GERGONNE POINT OF A TRIANGLE AND IT'S DISTANCE ANY POINT IN THE PLANE

José Ferreira de Queiroz Filho
jferreiraqueiroz@gmail.com
Olinda,Pernambuco, Brazil
April 26, 2021

Abstract In this article we give a metric relation which gives the distance between Gergonne Point to any point in the plane of the triangle.

Keywords: Gergonne Point, Stewart's Theorem, Gergonne cevian's.

1 Introduction

We call Gergonne point of the triangle to the meeting point of the lines containing the vertex of a triangle and the point of tangency with the inscribed circle. The Gergonne Point was discovered by Joseph Diaz Gergonne (1771-1859) French mathematician. The identity described here, gives us the distance between the Gergonne point of a triangle and any point on the plane that contains the triangle.

2 Notation

Let ABC be an acute triangle. We denote its side-lengths by $BC = a$, $AC = b$, $AB = c$, its semi perimeter by $s = \frac{1}{2}(a + b + c)$, its area by Δ, its circumradius by R and inradius by r. Its classical centres are the Centroid G, the Incenter I, the Circumcentre O, the Orthocentre H and Nagel point N_a.

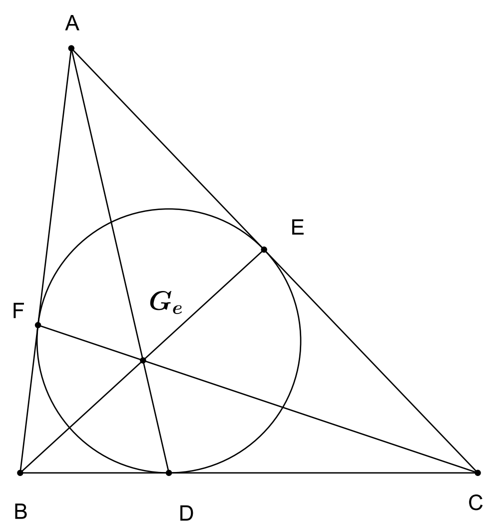

Figura 1: Gergonne Point

3 Prepositions

Preposition 3.1 *If AD, BE , CF are the Gergonne cevian's then $AE = AF = s-a$, $BF = BD = s-b$, $CE = CD = s-c$.*

Proof: Denoting $AF = AE = x$, $BF = BD = y$ and $CE = CD = z$. We have to $y+z=a$, $x+z=b$ and $x+y=c$. Solving the system we find $x = s-a$, $y = s-b$ and $z = s-c$.

Preposition 3.2 *If AD, BE , CF are the Gergonne cevian's of the triangle ABC then they are concurrent and the point of concurrence is the Gergonne Point G_e of the triangle ABC.*

Proof: By Preposition 3.1, we have $AF = AE = s-a$, $BF = BD = s-b$ and $CE = CD = s-c$.

Using Ceva's theorem and replacing AF with AE, BD with BF and CD with CE, we soon find that

$$\frac{AE}{CE}\cdot\frac{CD}{BD}\cdot\frac{BF}{AF} = 1$$

then

$$\frac{AE}{CE}\cdot\frac{CE}{BF}\cdot\frac{BF}{AE} = 1$$

Hence the three cevians that connect the vertices to the point of tangency of the circumference intersect at a point called the point of Gergonne G_e.

Preposition 3.3 *If AD, BE , CF are the Gergonne cevian's of the triangle ABC then the lenght of each cevian is given by $AD = \sqrt{(s-a)^2 + \frac{4\Delta^2}{as}}$, $BE = \sqrt{(s-b)^2 + \frac{4\Delta^2}{bs}}$ and $CF = \sqrt{(s-c)^2 + \frac{4\Delta^2}{cs}}$.*

Proof: We apply Stewart's theorem to triangle ABC in which AD is a cevian. See Figure 4. We get:

$$BC\cdot AD^2 = BD\cdot AC^2 + CD\cdot AB^2 = BC\cdot BD\cdot CD$$

$$a\cdot AD^2 = c^2(s-c) + b^2(s-b) - a(s-b)(s-c).$$

$$AD^2 = \frac{c^2(s-c)}{a} + \frac{b^2(s-b)}{a} - (s-b)(s-c). \quad (1)$$

After simplifying a few steps we obtain $AD = \sqrt{(s-a)^2 + \frac{4\Delta^2}{as}}$.

Similarly we can prove that $BE = \sqrt{(s-b)^2 + \frac{4\Delta^2}{bs}}$ and $CF = \sqrt{(s-c)^2 + \frac{4\Delta^2}{cs}}$.

Preposition 3.4 *The Gergonne Point G_e of the triangle ABC divides each cevian in the ratio given by*

$$\frac{AG_e}{G_eD} = \frac{a(s-a)}{(s-b)(s-c)},\ \frac{BG_e}{G_eE} = \frac{b(s-b)}{(s-a)(s-c)} \text{ and } \frac{CG_e}{G_eF} = \frac{c(s-c)}{(s-a)(s-b)}.$$

Proof: We have by Preposition 3.1, $BD = s-b$ and $CD = s-c$.

Now in the triangle ABD, the line CF as transversal. Apllying Menelaus Theorem we have

$$\frac{AF}{FB}\cdot\frac{BC}{CD}\cdot\frac{DG_e}{G_eA}=1$$
$$\frac{(s-a)}{(s-b)}\cdot\frac{(a)}{(s-c)}\cdot\frac{G_eD}{AG_e}=1$$
$$\frac{AG_e}{G_eD}=\frac{a(s-a)}{(s-b)(s-c)}. \tag{2}$$

Similarly we can prove that $\frac{BG_e}{G_eE}=\frac{b(s-b)}{(s-a)(s-c)}$ and $\frac{CG_e}{G_eF}=\frac{c(s-c)}{(s-a)(s-b)}$.

From expression (2) and the fact that $AD = AG_e + G_eD$, we will have

$$\frac{AG_e}{AD}=\frac{a(s-a)}{a(s-a)+(s-b)(s-c)}=\frac{a(s-a)}{bc-(s-a)^2}. \tag{3}$$

$$\frac{G_eD}{AD}=\frac{(s-b)(s-c)}{a(s-a)+(s-b)(s-c)}=\frac{(s-b)(s-c)}{bc-(s-a)^2}. \tag{4}$$

Preposition 3.5 *Let a, b and c the sides of an triangle ABC, and s, r, R and Δ are, respectively, its semiperimeter, inradius, circumradius and area of that triangle, then*

1. $ab + ac + bc = s^2 + r^2 + 4Rr$
2. $a^2 + b^2 + c^2 = 2s^2 - 2r^2 - 8Rr$
3. $a^3 + b^3 + c^3 = 2s^3 - 6r^2s - 12Rrs$

Proof: Using Heron's formula for the area of the triangle and the fact that $abc = 4R\Delta = 4Rrs$, we have

$$\Delta^2 = s(s-a)(s-b)(s-c)$$
$$s^2r^2 = s^2(-s^2 + ab + ac + bc - 4Rr)$$

hence

$$ab + ac + bc = s^2 + r^2 + 4Rr$$

From $2s = a + b + c$, we have

$$(2s)^2 = (a + b + c)^2$$
$$4s^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$$

hence

$$a^2 + b^2 + c^2 = 2s^2 - 2r^2 - 8Rr$$

Now we know that

$$a^3 + b^3 + c^3 = (a + b + c)(a^2 + b^2 + c^2 - ab - ac - bc) + 3abc$$
$$a^3 + b^3 + c^3 = (2s)[(2s^2 - 2r^2 - 8Rr) - (s^2 + r^2 + 4Rr) + 12Rrs$$

so

$$a^3 + b^3 + c^3 = 2s^3 - 6r^2s - 12Rrs$$

Theorem 3.6 *Let M be any point in the plane of an acute triangle ABC whith Gergonne point G_e. Then:*

$$MG_e^2 = \frac{1}{bc-(s-a)^2} \cdot [(s-b)(s-c) \cdot MA^2 + (s-a)(s-c) \cdot MB^2 + (s-a)(s-b) \cdot MC^2] - \frac{4rs^2(r+R)}{(r+4R)^2}.$$

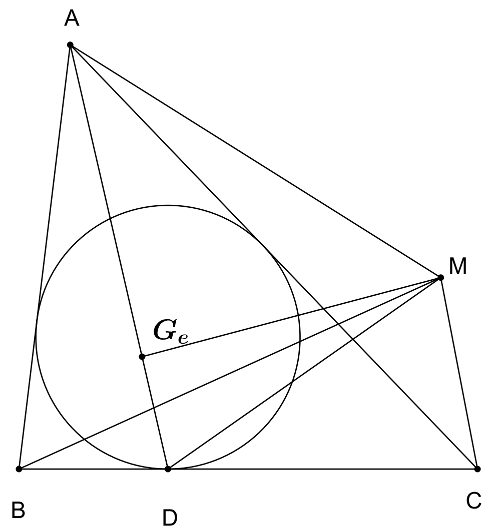

Figura 2: Gergonne Point

Proof: We apply Stewart's theorem in to triangle MBC in which MD is a cevian, according to figure 2. We get

$$a \cdot MD^2 = BD \cdot MC^2 + CD \cdot MB^2 - a \cdot BD \cdot DC$$

$$a \cdot MD^2 = (s-b) \cdot MC^2 + (s-c) \cdot MB^2 - a \cdot (s-b) \cdot (s-c)$$

$$MD^2 = \frac{(s-b)}{a} \cdot MC^2 + \frac{(s-c)}{a} \cdot MB^2 - (s-b)(s-c). \tag{5}$$

Now, we apply Stewart's Theorem in to triangle MAD in which MG_e is a cevian. We get

$$AD \cdot MG_e^2 = AG_e \cdot MD^2 + G_eD \cdot MA^2 - AD \cdot AG_e \cdot G_eD$$
$$MG_e^2 = \frac{AG_e}{AD} \cdot MD^2 + \frac{G_eD}{AD} \cdot MA^2 - AG_e \cdot G_eD.$$

Using the expressions (3), (4) and (5), we obtain

$$MG_e^2 = \frac{1}{bc-(s-a)^2} \cdot [(s-b)(s-c) \cdot MA^2 + (s-a)(s-c) \cdot MB^2 + (s-a)(s-b) \cdot MC^2] - \frac{(s-a)(s-b)(s-c)}{[bc-(s-a)^2]^2} \cdot \Big[4(s-a)(s-b)(s-c) + abc\Big].$$

Now observe that

$$a(s-a) + (s-b)(s-c) = bc - (s-a)^2 = \frac{1}{4}(-a^2 - b^2 - c^2 + 2ab + 2ac + 2bc) = \frac{1}{4}(4r^2 + 16Rr) = r^2 + 4Rr$$

and $abc = 4R\Delta = 4Rrs$.

After simplifying a few steps we obtain,

$$MG_e^2 = \frac{1}{bc-(s-a)^2} \cdot [(s-b)(s-c) \cdot MA^2 + (s-a)(s-c) \cdot MB^2 + (s-a)(s-b) \cdot MC^2] - \frac{4rs^2(r+R)}{(r+4R)^2}.$$

this proves the identity.

4 Main Result

Corollary 4.1 *Be G the Centroid of the triangle ABC and G_e the Gergonne point, then*

$$GG_e^2 = \frac{2}{9}(a^2+b^2+c^2) - \frac{8s^2(r^2+2R^2)}{3(r+4R)^2}.$$

Proof: In Theorem 3.6, replace M by the incenter G. We get

$$GG_e^2 = \frac{1}{bc-(s-a)^2} \cdot [(s-b)(s-c) \cdot GA^2 + (s-a)(s-c) \cdot GB^2 + (s-a)(s-b) \cdot GC^2] - \frac{4rs^2(r+R)}{(r+4R)^2}$$

We know that $GA^2 = \frac{1}{9}(2b^2+2c^2-a^2)$, $GB^2 = \frac{1}{9}(2a^2+2c^2-b^2)$ and $GC^2 = \frac{1}{9}(2a^2+2b^2-c^2)$, then

$$GG_e^2 = \frac{1}{bc-(s-a)^2} \cdot \frac{1}{9} \cdot [(s-b)(s-c)(2b^2+2c^2-a^2)+(s-a)(s-c)(2a^2+2c^2-b^2)+(s-a)(s-b)(2a^2+2b^2-c^2)] - \frac{4rs^2(r+R)}{(r+4R)^2}$$

$$GG_e^2 = \frac{1}{9\,(bc-(s-a)^2)} \cdot [(s-b)(s-c)+(s-a)(s-c)+(s-a)(s-b)](2a^2+2b^2+2c^2)$$
$$-\frac{1}{3\,(bc-(s-a)^2)} \cdot [a^2(s-b)(s-c)+b^2(s-a)(s-c)+c^2(s-a)(s-b)] - \frac{4rs^2(r+R)}{(r+4R)^2}.$$

$$GG_e^2 = \frac{(2a^2+2b^2+2c^2)}{9\,(bc-(s-a)^2)} \cdot [(bc-(s-a)^2)] - \frac{1}{3\,(bc-(s-a)^2)} \cdot [a^2(-s^2+as+bc)+b^2(-s^2+bs+ac)+c^2(-s^2+cs+ab)] - \frac{4rs^2(r+R)}{(r+4R)^2}.$$

$$GG_e^2 = \frac{2}{9}(a^2+b^2+c^2) - \frac{1}{3\,(bc-(s-a)^2)} \cdot [a^2(-s^2+as+bc)+b^2(-s^2+bs+ac)+c^2(-s^2+cs+ab)] - \frac{4rs^2(r+R)}{(r+4R)^2}.$$

$$GG_e^2 = \frac{2}{9}(a^2+b^2+c^2) - \frac{1}{3r(r+4R)}[-s^2(a^2+b^2+c^2)+s(a^3+b^3+c^3)+abc(a+b+c)] - \frac{4rs^2(r+R)}{(r+4R)^2}.$$

$$GG_e^2 = \frac{2}{9}(a^2+b^2+c^2) - \frac{1}{3r(r+4R)}[-s^2(2s^2-2r^2-8Rr)+s(2s^3-6r^2s-12Rrs)+8Rrs^2)] - \frac{4rs^2(r+R)}{(r+4R)^2}.$$

$$GG_e^2 = \frac{2}{9}(a^2+b^2+c^2) - \frac{4s^2(r-R)}{3(r+4R)} - \frac{4rs^2(r+R)}{(r+4R)^2}.$$

Therefore,

$$GG_e^2 = \frac{2}{9}(a^2+b^2+c^2) - \frac{8s^2(r^2+2R^2)}{3(r+4R)^2}.$$

Corollary 4.2 *Be I the Incenter of the triangle ABC and G_e the Gergonne point then*

$$IG_e^2 = r^2 - \frac{3r^2s^2}{(r+4R)^2}.$$

Proof: In Theorem 3.6, replace M by the incenter I. We get

$$IG_e^2 = \frac{1}{bc-(s-a)^2}\cdot[(s-b)(s-c)\cdot IA^2 + (s-a)(s-c)\cdot IB^2 + (s-a)(s-b)\cdot IC^2] - \frac{4rs^2(r+R)}{(r+4R)^2}.$$

Of the the triangle whose vertices are A, I and the point of contact of the incircle with side AB, we obtain

$$AI^2 = \frac{(s-a)^2}{cos^2(\widehat{A}/2)}.$$

Now, we Know that

$\cos(\widehat{A}/2) = \sqrt{\frac{s(s-a)}{bc}}$, $\cos(\widehat{B}/2) = \sqrt{\frac{s(s-b)}{ac}}$ and $\cos(\widehat{C}/2) = \sqrt{\frac{s(s-c)}{ab}}$, then

$$AI^2 = \frac{bc(s-a)}{s}.$$

Similarly we can prove $BI^2 = \frac{ac(s-b)}{s}$ and $CI^2 = \frac{ab(s-c)}{s}$.

$$IG_e^2 = \frac{1}{bc-(s-a)^2}\cdot[(s-b)(s-c)\frac{bc(s-a)}{s} + (s-a)(s-c)\frac{ac(s-b)}{s} + (s-a)(s-b)\frac{ab(s-c)}{s}] - \frac{4rs^2(r+R)}{(r+4R)^2}.$$

$$IG_e^2 = \frac{1}{bc-(s-a)^2}\cdot\frac{\Delta^2}{s^2}\cdot[bc+ac+ab] - \frac{4rs^2(r+R)}{(r+4R)^2}.$$

$$IG_e^2 = \frac{1}{r(r+4R)}\cdot\frac{s^2r^2}{s^2}\cdot[s^2+r^2+4Rr] - \frac{4rs^2(r+R)}{(r+4R)^2}.$$

$$IG_e^2 = r^2 + \frac{rs^2}{r+4R} - \frac{4rs^2(r+R)}{(r+4R)^2}.$$

Hence

$$IG_e^2 = r^2 - \frac{3r^2s^2}{(r+4R)^2}.$$

Corollary 4.3 *For any acute triangle ABC*

$$r+4R \geq s\sqrt{3}$$

whit equality when the triangle is equilateral.

Proof: This follows from Corollary 4.2. We know that $IG_e^2 \geq 0$, then

$$IG_e^2 = \frac{r^2(r+4R)^2 - 3r^2s^2}{(r+4R)^2} = \frac{r^2}{(r+4R)^2} \cdot [(r+4R) - s\sqrt{3}] \cdot [(r+4R) + s\sqrt{3}]$$

$$(r+4R) - s\sqrt{3} \geq 0$$

$$r + 4R \geq s\sqrt{3}.$$

Hence proved.

Corollary 4.4 *Be O the Circumcenter of the triangle ABC and G_e the Gergonne point then*

$$OG_e^2 = R^2 - \frac{4rs^2(r+R)}{(r+4R)^2}.$$

Proof: In Theorem 3.6, replace M by the Circumcenter O, and consider that $OA = OB = OC = R$. We get

$$OG_e^2 = \frac{R^2}{bc - (s-a)^2} \cdot [(s-b)(s-c) + (s-a)(s-c) + (s-a)(s-b)] - \frac{4rs^2(r+R)}{(r+4R)^2}.$$

$$OG_e^2 = \frac{R^2}{bc - (s-a)^2} \cdot [2ab + 2ac + 2bc - a^2 - b^2 - c^2] - \frac{4rs^2(r+R)}{(r+4R)^2}.$$

$$OG_e^2 = \frac{R^2}{bc - (s-a)^2} \cdot [bc - (s-a)^2] - \frac{4rs^2(r+R)}{(r+4R)^2}.$$

$$OG_e^2 = R^2 - \frac{4rs^2(r+R)}{(r+4R)^2}.$$

Corollary 4.5 *Be H the Orthocenter of the triangle ABC and G_e the Gergonne point then*

$$HG_e^2 = 4R^2 + \frac{8s^2(Rr - 2R^2)}{(r+4R)^2}.$$

Proof: In Theorem 3.6, replace M by the Circumcenter H, so

$$HG_e^2 = \frac{1}{bc - (s-a)^2} \cdot [(s-b)(s-c) \cdot HA^2 + (s-a)(s-c) \cdot HB^2 + (s-a)(s-b) \cdot HC^2] - \frac{4rs^2(r+R)}{(r+4R)^2}.$$

using the half angle formulas for the cosine funcion of an internal angle, we will then have

$$2\cos^2(\widehat{A}/2) = 1 + \cos\widehat{A}$$

$$\cos\widehat{A} = 2\cos^2(\widehat{A}/2) - 1$$

$$\cos\widehat{A} = \frac{2s(s-a)}{bc} - 1.$$

Similarly $\cos\widehat{B} = \dfrac{2s(s-b)}{ac} - 1$ and $\cos\widehat{C} = \dfrac{2s(s-c)}{ab} - 1.$ then

$$HG_e^2 = \frac{4R^2}{bc - (s-a)^2} \cdot [(s-b)(s-c)\cos^2(\widehat{A}) + (s-a)(s-c)\cos^2(\widehat{B}) + (s-a)(s-b)\cos^2(\widehat{C})] - \frac{4rs^2(r+R)}{(r+4R)^2}.$$

$$HG_e^2 = \frac{4R^2}{bc-(s-a)^2}\cdot\left[(s-b)(s-c)\left(\frac{2s(s-a)}{bc}-1\right)^2+(s-a)(s-c)\left(\frac{2s(s-b)}{ac}-1\right)^2+(s-a)(s-b)\left(\frac{2s(s-c)}{ab}-1\right)^2\right] -\frac{4rs^2(r+R)}{(r+4R)^2}.$$

After simplifying a few steps we obtain

$$HG_e^2 = 4R^2+\frac{16s\Delta^2R^2}{r(r+4R)}\cdot\left(\frac{(s-a)}{b^2c^2}+\frac{(s-b)}{a^2c^2}+\frac{(s-c)}{a^2b^2}\right)-\frac{16\Delta^2R^2}{r(r+R)}\left(\frac{1}{ab}+\frac{1}{ac}+\frac{1}{bc}\right)-\frac{4rs^2(r+R)}{(r+4R)^2}.$$

$$HG_e^2 = 4R^2+\frac{16s\Delta^2R^2}{r(r+4R)}\cdot\left(\frac{a^2(s-a)+b^2(s-b)+c^2(s-c)}{a^2b^2c^2}\right)-\frac{16\Delta^2R^2}{r(r+R)}\left(\frac{a+b+c}{abc}\right)-\frac{4rs^2(r+R)}{(r+4R)^2}.$$

$$HG_e^2 = 4R^2+\frac{4s}{(r+4R)}\cdot\left[s(a^2+b^2+c^2)-(a^3+b^3+c^3)\right]-\frac{8Rs^2}{(r+4R)}-\frac{4rs^2(r+R)}{(r+4R)^2}.$$

$$HG_e^2 = 4R^2+\frac{4s}{(r+4R)}\cdot\left[(2s^3-2r^2s-8Rrs)-(2s^3-6r^2s-12Rrs)\right]-\frac{8Rs^2}{(r+4R)}-\frac{4rs^2(r+R)}{(r+4R)^2}.$$

$$HG_e^2 = 4R^2+\frac{4s^2(r+R)}{(r+4R)}-\frac{8Rs^2}{(r+4R)}-\frac{4rs^2(r+R)}{(r+4R)^2}.$$

Further simplification gives

$$HG_e^2 = 4R^2+\frac{8s^2(Rr-2R^2)}{(r+4R)^2}.$$

Hence proved.

Corollary 4.6 *If N_a is Nagel point of the triangle ABC and G_e the Gergonne point then*

$$N_aG_e^2 = -16Rr+\frac{16s^2(Rr+R^2)}{(r+4R)^2}.$$

Proof: We know that

$$N_aA = \frac{a}{s}\sqrt{s^2-\frac{4\Delta^2}{a(s-a)}} \Longrightarrow N_aA^2 = a^2-\frac{4a}{s}(s-b)(s-c).$$

Similarly $N_aB^2 = b^2-\frac{4b}{s}(s-a)(s-c)$ and $N_aC^2 = c^2-\frac{4c}{s}(s-a)(s-b)$.

Using the Theorem 3.6, replace M by Nagel point N_a, then

$$N_aG_e^2 = \frac{1}{bc-(s-a)^2}\cdot[(s-b)(s-c)\cdot N_aA^2+(s-a)(s-c)\cdot N_aB^2+(s-a)(s-b)\cdot N_aC^2]-\frac{4rs^2(r+R)}{(r+4R)^2}.$$

now

$$N_aG_e^2 = \frac{1}{bc-(s-a)^2}\cdot\left[(s-b)(s-c)\cdot\left(a^2-\frac{4a}{s}(s-b)(s-c)\right)+(s-a)(s-c)\cdot\left(b^2-\frac{4b}{s}(s-a)(s-c)\right) +(s-a)(s-b)\cdot\left(c^2-\frac{4c}{s}(s-a)(s-b)\right)\right]-\frac{4rs^2(r+R)}{(r+4R)^2}.$$

$$N_aG_e^2 = \frac{1}{bc-(s-a)^2}\cdot\Big[a^2(s-b)(s-c)+b^2(s-a)(s-c)+c^2(s-a)(s-c)\Big]$$
$$-\frac{4}{s}[a(s-b)^2(s-c)^2+b(s-a)^2(s-c)^2+c(s-a)^2(s-c)^2\Big] - \frac{4rs^2(r+R)}{(r+4R)^2}.$$

$$N_aG_e^2 = \frac{1}{bc-(s-a)^2}\cdot\Big[a^2(-s^2+as+bc)+b^2(-s^2+bs+ac)+c^2(-s^2+cs+ab)\Big]$$
$$-\frac{4}{s(bc-(s-a)^2)}\cdot\Big[a(-s^2+as+bc)^2+b(-s^2+bs+ac)^2+c(-s^2+cs+ab)^2\Big] - \frac{4rs^2(r+R)}{(r+4R)^2}.$$

$$N_aG_e^2 = \frac{1}{bc-(s-a)^2}\cdot\Big[-s^2(a^2+b^2+c^2)+s(a^3+b^3+c^3)+abc(a+b+c)\Big]$$
$$-\frac{4}{s(bc-(s-a)^2)}\cdot\Big[a(s^4+a^2s^2+b^2c^2-2as^3-2bcs^2+2abcs)+b(s^4+b^2s^2+a^2c^2-2bs^3-2acs^2+2abcs)$$
$$+c(s^4+c^2s^2+a^2b^2-2cs^3-2abs^2+2abcs)\Big] - \frac{4rs^2(r+R)}{(r+4R)^2}.$$

$$N_aG_e^2 = \frac{1}{bc-(s-a)^2}\cdot\Big[-s^2(2s^2-2r^2-8Rr)+s(2s^3-6r^2s-12Rrs)+4Rrs(2s))\Big]$$
$$-\frac{4}{s(bc-(s-a)^2)}\cdot\Big[s^4(a+b+c)+s^2(a^3+b^3+c^3)+abc(ab+ac+bc)$$
$$-2s^3(a^2+b^2+c^2)-6abcs^2+2abcs(a+b+c)\Big] - \frac{4rs^2(r+R)}{(r+4R)^2}.$$

$$N_aG_e^2 = -\frac{1}{r(r+4R)}\cdot(-4r^2s^2+4Rrs^2)-\frac{4}{sr(r+4R)}\cdot\Big[2s^5+s^2(2s^3-6r^2s-12Rrs)+4Rrs(s^2+r^2+4Rr)$$
$$-2s^3(2s^3-6r^2s-12Rrs)-24Rrs^3+16Rr^3\Big] - \frac{4rs^2(r+R)}{(r+4R)^2}.$$

$$N_aG_e^2 = -\frac{1}{(r+4R)}\cdot(-4rs^2+4Rs^2)-\frac{4}{sr(r+4R)}\cdot\Big[2s^5+s^2(2s^3-6r^2s-12Rrs)+4Rrs(s^2+r^2+4Rr)$$
$$-2s^3(2s^3-6r^2s-12Rrs)-24Rrs^3+16Rr^3\Big] - \frac{4rs^2(r+R)}{(r+4R)^2}.$$

$$N_aG_e^2 = \frac{4s^2(R-r)}{(r+4R)} - \frac{4(16R^2r+4Rr^2-2rs^2)}{(r+4R)} - \frac{4rs^2(r+R)}{(r+4R)^2}.$$

$$N_aG_e^2 = -16Rr + \frac{8rs^2}{(r+4R)} + \frac{4s^2(R-r)}{(r+4R)} - \frac{4rs^2(r+R)}{(r+4R)^2}.$$

$$N_aG_e^2 = -16Rr + \frac{4s^2}{(r+4R)^2}.$$

Hence

$$N_aG_e^2 = -16Rr + \frac{16Rs^2(r+R)}{(r+4R)^2}.$$

5 Conclusion

In this article we find a metric relationship for the Gergonne point. With this relationship we can find the distance between the Gergonne point and other notable centers of the triangle. The proofs presented here only require basic knowledge of Geometry and its manipulation and application.

Reference

[1] **Romanian Mathematical Magazine - Interactive Journal**, www.ssmrmh.ro.

[2] COXETER, H. S. M. Greitzer, S. L. **Geometry Revisited**. Volume 19, Fifth Printing, Washington, D.C. The Mathematical Association of America, 1967.

[3] DASARI, Naga Vijay Krishna **Distance Between the Circumcenter and Any Point in the Plane of the Triangle**, GeoGebra International Journal of Romania (GGIJRO),volume-5, No. 2, 2016 art 92, pp 139-148.

[4] DASARI, Naga Vijay Krishna, **The fundamental property of Nagel point: A new proof**, Journal of Mathematical Sciences and Mathematics Education 12(2) (2017), 21-28.

[5] HONSBERGER, R. **Episodes in Nineteenth and Twentieth Century Euclidean Geometry**. The Mathematical Association of America, 1995.

[6] KIMBERLING,Clark. **Encyclopedia of Triangle Centers - ETC**.Available in: https:faculty.evansville.edu/ck6/encyclopedia/ETC.html.

[7] QUEIROZ FILHO, José Ferreira de. **A Reta de Nagel: Uma Abordagem Geométrica e Algébrica de um Alinhamento Notável**. Recife, 2017.125f. Dissertação de mestrado - UFRPE - Recife.