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If $a,b,x,y>0$ then:

$$32ab(ax+by)^4\le(a+b)^4(8a^2x^4+ab(x+y)^4+8b^2y^4)$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco, Solution 2 by Kunihiko Chikaya-Tokyo-Japan

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$(a+b)^4(8a^2x^4+ab(x+y)^4+8b^2y^4)\overset{AGM}{\ge}$$

$$\ge(a+b)^4\left(8a^2x^4+ab\left(2\sqrt{xy}\right)^4+8b^2y^4\right)=$$

$$=8(a+b)^2(a^2+2ab+b^2)(a^2x^4+2abx^2y^2+b^2y^4)\overset{CBS}{\ge}$$

$$\ge8(a+b)^2(a^2x^2+2abxy+b^2y^2)^2\overset{AGM}{\ge}$$

$$\ge8\cdot4ab(ax+by)^4=32ab(ax+by)^4$$

Therefore,

$$32ab(ax+by)^4\le(a+b)^4(8a^2x^4+ab(x+y)^4+8b^2y^4)$$

Equality holds for $a=b$.

Solution 2 by Kunihiko Chikaya-Tokyo-Japan

$$\text{Let: } f(t)=\frac{8a^2+ab(1+t)^4+8b^2t^4}{(a+bt)^4};\left(t=\frac{x}{y}>0\right)$$

$$f'(t)=\frac{(4ab(1+t)^3+32b^2t^3)(a+bt)^4-(8a^2+ab(1+t)^4+8b^2t^4)\cdot 4(a+bt)^3}{(a+bt)^8}$$

$$=4b\cdot\frac{(a+bt)(a(1+t)^3+8bt^3)-(8a^2+ab(1+t)^4+8b^2t^4)}{(a+bt)^5}=$$

$$=\frac{4ab(t-1)\left((a+7b)t^3+3(a-b)t^2+3(a-b)t-(7a+b)\right)}{(a+bt)^5}=$$

$$=\frac{4ab\left((a+7b)t^2+4(a+b)t+7a+b\right)}{(a+bt)^5}\cdot(t-1)$$

Since $a,b>0$, for $t>0$, the sign of $f'(t)$ coincides with $t-1$, thus $f(t)$ has a local minimum at $t=1$, which is also the minimum value. Therefore,

$(a+b)^2\geq 4ab$, **here by obtain**

$$f(t)\geq f(1)=\frac{8(a+b)^2}{(a+b)^4}\geq\frac{32ab}{(a+b)^4}.$$

Equality holds if and only if $t=1\Leftrightarrow a=b$.