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Prove that:

$$\int_0^{\frac{\pi}{2}} \log(x^2 + \log^2(\cos x))\, dx = \pi \log(\log 2)$$

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It is well-known that:

$$\int_0^{\frac{\pi}{2}} \frac{\cos\left(s \tan^{-1}\left(\frac{x}{-\log\cos x}\right)\right)}{(x^2 + \log^2 \cos x)^{\frac{s}{2}}} dx = \frac{\pi}{2} \cdot \frac{1}{\log^s 2}$$

Differentiable both sides w.r.t. s, we have:

$$\partial_s \left(\frac{\cos\left(s \tan^{-1}\left(\frac{x}{-\log\cos x}\right)\right)}{(x^2 + \log^2 \cos x)^{\frac{s}{2}}} \right)\Bigg|_{s=0} = -\frac{1}{2}\log(x^2 + \log^2 \cos x)$$

$$\partial_s \left(\frac{\pi}{2} \cdot \frac{1}{\log^s 2}\right)\Bigg|_{s=0} = -\frac{\pi}{2} \cdot \log(\log 2) \Rightarrow$$

$$-\frac{1}{2}\int_0^{\frac{\pi}{2}} \log(x^2 + \log^2(\cos x))\, dx = -\frac{\pi}{2} \cdot \log(\log 2)$$

Therefore,

$$\int_0^{\frac{\pi}{2}} \log(x^2 + \log^2(\cos x))\, dx = \pi \log(\log 2)$$