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BENCZE'S CRITERION

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Abstract: *In this paper are presented few applications of an outstanding result.*

THEOREM (***Mihàly Bencze***)

Let be $f:[0,1]\to(0,\infty)$ continuous function and $\alpha:\mathbb{R}\to[0,1]$ such that $\lim_{x\to\infty}\alpha(x)=0$. If exists the unique sequence $(x_n)_{n\geq 1}$ such that

$$\int_0^{x_n} f(x)\,dx=\alpha(n)\int_0^1 f(x)\,dx.$$

then:

$$\lim_{n\to\infty}\frac{x_n}{\alpha(n)}=\frac{1}{f(0)}\cdot\int_0^1 f(x)\,dx$$

Proof 1 by proposer.

Let be $g_n(x)=\int_0^{x_n} f(x)\,dx-\alpha(n)\int_0^1 f(x)\,dx$

The function g_n is differentiable, continuous and:

$$g_n(0)\cdot g_n(1)=-\alpha(n)\big(1-\alpha(n)\big)\left(\int_0^1 f(x)\,dx\right)^2<0,$$

so, from Rolle Theorem, exists $x_n\in(0,1)$ such that $g_n(x_n)=0$. But $g_n'(x)=f(x)>0$. Hence, g_n is decreasing, then g_n is injective and result the equation $g_n(x)=0$ have unique solution $x_n\in(0,1)$.

Let be: $m=\min_{x\in[0,1]} f(x)\,, M=\max_{x\in[0,1]} f(x)$ such that

$$m\cdot x_n\leq\int_0^{x_n} f(x)\,dx=\alpha(n)\int_0^1 f(x)\,dx\leq M\cdot\alpha(n)\text{ and }0\leq x_n\leq\frac{M}{m}\cdot\alpha(n).$$

Therefore, $0\leq\lim_{n\to\infty}x_n\leq\frac{M}{m}\lim_{n\to\infty}\alpha(n)=0$ and then, $\lim_{n\to\infty}x_n=0$. But

$$\alpha(n)\int_0^1 f(x)\,dx=\int_0^{x_n} f(x)\,dx=\frac{F(x_n)-F(0)}{x_n}\cdot x_n$$

$$\int_0^1 f(x)\,dx=\frac{F(x_n)-F(0)}{x_n}\cdot\frac{x_n}{\alpha(n)},\ \lim_{n\to\infty}\frac{F(x_n)-F(0)}{x_n}=f(0)$$

$$\lim_{n\to\infty}\frac{x_n}{\alpha(n)}=\frac{1}{f(0)}\int_0^1 f(x)\,dx$$

Proof 2 by Marius Olteanu

Because f −continuous function, then f admits primitives $F:[0,1]\to\mathbb{R}$ such that

$$\int_0^x f(t)\,dt=F(x),\forall x\in[0,1]$$

How F is continuous on $[0,1]$, then have Darboux property on $[0,1]$ and have values on the interval $\left[0, \int_0^1 f(x)\,dx\right]$. Because $\alpha(n) \in [0,1]$, then

$$\alpha(n) \cdot \int_0^1 f(x)\,dx \in \left[0, \int_0^1 f(x)\,dx\right] = J.$$

More, $f(x) > 0$ and $F'(x) = f(x) > 0$ imply that F is strictly increases. Therefore, F bijective function. So, $\forall y_0 \in J, \exists!\, x_0 \in [0,1]$ such that $F(x_0) = y_0$.

$$\text{If } y_0 = \alpha(n) \cdot \int_0^1 f(x)\,dx \text{ then } \exists x_0 = x_n \in [0,1] \text{ such that:}$$

$$F(x_n) = \alpha(n) \cdot \int_0^1 f(x)\,dx = \int_0^{x_n} f(x)\,dx$$

$$\lim_{n\to\infty} \int_0^{x_n} f(x)\,dx = \lim_{n\to\infty} \left[\alpha(n) \cdot \int_0^1 f(x)\,dx\right] = \left(\int_0^1 f(x)\,dx\right) \cdot \lim_{n\to\infty} \alpha(n) =$$

$$= \lim_{n\to\infty} [F(x_n) - F(0)] = 0 \Leftrightarrow \lim_{n\to\infty} F(x_n) = F(0) \Leftrightarrow \lim_{n\to\infty} (f(\xi_n) \cdot x_n) = 0; (0 < \xi_n < x_n)$$

because $F(x_n) - F(0) = (x_n - 0) \cdot F'(\xi_n) = x_n \cdot f(\xi_n)$

Because f is continuous function on $[0,1]$ then $f(x) \in [m, M]$ and then $f(\xi_n) \in [m, M], \forall n \in \mathbb{N}^*$. It follows that $\left(f(\xi_n)\right)_{n\geq 1}$ is bounded and

$$0 < x_n \cdot m \leq x_n \cdot f(\xi_n) \leq x_n \cdot M \Leftrightarrow$$

$$0 \leq \lim_{n\to\infty} (m \cdot x_n) \leq \lim_{n\to\infty} \left(x_n \cdot f(\xi_n)\right) = 0; (1)$$

$$\text{Now, } \int_0^{x_n} f(x)\,dx = x_n \cdot f(\xi_n); (0 < \xi_n < x_n) \Rightarrow \lim_{n\to\infty} \xi_n = 0$$

$$x_n \cdot f(\xi_n) = \alpha(n) \cdot \int_0^1 f(x)\,dx \Rightarrow \frac{x_n}{\alpha(n)} = \frac{1}{f(\xi_n)} \int_0^1 f(x)\,dx$$

$$\lim_{n\to\infty} \frac{x_n}{\alpha(n)} = \lim_{n\to\infty} \left(\frac{1}{f(\xi_n)} \int_0^1 f(x)\,dx\right) = \int_0^1 f(x)\,dx \cdot \lim_{n\to\infty} \frac{1}{f(\xi_n)} = \frac{1}{f(0)} \int_0^1 f(x)\,dx$$

Application 1.

If exists an unique $(x_n)_{n\geq 1}$ sequence of real numbers and $\alpha: \mathbb{R} \to [0,1], \lim_{n\to\infty} \alpha(n) = 0$ such that:

$$\int_0^{x_n} \frac{\tan^{-1} x}{x\sqrt{1-x^2}}\,dx = \alpha(n) \int_0^1 \frac{\tan^{-1} x}{x\sqrt{1-x^2}}\,dx$$

$$\textbf{then find: } \Omega = \lim_{n\to\infty} \frac{x_n}{\alpha(n)}$$

Solution.

$$\text{Let: } F(y) = \int_0^1 \frac{\tan^{-1}(xy)}{x\sqrt{1-x^2}}\,dx \text{ then } F'(y) = \int_0^1 \frac{dx}{(1+x^2y^2)\sqrt{1-x^2}} = \int_0^{\frac{\pi}{4}} \frac{dt}{1+y^2\cos^2 t} =$$

$$= \frac{1}{\sqrt{1+y^2}} \tan^{-1}\left(\frac{\tan t}{\sqrt{1+y^2}}\right) = \frac{\pi}{2\sqrt{1+y^2}}$$

$$\text{So}, F(y)=\frac{\pi}{2}\log\left(y+\sqrt{1+y^2}\right)+C.\text{Put } y=0\Rightarrow C=0\Rightarrow$$

$$\int_0^1\frac{\tan^{-1}x}{x\sqrt{1-x^2}}dx=\frac{\pi}{2}\log(1+\sqrt{2})$$

Using Bencze's Criterion for $f(x)=\frac{\tan^{-1}x}{x\sqrt{1-x^2}}$, we get:

$$\Omega=\lim_{n\to\infty}\frac{x_n}{\alpha(n)}=\frac{1}{f(0)}\int_0^1\frac{\tan^{-1}x}{x\sqrt{1-x^2}}dx=\frac{\pi}{2}\log(1+\sqrt{2})$$

Application 2.

If exists an unique $(x_n)_{n\geq 1}$ sequence of real numbers and

$\alpha:\mathbb{R}\to[0,1],\lim_{n\to\infty}\alpha(n)=0$ such that:

$$\int_0^{x_n}\sqrt{1+\sqrt{x}}\,dx=\alpha(n)\int_0^1\sqrt{1+\sqrt{x}}\,dx$$

$$\textbf{Find: }\Omega=\lim_{n\to\infty}\frac{x_n}{\alpha(n)}$$

Solution. Let $f(x)=\sqrt{1+\sqrt{x}}$ continuous function. On the interval $[\varepsilon,1]\subset[0,1],\varepsilon>0$, let

$$g:[\varepsilon,1]\to\mathbb{R},g(x)=\sqrt{1+\sqrt{x}}\left(1+\sqrt{x}\right)'\left(2\sqrt{x}+2-2\right)=$$

$$=2\varphi^{\frac{3}{2}}(x)\varphi'(x)-2\varphi^{\frac{1}{2}}(x)\varphi'(x),\text{where }\varphi(x)=1+\sqrt{x},\varphi'(x)=\frac{1}{2\sqrt{x}}.$$

On $[\varepsilon,1]$ function g is continuous and $G_\varepsilon(x)=\frac{4}{5}\left(1+\sqrt{x}\right)^{\frac{5}{2}}-\frac{4}{3}\left(1+\sqrt{x}\right)^{\frac{3}{2}}$

$$I_\varepsilon=G_\varepsilon(1)-G_\varepsilon(\varepsilon),\lim_{\substack{\varepsilon\to0\\\varepsilon>0}}I_\varepsilon=\frac{8\left(\sqrt{2}+1\right)}{15}\text{ then }I=\int_0^1 f(x)\,dx=\lim_{\substack{\varepsilon\to0\\\varepsilon>0}}I_\varepsilon=\frac{8\left(\sqrt{2}+1\right)}{15}$$

Using Bencze's Criterion for $f(x)=\sqrt{1+\sqrt{x}}$, we get:

$$\Omega=\lim_{n\to\infty}\frac{x_n}{\alpha(n)}=\frac{1}{f(0)}\int_0^1 f(x)\,dx=\frac{8\left(\sqrt{2}+1\right)}{15}$$

Observation. For $f(x)=\sqrt{1+\sqrt{x}}\,,f:[0,1]\to[1,\sqrt{2}]$ continuous function we take

$$t=1+\sqrt{x}\Rightarrow x=\varphi(t)=(t-1)^2,\text{where }\varphi:[1,2]\to[0,1],\text{with }\varphi^{-1}(0)=1,\varphi^{-1}(1)=2$$

$$\varphi'(t)=2(t-1)\Rightarrow\int_0^1 f(x)\,dx=\int_1^2\sqrt{2}(2t-2)\,dt=\frac{8\left(\sqrt{2}+1\right)}{15}.$$

Application 3.

If exists an unique $(x_n)_{n\geq 1}$ sequence of real numbers and $\alpha: \mathbb{R} \to [0,1]$, $\lim\limits_{n\to\infty} \alpha(n) = 0$ such that:

$$\int_0^{x_n} \frac{\log(1-x^2)}{x} dx = \alpha(n) \int_0^1 \frac{\log(1-x^2)}{x} dx.$$

then find: $\Omega = \lim\limits_{n\to\infty} \frac{x_n}{\alpha(n)}$

Solution.

$$\text{Let } I = \int_0^1 \frac{\log(1-x)}{x} dx + \int_0^1 \frac{\log(1+x)}{x} dx = I_1 + I_2$$

$$\text{We know: } \log(1+x) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^n}{n+1}; x \in (-1,1]$$

$$\log(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}; x \in (-1,1)$$

$$\log\frac{1+x}{1-x} = 2 \cdot \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}; x \in (-1,1)$$

$$\text{We have: } I_1 = \int_0^1 \left(\sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^n}{n+1}\right) dx = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{(n+1)^2} = \frac{\pi^2}{12}$$

$$I_2 = \int_0^1 \frac{\log(1-x)}{x} dx = \lim_{x\to 1} \int_0^x \frac{\log(1-t)}{t} dt = -\sum_{n=0}^{\infty} \frac{1}{(n+1)^2} = -\frac{\pi^2}{6}$$

$$\text{Hence, } \int_0^1 \frac{\log(1-x)}{x} dx = I_1 + I_2 = -\frac{\pi^2}{12}$$

Using Bencze's Criterion for $f(x) = \frac{\log(1-x)}{x}$, we get:

$$\Omega = \lim_{n\to\infty} \frac{x_n}{\alpha(n)} = \frac{1}{f(0)} \int_0^1 \frac{\log(1-x)}{x} dx = \frac{\pi^2}{12}, \text{where } f(0) = \lim_{t\to 0} \frac{\log(1-t)}{t} = -1$$

Application 4.

If exists an unique $(x_n)_{n\geq 1}$ sequence of real numbers, $\alpha: \mathbb{R} \to [0,1]$,

$\lim\limits_{n\to\infty} \alpha(n) = 0$ such that:

$$\int_0^{x_n} \frac{dx}{1 + \sqrt[n]{x} + \sqrt[n]{x^2} + \cdots + \sqrt[n]{x^n}} = \alpha(n) \cdot \int_0^1 \frac{dx}{1 + \sqrt[n]{x} + \sqrt[n]{x^2} + \cdots + \sqrt[n]{x^n}}$$

then find: $\Omega = \lim\limits_{n\to\infty} \frac{n \cdot x_n}{\alpha(n)}$

Solution. Let $\sqrt[n]{x} = t \Rightarrow x = t^n, dx = nt^{n-1}dt$ then

$$I_n = \int_0^1 \frac{dx}{1 + \sqrt[n]{x} + \sqrt[n]{x^2} + \cdots + \sqrt[n]{x^n}} = n\int_0^1 \frac{t^{n-1}\,dt}{1 + t + t^2 + \cdots + t^n}$$

$$\because (1 - t^{n+1})\left(\sum_{i=0}^{p} t^{i(n+1)}\right) + t^{(p+1)(n+1)} = 1 \Rightarrow$$

$$\frac{1}{1 + t + \cdots + t^n} = (1-t)\left(\sum_{i=0}^{p} t^{i(n+1)}\right) + \frac{t^{(p+1)(n+1)}}{1 + t + \cdots + t^n} \Leftrightarrow$$

$$\frac{t^{n-1}}{1 + t + t^2 + \cdots + t^n} = \sum_{i=0}^{p}\left[t^{i(n+1)+n-1} - t^{i(n+1)+n}\right] + \frac{t^{(p+1)(n+1)+n-1}}{1 + +t + \cdots + t^n} \Rightarrow$$

$$\int_0^1 \frac{t^{n-1}\,dt}{1 + t + t^2 + \cdots + t^n} = \sum_{i=0}^{p}\left[\frac{1}{i(n+1)+n} - \frac{1}{i(n+1)+n+1}\right] + \int_0^1 \frac{t^{(p+1)(n+1)+n-1}}{1 + +t + \cdots + t^n}dt$$

$$I_n = a_{pn} + b_{pn}, \forall n \in \mathbb{N}, \forall p \in \mathbb{N}; (1), \text{ where}$$

$$a_{np} = \frac{n}{n+1}\sum_{i=0}^{p} \frac{1}{(i+1)[i(n+1)-n]}$$

$$b_{pn} = n\int_0^1 \frac{t^{(p+1)(n+1)+n-1}}{1 + +t + \cdots + t^n}dt \le n\int_0^1 t^{(p+1)(n+1)}\,dt = \frac{n}{(p+1)(n+1)+1}$$

How, $\lim_{n\to\infty} b_{pn} = 0, \forall n \in \mathbb{N}$ from (1) it follows that:

$$I_n = \lim_{p\to\infty} a_{pn} = \frac{n}{n+1}\sum_{i=0}^{\infty} \frac{1}{(i+1)[i(n+1)+n]}$$

Let: $g_i: [0,\infty) \to \mathbb{R}, g_i(x) = \frac{1}{(i+1)(i+1-x)}$, we have: $I_n = \frac{n}{(n+1)^2}\sum_{i=0}^{\infty} g_i\left(\frac{1}{n+1}\right); (2)$

But $0 \le g_i(x) \le \frac{1}{(i+1)^2}; \forall x \ge 0$ and $\sum_{i=0}^{\infty}\frac{1}{(i+1)^2} < \infty$ from $Weiestrass$, we get that

$$\sum_{i=0}^{\infty} g_i \text{ converges uniform on } [0,\infty)$$

Hence, $nI_n = \left(\frac{n}{n+1}\right)^2 f\left(\frac{1}{n+1}\right)$ and using Bencze's Criterion, we get:

$$\Omega = \lim_{n\to\infty}\frac{n \cdot x_n}{\alpha(n)} = \lim_{n\to\infty} n \cdot I_n = \lim_{n\to\infty} g\left(\frac{1}{n+1}\right) = g(0) = \sum_{i=0}^{\infty}\frac{1}{(i+1)^2} = \frac{\pi^2}{6}$$

Application 5.

For $x_n \in (0,1)$ let $\lim\limits_{n\to\infty} n\int_0^{x_n} x^n f(x)\,dx = 0$, where
$f: [0,1] \to \mathbb{R}$ integrable on $[0,1]$
and continuous in $x = 1$. Prove that:

$$\lim_{n\to\infty} n\int_0^1 x^n f(x)\,dx = f(1)$$

Solution. How f integrable function, then f bounded function. So, $\exists M > 0$ such that $|f(x)| \le M, \forall x \in [0,1]$. We have:

$$\left|n\int_0^{x_n} x^n f(x)\,dx\right| \le n\int_0^{x_n} x^n |f(x)|\,dx \le nM \cdot \frac{x_n^{n+1}}{n+1}$$

But $x_n \in (0,1) \Rightarrow \lim\limits_{n\to\infty} x_n^n = 0$, so $\lim\limits_{n\to\infty}\left(nM \cdot \frac{x_n^{n+1}}{n+1}\right) = 0$

Hence, $\lim\limits_{n\to\infty} n\int_0^{x_n} x^n f(x)\,dx = 0$

Function f continuous at point $x = 1$, then we have:

$$n\int_0^1 x^n f(x)\,dx - f(1) = n\int_0^1 x^{n-1}[f(x) - f(1)]\,dx =$$

$$= n\int_0^{x_n} x^{n-1}[xf(x) - f(1)]\,dx + n\int_{x_n}^1 x^{n-1}[xf(x) - f(1)]\,dx; (1)$$

Now, f −continous at point $x = 1$ then $\exists x_n \in (0,1)$ such that $|xf(x) - f(1)| < \varepsilon, \forall x \in [x_n, 1], \varepsilon > 0\ (fixed); (2)$

Applying up these strategy for function $x \to xf(x) - f(1), x \in [0,1], \exists N_\varepsilon \ge 1$ such that

$$\left|n\int_0^{x_n} x^{n-1}[xf(x) - f(1)]\,dx\right| < \varepsilon, \forall n \ge N_\varepsilon; (3)$$

Hence, we obtain:

$$\left|n\int_{x_n}^1 x^{n-1}[xf(x) - f(1)]\,dx\right| \le n\int_{x_n}^1 \varepsilon x^{n-1}\,dx = \varepsilon(1 - a^n) < \varepsilon; (4)$$

From (1),(3),(4) we get:

$$\left|n\int_0^1 x^n f(x)\,dx - f(1)\right| \le 2\varepsilon \Rightarrow \lim_{n\to\infty} n\int_0^1 x^n f(x)\,dx = f(1)$$

REFERENCES:

[1] ROMANIAN MATHEMATICAL MAGAZINE CHALLENGES 1-500-D. Sitaru, M. Ursărescu-Studis 2021

[2] QUANTUM MATHEMATICAL POWER- M. Bencze, D. Sitaru-Studis, 2018

[3] OLYMPIC MATHEMATICAL ENERGY- M. Bencze, D. Sitaru-Studis, 2018

[4] 699 OLYMPIC MATHEMATICAL CHALLENGES- M. Bencze, D. Sitaru-Studis, 2017

[5] ANALYTIC PHENOMENON- D. Sitaru -Studis,2018

[6] OLYMPIC MATHEMATICAL POWER-M. Bencze, D. Sitaru, M. Ursărescu-Studis, 2018

[7] ANALIZĂ MATEMATICA –D.M.Bătineţu-Giurgiu & Co. –MatrixRom,2003

[8] OLIMPIC MATHEMATICAL BEAUTIES- M. Bencze, D. Sitaru, M. Ursărescu-Studis, 2018

[9] ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro

[10] OCTOGON MATHEMATICAL MAGAZINE-www.uni-miskolk.hu/~mastefi/Octogon/