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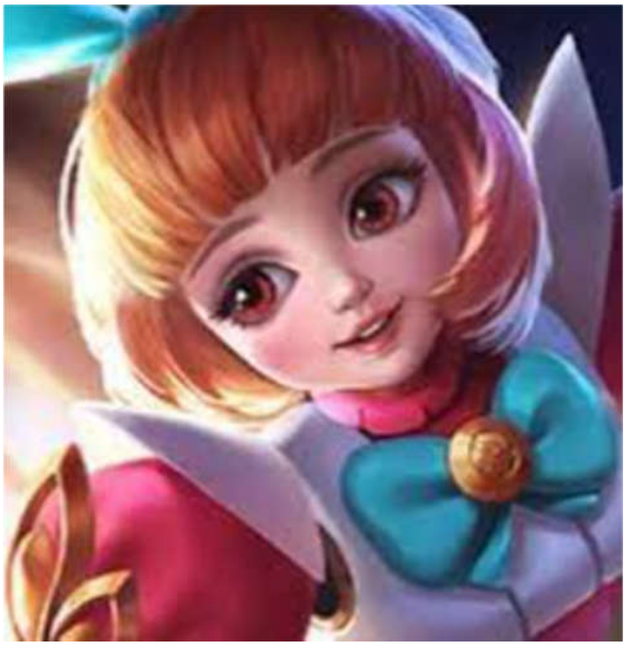

Let $n \in \mathbb{N}^*$ and $\lambda \geq 0$. Find:

$$\int_0^{\frac{\pi}{2}} \frac{\cos^n x + \lambda \sin^2 x}{\lambda + \sin^n x + \cos^n x} dx$$

Proposed by Marin Chirciu – Romania

Solution 1 by George Florin Șerban-Romania, Solution 2 by Ankush Kumar Parcha-India

Solution 1 by George Florin Serban-Romania

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos^n x + \lambda \sin^2 x}{\lambda + \sin^n x + \cos^n x} dx$$

$$\frac{\pi}{2} - x = y \Rightarrow dx = dy$$

$$x = 0 \Rightarrow y = \frac{\pi}{2}, x = \frac{\pi}{2} \Rightarrow y = 0$$

$$\Rightarrow I = -\int_{\frac{\pi}{2}}^{0} \frac{\cos^n\left(\frac{\pi}{2} - y\right) + \lambda \sin^2\left(\frac{\pi}{2} - y\right)}{\lambda + \sin^n\left(\frac{\pi}{2} - y\right) + \cos^n\left(\frac{\pi}{2} - y\right)} dy$$

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin^n y + \lambda \cos^2 y}{\lambda + \cos^n y + \sin^n y} dx$$

$$\Rightarrow 2I=\int_0^{\frac{\pi}{2}}\frac{\cos^n x+\lambda\sin^2 x+\sin^n x+\lambda\cos^2 x}{\lambda+\sin^n x+\cos^n x}dx$$

$$2I=\int_0^{\frac{\pi}{2}}\frac{-\sin^n x+\cos^n x+\lambda(\sin^2 x+\cos^2 x)}{\lambda+\sin^n x+\cos^n x}dx$$

$$2I=\int_0^{\frac{\pi}{2}}\frac{\sin^n x+\cos^n x+\lambda}{\sin^n x+\cos^n x+\lambda}dx=\int_0^{\frac{\pi}{2}}1\,dx$$

$$2I=x|_0^{\frac{\pi}{2}}=\frac{\pi}{2}-0=\frac{\pi}{2}\Rightarrow I=\frac{\pi}{4}$$

$$\Rightarrow\int_0^{\frac{\pi}{2}}\frac{\cos^n x+\lambda\sin^2 x}{\lambda+\sin^n x+\cos^n x}dx=\frac{\pi}{4}$$

Solution 2 by Ankush Kumar Parcha-India

$$I=\int_0^{\frac{\pi}{2}}\frac{\cos^n(x)+\lambda\sin^2(x)}{\lambda+\sin^n(x)+\cos^n(x)}dx\quad(1)$$

$$I=\int_0^{\frac{\pi}{2}}\frac{\cos^n(x)+\lambda\sin^2(x)}{\lambda+\sin^n(x)+\cos^n(x)}dx=\int_0^{\frac{\pi}{2}}\frac{\sin^n(x)+\lambda\cos^2(x)}{\lambda+\sin^n(x)+\cos^n(x)}dx$$

$$\left(\because\int_a^b f(x)\,dx=\int_a^b f(a+b-x)dx\right)$$

Adding above equation with equation (1)

$$2I=\int_0^{\frac{\pi}{2}}\frac{\sin^n(x)+\cos^n(x)+\lambda(\sin^2(x)+\cos^2(x))}{\lambda+\sin^n(x)+\cos^n(x)}dx$$

$$I=\int_0^{\frac{\pi}{2}}\frac{\cos^n(x)+\lambda\sin^2(x)}{\lambda+\sin^n(x)+\cos^n(x)}dx=\frac{\pi}{4}$$