEMATICAL MAGAZINE

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If $0 \le a \le \frac{\pi}{12}$ then:

$$\int_0^a \sin x \, . \cos(6x) \, . \cos^6(4x) \, . \cos^{15}(2x) \, dx \le \frac{1}{193}(1 - \cos^{193} a)$$

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Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Let\ x \in [0, a], since\ a \le \frac{\pi}{12} \to \cos x, \cos(2x), \cos(4x), \cos(6x), \sin x \ge 0$$

$$We\ have: \cos(2x) = 2\cos^2 x - 1 \overset{AM-GM}{\leq} (\cos^4 x + 1) - 1 = \cos^4 x \to \cos(2x) \le \cos^4 x \ (1)$$

$$\to \cos(4x) \overset{(1)}{\leq} \cos^4(2x) \overset{(1)}{\leq} (\cos^4 x)^4 = \cos^{16} x \to \cos(4x) \le \cos^{16} x \ (2)$$

$$\cos(3x) = \cos x \, . (4\cos^2 x - 3) \overset{AM-GM}{\leq} \cos x \, . [(\cos^8 x + 1 + 1 + 1) - 3] = \cos^9 x \to \cos(3x) \le \cos^9 x \ (i)$$

$$\to \cos(6x) \overset{(1)}{\leq} \cos^4(3x) \overset{(i)}{\leq} (\cos^9 x)^4 = \cos^{36} x \to \cos(6x) \le \cos^{36} x \ (3)$$

$$(1), (2), (3) \to \cos(6x) \, . \cos^6(4x) \, . \cos^{15}(2x) \le \cos^{36} x \, . (\cos^{16} x)^6 . (\cos^4 x)^{15} = \cos^{192} x$$

$$\rightarrow \int_0^a \sin x \,.\, \cos(6x)\,.\,\cos^6(4x)\,.\,\cos^{15}(2x)\,dx \le \int_0^a \sin x\,.\,\cos^{192} x\,dx = \left[\frac{-1}{193}\cos^{193} x\right]_0^a$$

$$= \frac{1}{193}\left(1 - \cos^{193} a\right).$$