ABOUT AN INEQUALITY BY KOSTAS GERONIKOLAS-V

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1) In ΔABC the following relationship holds

$$\frac{a^2}{m_a^2}+\frac{b^2}{m_b^2}+\frac{c^2}{m_c^2}\le\frac{2R(R-r)}{r^2}$$

Proposed by Kostas Geronikolas-Greece

Solution. **Lemma 2.** In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{a^2}{m_a^2}\le\frac{4(R-r)}{r}$$

Proof. Using $m_a\ge\sqrt{s(s-a)}$ we get $\sum\frac{a^2}{m_a^2}\le\sum\frac{a^2}{s(s-a)}=\frac{4(R-r)}{r}$, which follows from $\sum\frac{a^2}{s-a}=\frac{4s(R-r)}{r}$. Let's get back to the main problem, using Lemma we get:

$LHS=\sum\frac{a^2}{m_a^2}\le\frac{4(R-r)}{r}\overset{(1)}{\le}\frac{2R(R-r)}{r^2}=RHS$, where $(1)\Leftrightarrow\frac{4(R-r)}{r}\le\frac{2R(R-r)}{r^2}\Leftrightarrow$ $R\ge 2r$ $(Euler)$. Equality holds if and only if triangle is equilateral.

Remark. Let's find an opposite inequality.

3) In ΔABC the following relationship holds:

$$\frac{a^2}{m_a^2}+\frac{b^2}{m_b^2}+\frac{c^2}{m_c^2}\ge 4\left(\frac{2r}{R}\right)^{\frac{2}{3}}$$

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Solution. Using AM-GM, we get: $\sum\frac{a^2}{m_a^2}\ge 3\sqrt[3]{\prod\frac{a^2}{m_a^2}}\overset{(2)}{\ge}3\sqrt[3]{\frac{16R^2r^2s^2}{\left(\frac{Rs^2}{2}\right)^2}}=3\sqrt[3]{64\frac{r^2}{s^2}}=$

$=3\cdot 4\sqrt[3]{\frac{r^2}{s^2}}\overset{Gerretsen}{\ge}4\left(\frac{2r}{R}\right)^{\frac{2}{3}}$, where $(2)\Leftrightarrow m_am_bm_c\le\frac{Rs^2}{2}$, which follows from Lemma:

Lemma. **In ΔABC the following relationship holds:**

$$m_am_bm_c\le\frac{Rs^2}{2}$$

Proof. Using the following identity:

$$\prod m_a^2=\frac{s^6+s^4(33r^2-12Rr)-s^2r^2(60R^2+120Rr+33r^2)-r^3(4R+r)^3}{16}$$

Inequality becomes: $\frac{s^6+s^4(33r^2-12Rr)-s^2r^2(60R^2+120Rr+33r^2)-r^3(4R+r)^3}{16}\le\left(\frac{Rs^2}{2}\right)^2\Leftrightarrow$

$$s^6+s^4(33r^2-12Rr-4R^2)-s^2r^2(60R^3+120Rr+33r^2)-r^3(4R+r)^3\le 0;(1)$$

Using Gerretsen inequality: $s^2\le 4R^2+4r+3r^2$, we get:

$s^6+s^4(33r^2-12Rr-4R^2)\le s^4(36r^2-8Rr)$; (2) which follows from

$s^6+s^4(33r^2-12Rr-4R^2)=s^4(s^2+33r^2-12Rr-4R^2)\le s^4(4R^2+4Rr+3r^2+$ $33r2-12Rr-4R2\le s436r2-8Rr$. From (1),(2) it is enough to prove that:

$$s^4(36r^2-8Rr)-s^2r^2(60R^2+120Rr+33r^2)-r^3(4R+r)^3\le 0\Leftrightarrow$$
$$s^4(36r-8R)-s^2r(60R^2+120Rr+33r^2)-r^2(4R+r)^3\le 0\Leftrightarrow$$
$s^4(8R-16r)+s^2r(60R^2+120Rr+33r^2)+r^2(4R+r)^3\ge 20rs^4$, which follows from $(16Rr-5r^2\le s^2\le 4R^2+4Rr+3r^2$ $(Gerretsen)$. Remains to prove that:
$$s^2(16Rr-5r^2)(8R-16r)+s^2r(60r^2+120Rr+33r^2)+r^2(4R+r)^2$$
$$\ge 20rs^2(4R^2+4Rr+3r^2)\Leftrightarrow$$
$$s^2(16R-5r)(8R-16r)+s^2(60R^2+120Rr+33r^2)+r(4R+r)^3$$
$$\ge 20s^2(4R^2+4Rr+3r^2)\Leftrightarrow$$
$$s^2(108R^2-256Rr+53r^2)+r(4R+r)^3\ge 0$$
We distinguish the following cases:
Case 1) If $108R^2-256Rr+53r^2\ge 0$, inequality is obviously true.
Case 2) If $108R^2-256Rr+53r^2<0$, inequality becomes:
$r(4R+r)^3\ge s^2(-108R^2+256Rr-53r^2)$, which follows from Blundon-Gerretsen inequality: $s^2\le\frac{R(4R+r)^2}{2(2R-r)}\le 4R^2+4Rr+3r^2$. Remains to prove that:
$$r(4R+r)^3\ge\frac{R(4R+r)^2}{2(2R-r)}<(-108R^2+256Rr-53r^2)\Leftrightarrow$$
$$2r(2R-r)(4R+r)\ge R(-108R^2+256Rr-53r^2)\Leftrightarrow$$
$108R^3-240R^2r+49Rr^2-2r^3\ge 0\Leftrightarrow(R-2r)(108R^2-24Rr+r^2)\ge 0$, which is true from $R\ge 2r$ $(Euler)$. Equality holds if and only if triangle is equilateral.
Remark. Inequality can be doubled.

4) In ΔABC the following relationship holds:

$$4\left(\frac{2r}{R}\right)^{\frac{2}{3}}\le\sum_{cyc}\frac{a^2}{m_a^2}\le 4\left(\frac{R}{r}-1\right)$$

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Solution. See up these inequalities. Equality if and only if triangle is equilateral. **Remark.** In same class of the problem.

5) In ΔABC the following relationship holds:

$$\frac{32r}{R}\left(\frac{2r}{R}\right)^{\frac{1}{3}}\le\sum_{cyc}\frac{(b+c)^2}{m_a^2}\le 4\left(\frac{R}{r}+2\right)$$

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Solution. For RHS, using $m_a\ge\sqrt{s(s-a)}$, we get: $\sum\frac{(b+c)^2}{m_a^2}\le\sum\frac{(b+c)^2}{s(s-a)}=\frac{4(R+2r)}{r}=$ $4\left(\frac{R}{r}+2\right)$, which follows from $\sum\frac{(b+c)^2}{s(s-a)}=\frac{4(R+2r)}{r}$. Equality holds if and only if triangle is equilateral.

For LHS, using AM-GM we have: $\sum\frac{(b+c)^2}{m_a^2}\ge 3\sqrt[3]{\prod\frac{(b+c)^2}{m_a^2}}\overset{Lemma}{\ge}3\sqrt[3]{\frac{4s^2(s^2+r^2+2Rr)^2}{\left(\frac{Rs^2}{2}\right)^2}}=$

$$= 3\sqrt[3]{\frac{16(s^2+r^2+2Rr)^2}{r^2s^2}} \overset{Gerretsen}{\geq} 3\sqrt[3]{\frac{16(16Rr-5r^2+r^2+2Rr)^2}{R^2\cdot\frac{27R^2}{4}}} =$$

$$= 3\sqrt[3]{\frac{64(18Rr-4r^2)^2}{27R^4}} + 3\cdot\frac{4}{3R}\sqrt[3]{\frac{(18Rr-4r^2)^2}{R}} \overset{Euler}{\geq} \frac{4r}{R}\sqrt[3]{\frac{2^{10}r}{R}} = \frac{32r}{R}\left(\frac{2r}{R}\right)^{\frac{1}{3}} = 4\left(\frac{2r}{R}\right)^{\frac{2}{3}},$$

where:

Lemma: In ΔABC the following relationship holds:

$$m_am_bm_c \leq \frac{Rs^2}{2}$$

Proof. Using the following identity:

$$\prod m_a^2 = \frac{s^6+s^4(33r^2-12Rr)-s^2r^2(60R^2+120Rr+33r^2)-r^3(4R+r)^3}{16}$$

Inequality becomes: $\frac{s^6+s^4(33r^2-12Rr)-s^2r^2(60R^2+120Rr+33r^2)-r^3(4R+r)^3}{16} \leq \left(\frac{Rs^2}{2}\right)^2 \Leftrightarrow$

$$s^6+s^4(33r^2-12Rr-4R^2)-s^2r^2(60R^3+120Rr+33r^2)-r^3(4R+r)^3 \leq 0; (1)$$

Using Gerretsen inequality: $s^2 \leq 4R^2+4r+3r^2$, we get:

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$s^6+s^4(33r^2-12Rr-4R^2) = s^4(s^2+33r^2-12Rr-4R^2) \leq s^4(4R^2+4Rr+3r^2+$ 33r2−12Rr−4R2≤s436r2−8Rr. From (1),(2) it is enough to prove that:

$$s^4(36r^2-8Rr)-s^2r^2(60R^2+120Rr+33r^2)-r^3(4R+r)^3 \leq 0 \Leftrightarrow$$

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$$\geq 20s^2(4R^2+4Rr+3r^2) \Leftrightarrow$$
$$s^2(108R^2-256Rr+53r^2)+r(4R+r)^3 \geq 0$$

We distinguish the following cases: Case 1) If $108R^2-256Rr+53r^2 \geq 0$, inequality is obviously true. Case 2) If $108R^2-256Rr+53r^2 < 0$, inequality becomes: $r(4R+r)^3 \geq s^2(-108R^2+256Rr-53r^2)$, which follows from Blundon-Gerretsen inequality: $s^2 \leq \frac{R(4R+r)^2}{2(2R-r)} \leq 4R^2+4Rr+3r^2$. Remains to prove that:

$$r(4R+r)^3 \geq \frac{R(4R+r)^2}{2(2R-r)} < (-108R^2+256Rr-53r^2) \Leftrightarrow$$
$$2r(2R-r)(4R+r) \geq R(-108R^2+256Rr-53r^2) \Leftrightarrow$$

$108R^3-240R^2r+49Rr^2-2r^3 \geq 0 \Leftrightarrow (R-2r)(108R^2-24Rr+r^2) \geq 0$, which is true from $R \geq 2r$ $(Euler)$. Equality holds if and only if triangle is equilateral.

Reference:

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