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If $x, y, z > 0$ such that

$$\sum_{cyc} \frac{x^3}{1+x^3} = 2 \text{ then } \sum_{cyc} x\sqrt{x} \le \frac{3}{2} xyz\sqrt{xyz}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco, Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have: \sum_{cyc} \frac{x^3}{1+x^3} = 2 \leftrightarrow 1 = \sum_{cyc}\left(1 - \frac{x^3}{1+x^3}\right) = \sum_{cyc} \frac{1}{1+x^3}$$

$$Let\ a = \frac{1}{1+x^3}, b = \frac{1}{1+y^3}, c = \frac{1}{1+z^3} \to a+b+c = 1, x^3 = \frac{1-a}{a} = \frac{b+c}{a}, y^3 = \frac{c+a}{b}, z^3 = \frac{a+b}{c}$$

$$We\ need\ to\ prove: \sum_{cyc} x\sqrt{x} \le \frac{3}{2} xyz\sqrt{xyz}\ \ or\ \sum_{cyc} \sqrt{\frac{1}{y^3 z^3}} \le \frac{3}{2}$$

$$\sum_{cyc} \sqrt{\frac{1}{y^3 z^3}} = \sum_{cyc} \sqrt{\frac{bc}{(c+a)(a+b)}} \overset{AM-GM}{\le} \sum_{cyc} \frac{1}{2}\left(\frac{b}{a+b} + \frac{c}{c+a}\right) =$$

$$\frac{1}{2} \sum_{cyc} \left(\frac{b}{a+b} + \frac{a}{a+b}\right) = \frac{3}{2}.$$

$$\textit{Therefore},\quad \sum_{cyc} x\sqrt{x} \le \frac{3}{2}xyz\sqrt{xyz}.$$

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\frac{x^3}{x^3+1}+\frac{y^3}{y^3+1}+\frac{z^3}{z^3+1}=2 \Rightarrow \frac{x^3}{x^3+1}-1+\frac{y^3}{y^3+1}-1+\frac{z^3}{z^3+1}-1=-1$$

$$\frac{1}{x^3+1}+\frac{1}{y^3+1}+\frac{1}{z^3+1}=1 \Rightarrow \frac{1}{2x\sqrt{x}}+\frac{1}{2y\sqrt{y}}+\frac{1}{2z\sqrt{z}} \le 1$$

$$\frac{1}{x\sqrt{x}}+\frac{1}{y\sqrt{y}}+\frac{1}{z\sqrt{z}} \le 2 \Rightarrow \left(\frac{1}{x\sqrt{x}}+\frac{1}{y\sqrt{y}}+\frac{1}{z\sqrt{z}}\right)^2 \le 4$$

$$3\left(\frac{1}{xy\sqrt{xy}}+\frac{1}{yz\sqrt{yz}}+\frac{1}{zx\sqrt{zx}}\right) \le 4 \Leftrightarrow \frac{1}{xy\sqrt{xy}}+\frac{1}{yz\sqrt{yz}}+\frac{1}{zx\sqrt{zx}} \le \frac{4}{3} \le \frac{3}{2}$$

$$xyz\sqrt{xyz}\left(\frac{1}{xy\sqrt{xy}}+\frac{1}{yz\sqrt{yz}}+\frac{1}{zx\sqrt{zx}}\right) \le \frac{3}{2}xyz\sqrt{xyz}$$

Therefore,

$$\sum_{cyc} x\sqrt{x} \le \frac{3}{2}xyz\sqrt{xyz}$$