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ABOUT AN INEQUALITY BY ERTAN YILDIRIM-XV

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In ΔABC the following inequality holds:

$$4R \le \frac{1}{a+b+c}\sum_{cyc} bc\left(\tan\frac{A}{2}+\cot\frac{A}{2}\right) \le \frac{2R^2}{r}$$

Proposed by Ertan Yildirim-Izmir-Turkyie

Solution. Lemma. **In ΔABC the following relationship holds:**

$$\sum_{cyc} bc\left(\tan\frac{A}{2}+\cot\frac{A}{2}\right) = \frac{s^2(s^2+2r^2-8Rr)+r^2(4R+r)^2}{F}$$

Proof. We have: $\sum bc\left(\tan\frac{A}{2}+\cot\frac{A}{2}\right) = \sum bc\left(\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}+\frac{\cos\frac{A}{2}}{\sin\frac{A}{2}}\right) + \sum bc\left(\frac{\sin^2\frac{A}{2}+\cos^2\frac{A}{2}}{\sin\frac{A}{2}\cos\frac{A}{2}}\right) =$

$$= \sum bc\cdot\frac{1}{\sin\frac{A}{2}\cos\frac{A}{2}} = \sum 2bc\cdot\frac{1}{2\sin\frac{A}{2}\cos\frac{A}{2}} = \sum\frac{2bc}{\sin A} = \sum\frac{2bc}{\frac{a}{2R}} = 4R\frac{\sum b^2c^2}{abc} =$$

$$= 4R\frac{s^2(s^2+2r^2-8Rr)+r^2(4R+r)^2}{4Rrs} = \frac{s^2(s^2+2r^2-8Rr)+r^2(4Rr+r)^2}{rs}$$

Let's get back to the main problem. For RHS, using Lemma, we get:

$$\frac{1}{a+b+c}\sum bc\left(\tan\frac{A}{2}+\cot\frac{A}{2}\right) = \frac{1}{2s}\cdot\frac{s^2(s^2+2r^2-8Rr)+r^2(4R+r)^2}{F} \overset{(1)}{\le} \frac{2R^2}{r}$$

Where, (1) $\Leftrightarrow \frac{1}{2s}\cdot\frac{s^2(s^2+2r^2-8Rr)+r^2(4R+r)^2}{F} \le \frac{2R^2}{r} \Leftrightarrow \frac{s^2(s^2+2r^2-8Rr)+r^2(4R+r)^2}{2s^2r} \le \frac{2R^2}{r} \Leftrightarrow$

$$s^2(s^2+2r^2-8Rr)+r^2(4R+r)^2 \le 4R^2s^2 \Leftrightarrow$$

$r^2(4R+r)^2 \le s^2(4R^2+8Rr-2r^2-s^2)$, which follows from Gerretsen inequality

$\frac{r(4R+r)^2}{R+r} \le 16Rr-5r^2 \le s^2 \le 4R^2+4Rr+3r^2$. Remains to prove that:

$r(R+r) \le r(4R-5r) \Leftrightarrow R \ge 2r$ $(Euler)$. Equality holds if and only if triangle is equilateral.

For LHS, we have: $\frac{1}{a+b+c}\sum bc\left(\tan\frac{A}{2}+\cot\frac{A}{2}\right) = \frac{1}{2s}\cdot\frac{s^2(s^2+2r^2-8Rr)+r^2(4R+r)^2}{F} \overset{(2)}{\ge} 4R$, where

$$(2) \Leftrightarrow \frac{1}{2s} \cdot \frac{s^2(s^2+2r^2-8Rr)+r^2(4R+r)^2}{F} \ge 4R \Leftrightarrow$$

$$\frac{s^2(s^2+2r^2-8Rr)+r^2(4R+r)^2}{2s^2r} \ge 4R \Leftrightarrow$$

$$s^2(s^2+2r^2-8Rr)+r^2(4R+r)^2 \ge 8Rrs^2 \Leftrightarrow s^2(s^2+2r^2-16Rr)+r^2(4R+r)^2 \ge 0$$

Distinguish the following cases:

Case 1) If $s^2+2r^2-16Rr \ge 0$, inequality is obviously true.

Case 2) If $s^2+2r^2-16Rr < 0$, inequality becomes: $r^2(4R+r)^2 \ge s^2(16Rr-2r^2-s^2)$,

which follows from Blundon-Gerretsen: $16Rr-5r^2 \le s^2 \le 4R^2+4Rr+3r^2 \le \frac{R(4R+r)^2}{2(2R-r)}$.

Remains to prove that: $r^2(4R+r)^2 \ge \frac{R(4R+r)^2}{2(2R-r)}(16Rr-2r^2-16Rr+5r^2) \Leftrightarrow$

$$2r^2(2R-r) \ge 3r^2R \Leftrightarrow 2(2R-r) \ge 3R \Leftrightarrow R \ge 2r\ (Euler).$$

Equality holds if and only if triangle is equilateral.

Reference:

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