A MATHEMATICAL CORRESPONDENCE

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A mathematical correspondence between Al. Lupaş and D.M. Bătineţu-Giurgiu

At 120 years from the publication of Lalescu sequence
On the 85th anniversary of D.M. Bătineţu - Giurgiu's birth

The object of this note is a unique correspondence between two well-known Romanian. The two mathematical personalities in the title are well known in the landscape of Romanian mathematics (- and not only) from the last 50-60 years mathematicians. Both are very prolific creators in mathematics, with numerous books, articles, notes, problems.
Mr. Alexandru Lupaş, the Romanian mathematician with two doctorates in mathematics (one in Germany and the other in Romania) is probably the most cited Romanian author in mathematical literature abroad (along with his mentor Tiberiu Popoviciu), when Mr. D. M. Bătineţu-Giurgiu is probably the greatest Romanian problemist in life (at least in the statistics of the Mathematical Gazette - online).
Although with different mathematical interests, there is still a common subject that interested them equally, namely the sequence of Traian Lalescu (the one who was the object of this correspondence). Each of them had important contributions in the study of this sequence. The scientific portraits of the two mathematicians can be traced, for example, in [1], [2], as well as in the related bibliography there.
By the kindness of Professor D.M. Bătineţu-Giurgiu, we are in possession of a very interesting postal item to his lordship, from Alexandru Lupaş.
This 'mathematical envelope' actually contains three components:
1) a letter from Alexandru Lupaş to D.M. Bătineţu-Giurgiu;
2) the paper "Another method of determining the limit of Traian Lalescu's sequence developed by D.M. Bătineţu-Giurgiu, but annotated by Alexandru Lupaş;
3) an essay of this paper signed by Al. Lupaş in which publication is recommended in the Mathematical Gazette - series A.
Mr. Lupaş's letter from 1-02-1990 is a letter in response to a letter from 11-11-1989 of Mr. Bătineţu-Giurgiu and is reproduced next to it in the first pages.

The letter - although written immediately after the revolution of December 1989, is exemplary in many ways. Besides mathematical concerns of the two mathematicians, captured in the letter, we also note: the availability - even the pleasure to communicate of Mr. Lupaş, respect for the interlocutor, frank expression, open communication, even with generous suggestions or recommendations (which other mathematicians - perhaps, would have kept secrets, for their own research), etc.
We hope that younger mathematicians - and not only them - will find here a model of relationship, both scientific and human.
The end of the letter also reveals Al Lupaş's mathematical preoccupations at that

time. In the post-scriptum is made the invitation to participate in the IV Symposium of inequalities that was to take place that year in Sibiu.
The next three pages are a mathematical note by Mr. D.M. Bătineţu- Giurgiu submitted to the attention of Mr. Al. Lupaş. The object of the paper is the Lalescu sequence - a favorite subject of Mr. Bătineţu-Giurgiu - to which he dedicated dozens of notes or articles. More precisely, the limit of a function expressed as differences of Gamma functions is calculated - which naturally extend Lalescu's sequence. Al Lupaş's few suggestions, regarding the note are recorded directly on the paper and are rather stylistic recommendations.
Finally, the last page represents the report on the mentioned note.

Dear Professor Bătineţu,
I was very happy to receive your letter from 11.11 1989. I have followed and appreciated your scientific activity over the years. With special interest, I also read the articles, the notes, the problems you elaborated regarding the Lalescu seqence, a matter which - at the urging of Tiberiu Popoviciu - I personally dealt with.
Due to service issues, and then the events of 1989, I have not been able to answer you so far. I enclose an essay on your paper "Another method of determining the boundary of Traian Lalescu's sequence" for the Mathematical Gazette.
Allow me to make some insignificant suggestions. For this purpose, in order to improve the material, I will return the typed manuscript, my proposals being written on the text. It is up to you to decide on any changes.
I consider that your work is welcome for the column "Commented problems" from Gazeta Matematica - series A.
In essence, it bothers me to find a simple demonstration (without Stirling) that:

$$(*)\qquad \lim_{x\to\infty}\left(\frac{1}{x}[\Gamma(x)]^{\frac{1}{x}}\right)=\frac{1}{e}$$

(see (1) from your paper)
By a simple demonstration, I would understand the following: let's start from the elegant definition of the Gamma function, namely (H. Bohr and I. Mollerup):
"There is only one function $G:(0,+\infty)\to(0,+\infty)$ with the properties:
1) $G(x+1)=xG(x),\forall x>0$
2) $G(1)=1$
3) $\ln G$ is convex on $(0,+\infty)$ (namely logarithmic convex)
By definition, we denote $\Gamma=G$, is it possible by using (1)-(3) to prove (*)?
Let's notice that if $x>0, 0\le a<b<c$, then from 1) - 3) it follows:

$$(**)\qquad \Gamma(x+b)^{c-a}\le\Gamma(x+a)^{c-b}\cdot\Gamma(x+c)^{b-a}$$

Choosing for example, $a=0,b=\frac{1}{2},c=1$, and then $a=\frac{1}{2},b=1,c=\frac{3}{2}$, we obtain the double inequality (used in statistics)

$$\frac{1}{\sqrt{x+\frac{1}{2}}}\le\frac{\Gamma(x+\frac{1}{2})}{\Gamma(x)}\le\frac{1}{\sqrt{x}}$$

Is it not possible to choose in (**) a,b,c conveniently, so that it results (*)?
Currently, my concerns are related to the following areas:
- approximation of functions
- special functions. I find this area interesting. In essence, I am concerned with

finding sufficient conditions to verify a sequence (a_n) of real numbers, so that for $n = 1, 2, \dots,$ and $x \in [-1, 1]$

$$\sum_{n=0}^{n} a_k P_k^{\alpha,\beta}(x) \geq 0 \tag{2}$$

By $P_k^{\alpha,\beta}$ we've noted Jacobi's polynomial with the parameters $(\alpha, \beta), \alpha > -1$, $\beta > -1$. Such a problem, namely that

$$\sum_{k=0}^{n} P_k^{\alpha,0}(x) \geq 0, x \in [-1, 1], \alpha > -1$$

led to DeBrouges' demonstration of Bieberbach's conjuncture. Another problem: fixing $a_k = a_k(n)$ and assuming that (2) takes place for $x \in [-1, 1]$, to find the range of variation of the parameters α, β.
threshold calculation applied to classical polynomials. In this direction I published in RNT two articles and also an article in Gazeta Matematica - Series A. This field is very attractive. We work, on the algebra of polynomials, with a generalization of the derivation operator, namely with a so-called "delta operator", ie with linear operator Q with the properties:
- Q it is invariant towards translation
- $Qe_1 = \text{constant} \neq 0, e_1(x) = x$.

Classical examples of delta operators:

$$(Qf)(x) = f'(x), (Qf)(x) = f'(x + a)$$
$$(Qf)(x) = f'(x) + \lambda f''(x), \text{ etc.}$$

It is shown (Gian - Carlo Rota) that any delta operator is uniquely attached to a polynomial sequence (p_n) such that $p_o(x) = 1$ and

$$\begin{cases} (Qp_n)(x) = np_{n-1}(x) \\ p_n(0) = 0, n = 1, 2, \dots \end{cases}$$

In addition, this string is binomial, that is

$$p_n(x + y) = \sum_{k=0}^{n} (n_k) p_k(x) p_{n-k}(y)$$

Binomial polynomial sequence are of particular interest in combinatorics.
Excuse me for extending the letter,
With esteem and appreciation,
Alexandru Lupaş
P.S. In June we organize in Sibiu the "4th Symposium on Inequalities". It would be an honor for me to meet each other personally.

Another method of determining the limit of Traian Lalescu's sequence.

By D.M. Bătineţu-Giurgiu

Methods for determining the limit of the sequence $(L_n)_{n \geq 2}$ of Traina Lalescu were given in chronological order in the works cited in the bibliography: [4], [3], [1], [2] and [5].
In this paper we will expose in a more general framework the method used by the

author of problem C: 844 from Gazeta Matematica no. 11-12, pp. 46.
We will use in the paper some properties of Euler's Γ function, among which we list:

$$\Gamma(x+1) = x \cdot \Gamma(x), (\forall) x \in \mathbb{R}_+^*. \quad \Gamma(x+1) = x!, (\forall) x \in \mathbb{R}^* \text{ and}$$

$$\lim_{x\to\infty}\left(x^{-1}\cdot\left(\Gamma(x)\right)^{\frac{1}{x}}\right) = \frac{1}{e} \tag{1}$$

wherefrom we deduce if we denote

$$f(x) = (\Gamma(x+1))^{\frac{1}{x+1}} \cdot (\Gamma(x))^{-\frac{1}{x}} \text{ then}$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{(\Gamma(x+1))^{\frac{1}{x+1}}}{x+1} \cdot \frac{x}{(\Gamma(x))^{\frac{1}{x}}} \cdot \frac{x+1}{x} = \frac{1}{e} \cdot e \cdot 1 = 1 \tag{2}$$

Also, we will use the fact that if $g : D \subset \mathbb{R} \to \mathbb{R})_+^*$ and $s \in \mathbb{R}$ is an accumulation point of D such that $\lim_{x\to a} g(x) = 1$ then:

$$\lim_{x\to\infty} \frac{g(x)-1}{\ln g(x)} = 1 \tag{3}$$

Also, we will use the fact that if $(u_n)_{n\geq 1}$ is a sequence of real strictly positive numbers such that:

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = u > 0 \text{ then } \lim_{n\to\infty} \sqrt[n]{u_n} = u \tag{4}$$

$$\text{then } \lim_{n\to\infty} \frac{\sqrt[n+1]{u_{n+1}}}{\sqrt[n]{u_n}} = \frac{u}{u} = 1 \tag{5}$$

Next, we prove:

Theorem. Let be $h(x) = (\Gamma(x+2))^{\frac{1}{x+1}} - (\Gamma(x+1))^{\frac{1}{x}}$; then:

$$\lim_{x\to\infty} h(x) = \frac{1}{e} \tag{6}$$

Proof. Let's denote

$$g(x) = (\Gamma(x+2))^{\frac{1}{x+1}} \cdot (\Gamma(x+1))^{-\frac{1}{x}} \text{ and then}$$

$$h(x) = (\Gamma(x+1))^{\frac{1}{x+1}} \cdot \frac{g(x)-1}{\ln g(x)} \cdot \ln g(x) = \frac{(\Gamma(x+1))^{\frac{1}{x}}}{x} \cdot \frac{g(x)-1}{\ln g(x)} \cdot \ln(g(x))^x$$

where by passing to the limit with $x \to \infty$ and taking into account the relationships (1), (2), (3) we deduce that:

$$\lim_{x\to\infty} h(x) = \frac{1}{e}\cdot 1\cdot \ln\left(\lim_{x\to\infty} \frac{(\Gamma(x+2))^{\frac{x}{x+1}}}{\Gamma(x+1)}\right) = \frac{1}{e}\ln\left(\lim_{x\to\infty}\frac{\Gamma(x+2)}{\Gamma(x+1)}\cdot\frac{1}{(\Gamma(x+2))^{\frac{1}{x+1}}}\right) =$$

$$= \frac{1}{e}\ln\left(\lim_{x\to\infty}\frac{x+1}{(\Gamma(x+2))^{\frac{1}{x+1}}}\right) = \frac{1}{e}\cdot \ln e = \frac{1}{e}$$

and the theorem is proved. □

Remark. If $n \in \mathbb{N}^* - \{1\}$ and $x = n$ then $h(n) = L_n$ so the theorem shows us that

$$\lim_{n\to\infty} L_n = \frac{1}{e}, \text{ where } L_n = \ldots$$

In the following we highlight the method set out in the proved theorem:
Sentence. (Problem C:844 from Gazeta Matematică, no. 11-12/1988, pp. 488). Let be $(a_n)_{n\geq 1}, (b_n)_{n\geq 1}$ sequences of real strictly positive numbers such that:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = a > 0 \text{ and } \lim_{n\to\infty}(b_{n+1} - b_n) = b \in \mathbb{R}$$

Prove that:

$$\lim_{n\to\infty}(b_{n+1} \sqrt[n+1]{a_{n+1}} - b_n \sqrt[n]{a_n}) = ab$$

Proof.

$$\text{It is obviously that } \lim_{n\to\infty} \sqrt[n]{a_n} = a \text{ and that } \lim_{n\to\infty}(b_{n+1} - b_n) = b$$

$$\text{implies } \lim_{n\to\infty} \frac{b_n}{n} = b \text{ (we apply Cesaro-Stolz theorem).}$$

It follows that if we denote $y_n = \frac{b_{n+1}}{b_n} \cdot \frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}}, n \geq 2$ we have:

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty}\Big(\frac{b_{n+1}}{n+1} \cdot \frac{n}{b_n} \cdot \frac{n+1}{n} \cdot \frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}}\Big) = b \cdot \frac{1}{b} \cdot 1 \cdot \frac{a}{a} = 1 \tag{7}$$

Let be $x_n = b_{n+1} \sqrt[n+1]{a_{n+1}} - b_n \sqrt[n]{a_n}, n \geq 2$. It follows that:

$$x_n = b_n \sqrt[n]{a_n} \cdot \frac{y_n - 1}{\ln y_n} \cdot \ln y_n = \frac{b_n}{n} \cdot \sqrt[n]{a_n} \cdot \frac{y_n - 1}{\ln y_n} \cdot \ln y_n^n$$

wherefrom by passing to the limit with $n \to \infty$ taking into account (3) and (4) we deduce:

$$\lim_{n\to\infty} x_n = ab \cdot \ln(\lim_{n\to\infty} y_n^n) = ab \cdot \ln\Big(\lim_{n\to\infty}\Big(\frac{b_{n+1}}{b_n}\Big)^n \cdot \frac{\sqrt[n+1]{a_{n+1}^n}}{a_n}\Big) =$$

$$= ab \cdot \ln\Big(\lim_{n\to\infty}\Big(\Big(1 + \frac{b_{n+1} - b_n}{b_n}\Big)^{\frac{b_n}{b_{n+1}-b_n}}\Big)^{(b_{n+1}-b_n)\frac{n}{b_n}} \cdot \frac{a_{n+1}}{a_n} \cdot \frac{1}{\sqrt[n+1]{a_{n+1}}}\Big)$$

$$= ab \cdot \lim_{n\to\infty}(b_{n+1} - b_n)\frac{n}{b_n} = ab \cdot b \cdot b^{-1} = ab$$

and the sentence is proved.

□

Remark. If $b_n = n, (\forall n \in \mathbb{N}^*, a_n = \frac{n!}{n^n}, (\forall) n \in \mathbb{N}^*$ we obtain Traian Lalescu's sequence, whereform we take into account that $b_{n+1} - b_n = 1, (\forall) n \in \mathbb{N}^*$, $\frac{a_{n+1}}{a_n} = \Big(\frac{n}{n+1}\Big)^n, (\forall) n \in \mathbb{N}^*$ we deduce that:

$$\lim_{n\to\infty}(b_{n+1} - b_n) = 1 \cdot \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{e}$$

it follows that in this case $x_n = L_n, (\forall) n \geq 2$ and so the limit of Traian Lalescu sequence is $\frac{1}{e}$

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Sibiu, 1.02.1990

REPORT
on professor D.M. Bătineţu-Giurgiu's work with the title
"Another method of determining the limit of Traian Lalescu's sequence"

Using some known, elementary properties of the Gamma function, the author proves that

$$\lim_{x\to\infty} h(x) = \frac{1}{e}, \text{ where } h(x) = (\Gamma(x+2))^{\frac{1}{x+1}} - (\Gamma(x+1))^{\frac{1}{x}}$$

This result is illustrated by presenting a solution to the problem C: 844 from Gazeta Matematică no. 11-12 / 1988. There is also the solution to the famous problem of Traian Lalescu by which it is required to show that

$$\lim_{n\to\infty} \left(\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!} \right) = \frac{1}{e}$$

The paper is a continuation of some interesting results belonging to Professor Bătineţu, regarding Lalescu's series. I consider that the paper deserves to be published in Gazeta Matematică - Series A, possibly in the column of Commented Problems.

Alexadru Lupaş

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Happy birthday, dear professor
Dumitru M. Bătineţu-Giurgiu!

Dumitru M. Bătineţu-Giurgiu was born on January 27, 1936, in Pietroşani commune, Vlaşca county, where he attended the General School; then he studied of the "Ion Maiorescu" National College from Giurgiu. In the autumn of 1960, he became a student of the University of Bucharest, Faculty of Mathematics (specialization in mathematical analysis).

He was an assistant professor at the Department of Mathematics of the Polytechnic Institute " Gh. Asachi " from Iaşi (1965-1968); scientific researcher at the Forest Research Institute in Bucharest (1968-1970); assistant professor at the mathematics-physics department of the "Nicolae Bălcescu" Agronomic Institute in Bucharest (1970-1972); professor at the "Ion Creangă" National College in Bucharest (1985-1989) and at the "Matei Basarab" National College in Bucharest (1976-1985; 1989-2003).

He dealt with: Lalescu's sequences, introducing various extensions and concepts of Lalescu functions and Euler-Lalescu functions; the Ioachimescu sequences, introducing the notion of Ioachimescu type constant, the Euler-Ioachimescu type sequences and generalizations of the Ioachimescu sequences; Ghermănescu's sequences and Euler-Ghermănescu constants.

The scientific activity of Professor D.M. Bătineţu-Giurgiu is extremely rich and varied through reference works, articles and studies, published books, notes and teaching materials. He has been and he is part of the editorial boards of several national and international math magazines.

He participated and presented his results of scientific research at international and national conferences, symposia and sessions of scientific communications.

Throughout his career, he has been honored with numerous medals and diplomas.

On January 27th, 2021, Dumitru M. Bătineţu-Giurgiu – DMBG, as we calls him - turned 85. His long career was, and continue to be, extremely fruitful and influential for students and teachers - lovers of mathematics from all over the world. DMBG always regarded and practical mathematics as a whole – this is a lesson he served us constantly. The activity of DMBG (for more details see the references below) proves that he didn't preach in the desert.

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