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# 100

# RMM MATH

# PROBLEMS

***By Florică Anastase-Romania***

**Pbl. 1** In acute $\Delta ABC$ the following relationship holds:

$$\frac{s}{3}(\cos A+\cos B+\cos C)\frac{S}{R}$$

*Florică Anastase*

**Solution.**

$$If\ a\ge b\ge c\ \Rightarrow\begin{cases}A\ge B\ge C\\ \cos A\le\cos B\le\cos C\end{cases}\overset{Chebyshev}{\Rightarrow}$$

$$\frac{s}{3}(\cos A+\cos B+\cos C)=\frac{a+b+c}{6}\cdot(\cos A+\cos B+\cos C)$$

$$\ge\frac{a\cos A+b\cos B+c\cos C}{2}=$$

$$=\frac{R}{2}(2\sin A\cos A+2\sin B\cos B+2\sin C\cos C)=\frac{R}{2}(\sin 2A+\sin 2B+\sin 2C)=\frac{abc}{4R^2}$$

$$=\frac{S}{R}$$

**Pbl. 2** In acute $\Delta ABC$ the following relationship holds:

$$3R(s^2-r^2-4Rr)\ge(R+r)(r^2+4rR+s^2)$$

*Florică Anastase*

**Solution.**

$$Let: a\ge b\ge c\Rightarrow\begin{cases}\cos A\le\cos B\le\cos C\\ \frac{1}{\sin A}\le\frac{1}{\sin B}\le\frac{1}{\sin C}\end{cases}\overset{Chebyshev}{\Rightarrow}$$

$$\frac{a^2+b^2+c^2}{4S}=\cot A+\cot B+\cot C$$

$$\ge\frac{1}{3}(\cos A+\cos B+\cos C)\left(\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}\right)$$

$$\cos A+\cos B+\cos C=1+\frac{r}{R}\ and\ sinus\ theorem\ result:$$

$$\frac{a^2+b^2+c^2}{4S}\ge\frac{1}{3}\left(1+\frac{r}{R}\right)\left(\frac{2R}{a}+\frac{2R}{b}+\frac{2R}{c}\right)\Leftrightarrow$$

$$\begin{cases} \frac{a^2+b^2+c^2}{4S} \geq \frac{2}{3}(R+r)\left(\frac{ab+bc+ca}{abc}\right) \\ a^2+b^2+c^2 = 2(s^2-r^2-4Rr) \\ ab+bc+ca = r^2+4Rr+s^2 \end{cases} \Rightarrow$$

$$\frac{s^2-r^2-4Rr}{4S} \geq \frac{1}{3}(R+r)\frac{r^2+4rR+s^2}{4RS} \Leftrightarrow$$

$$3R(s^2-r^2-4rR) \geq (R+r)(r^2+4rR+s^2)$$

**Pbl. 3** ***In acute $\Delta ABC$ the following relationship holds:***

$$\frac{9a^3b^3c^3}{b^2+c^2-a^2}+\frac{9a^3b^3c^3}{a^2+c^2-b^2}+\frac{9a^3b^3c^3}{a^2+b^2-c^2} \geq 2p[8S(R+r)]^2$$

*Florică Anastase*

***Solution.***

$$If\ a \geq b \geq c \Rightarrow A \geq B \geq C \Rightarrow \begin{cases} sinA \geq sinB \geq sinC \\ \frac{1}{cosA} \geq \frac{1}{cosB} \geq \frac{1}{cosC} \end{cases} \overset{Chebyshev}{\Rightarrow}$$

$$tgA+tgB+tgC \geq \frac{1}{3}(sinA+sinB+sinC)\left(\frac{1}{cosA}+\frac{1}{cosB}+\frac{1}{cosC}\right)$$

$$= \frac{1}{3}(sinA+sinB+sinC)\left(\frac{cosAcosB+cosBcosC+cosCcosA}{cosAcosBcosC}\right) \quad (1)$$

$$Cu\ relațiile: \begin{cases} sinA+sinB+sinC = \frac{p}{R} \\ \frac{1}{cosAcosBcosC} \geq 8 \\ tgA = \frac{4S}{b^2+c^2-a^2}\ și\ analoagele \end{cases} \overset{(1)}{\Rightarrow}$$

$$\frac{4S}{b^2+c^2-a^2}+\frac{4S}{a^2+c^2-b^2}+\frac{4S}{a^2+b^2-c^2}$$

$$\geq \frac{8p}{3R}(cosAcosB+cosBcosC+cosCcosA)$$

$$\frac{4S}{b^2+c^2-a^2}+\frac{4S}{a^2+c^2-b^2}+\frac{4S}{a^2+b^2-c^2} \geq \frac{8p}{9R}(cosA+cosB+cosC)^2$$

$$\frac{9\cdot 4RS}{b^2+c^2-a^2}+\frac{9\cdot 4RS}{a^2+c^2-b^2}+\frac{9\cdot 4RS}{a^2+b^2-c^2} \geq 8p\left(1+\frac{r}{R}\right)^2 \Leftrightarrow$$

$$\frac{9\cdot abcR^2}{b^2+c^2-a^2}+\frac{9\cdot abcR^2}{a^2+c^2-b^2}+\frac{9\cdot abcR^2}{a^2+b^2-c^2}\ge 8p(R+r)^2\Leftrightarrow$$

$$\frac{9a^3b^3c^3}{b^2+c^2-a^2}+\frac{9a^3b^3c^3}{a^2+c^2-b^2}+\frac{9a^3b^3c^3}{a^2+b^2-c^2}\ge 2p[8S(R+r)]^2$$

**Pbl. 4** *In acute $\Delta ABC$ the following relationship holds:*

$$2s(s^2-r^2-4rR)(r+R)\ge 3rR[3s^2-(2R+r)(4R+r)]$$

*Florică Anastase*

*Solution.*

$$a^2cosA+b^2cosB+c^2cosC=\frac{r}{s}[3s^2-(2R+r)(4R+r)]$$

$$\begin{cases}a^2\ge b^2\ge c^2\\ cosA\le cosB\le cosC\\ a^2+b^2+c^2=2(s^2-r^2-4rR)\\ cosA+cosB+cosC=1+\dfrac{r}{R}\end{cases}\overset{Chebyshev}{\Rightarrow}$$

$$a^2cosA+b^2cosB+c^2cosC\le\frac{a^2+b^2+c^2}{3}(cosA+cosB+cosC)\Rightarrow$$

$$2s(s^2-r^2-4rR)(r+R)\ge 3rR[3s^2-(2R+r)(4R+r)]$$

**Pbl. 5** *In acute $\Delta ABC$ the following relationship holds:*

$$9R(bch_a+ach_b+abh_c)\ge 2r^2(s^2+r^2+4rR)$$

*Florică Anastase*

*Solution.*

$$If\ a\ge b\ge c\Rightarrow\begin{cases}h_a\le h_b\le h_c\\ \dfrac{1}{sinA}\le\dfrac{1}{sinB}\le\dfrac{1}{sinC}\end{cases}\overset{Chebyshev}{\Rightarrow}$$

$$\frac{h_a}{sinA}+\frac{h_b}{sinB}+\frac{h_c}{sinC}\ge\left(\frac{h_a+h_b+h_c}{3}\right)\left(\frac{1}{sinA}+\frac{1}{sinB}+\frac{1}{sinC}\right)\Leftrightarrow$$

$$\frac{h_a}{sinA}+\frac{h_b}{sinB}+\frac{h_c}{sinC}\ge\left(\frac{h_a+h_b+h_c}{3}\right)\left(\frac{sinAsinB+sinBsinC+sinCsinA}{sinAsinBsinC}\right)\overset{Chebyshev}{\ge}$$

$$\geq \left(\frac{h_a+h_b+h_c}{9}\right)\frac{(sinA+sinB+sinC)^2}{sinAsinBsinC}\Rightarrow$$

$$2R\left(\frac{h_a}{a}+\frac{h_b}{b}+\frac{h_c}{c}\right)\geq \frac{1}{9}\left(\frac{s^2+r^2+4rR}{2R}\right)\left(\frac{8R^3}{abc}\right)\left(\frac{r}{R}\right)^2\Leftrightarrow$$

$$9abcR\left(\frac{h_a}{a}+\frac{h_b}{b}+\frac{h_c}{c}\right)\geq 2r^2(s^2+r^2+4rR)\Leftrightarrow$$

$$9R(bch_a+ach_b+abh_c)\geq 2r^2(s^2+r^2+4rR)$$

**Pbl. 6** ***In $\Delta ABC$, P, Q$\subset (ABC)$ such that:***

$$\beta\overrightarrow{AB}+\gamma\overrightarrow{BP}+\overrightarrow{PC}=0 \text{ and } \overrightarrow{AQ}+\alpha\overrightarrow{QB}+\overrightarrow{BC}=0, \alpha,\beta,\gamma\in\mathbb{R}, \alpha,\gamma\neq 1$$

***Prove that A, P, Q are collinear if and only if $\alpha+\gamma=\beta+1$.***

*Florică Anastase*

***Solution.***

$$\overrightarrow{AQ}+\alpha\overrightarrow{QB}+\overrightarrow{BC}=0\Leftrightarrow\left(\overrightarrow{AQ}+\overrightarrow{QB}+\overrightarrow{BC}\right)=(\alpha-1)\overrightarrow{BQ}\Leftrightarrow\overrightarrow{AC}=(\alpha-1)\overrightarrow{BQ}$$

$$\overrightarrow{AQ}=\overrightarrow{AB}+\overrightarrow{BQ}=\overrightarrow{AB}+\frac{1}{\alpha-1}\overrightarrow{AC};\ (1)$$

$$\beta\overrightarrow{AB}+\gamma\overrightarrow{BP}+\overrightarrow{PC}=0\Leftrightarrow\beta\overrightarrow{AB}+\gamma\left(\overrightarrow{BA}+\overrightarrow{AP}\right)+\overrightarrow{PC}=0\Leftrightarrow$$

$$(\beta-\gamma)\overrightarrow{AB}+\gamma\overrightarrow{AP}+\overrightarrow{PC}=0\Leftrightarrow$$

$$(\beta-\gamma)\overrightarrow{AB}+(\gamma-1)\overrightarrow{AP}+\overrightarrow{AC}=0\Leftrightarrow$$

$$\overrightarrow{AP}=-\frac{1}{\gamma-1}\left((\beta-\gamma)\overrightarrow{AB}+\overrightarrow{AC}\right)=\frac{1}{1-\gamma}\left((\beta-\gamma)\overrightarrow{AB}+\overrightarrow{AC}\right);\ (2)$$

***From (1) and (2)*** $A, P, Q$ ***are collinear if and only if exist*** $\lambda\in\mathbb{R}$ ***such that***

$$\overrightarrow{AP}=\lambda\overrightarrow{AQ}\Leftrightarrow\frac{1}{1-\gamma}\left((\beta-\gamma)\overrightarrow{AB}+\overrightarrow{AC}\right)=\lambda\left(\overrightarrow{AB}+\frac{1}{\alpha-1}\overrightarrow{AC}\right)\Leftrightarrow$$

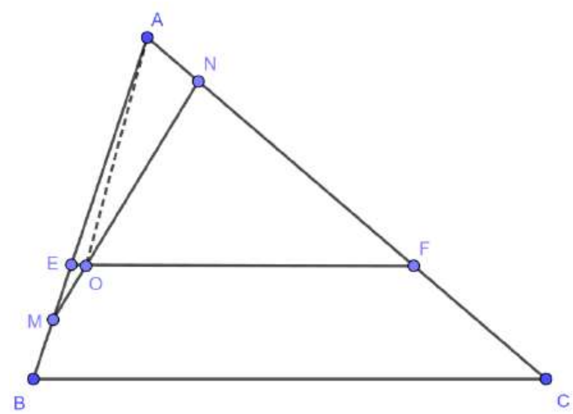
$$\begin{cases}\dfrac{\beta-\gamma}{1-\gamma}=\lambda\\ \dfrac{1}{1-\gamma}=\dfrac{\lambda}{\alpha-1}\end{cases}\Leftrightarrow\begin{cases}\beta-\gamma=\lambda(1-\gamma)\\ \alpha-1=\lambda(1-\gamma)\end{cases}\Leftrightarrow\alpha+\gamma=\beta+1$$

**Pbl.7** **In $\Delta ABC; M, E \in (AB); N, F \in (AC)$ such that $\overrightarrow{AE} = m\overrightarrow{EB}, \overrightarrow{AF} = n\overrightarrow{FC}, \overrightarrow{MO} = p\overrightarrow{ON}$ and $\frac{MB}{MA} = \frac{NA}{NC} = \lambda; m, n, p, \lambda \in \mathbb{R}^*; p \neq -1, \lambda \neq 1; m \cdot p = 1$.**

**Prove that: $E, O, F$ are collinear if and only if $p = n$.**

*Florică Anastase*

**Solution.**




$$\frac{MB}{MA} = \frac{NA}{NC} = \lambda \Rightarrow \begin{cases} \overrightarrow{MB} = \lambda\overrightarrow{NA} \\ \overrightarrow{MA} = \lambda\overrightarrow{NC} \end{cases}$$

$$\Rightarrow \begin{cases} \overrightarrow{MA} = -\dfrac{1}{1-\lambda}\overrightarrow{AB} \\ \overrightarrow{AN} = -\dfrac{\lambda}{1-\lambda}\overrightarrow{AC} \end{cases}; \ (1)$$

$$\begin{cases} \overrightarrow{AE} = m\overrightarrow{EB} \\ \overrightarrow{AF} = n\overrightarrow{FC} \end{cases} \Rightarrow \begin{cases} \overrightarrow{AB} = \dfrac{m+1}{m}\overrightarrow{AE} \\ \overrightarrow{AC} = \dfrac{n+1}{n}\overrightarrow{AF} \end{cases}; \ (2)$$

$$\overrightarrow{MO} = p\overrightarrow{ON} \Rightarrow \overrightarrow{MA} + \overrightarrow{AO} = p(\overrightarrow{OA} + \overrightarrow{AN})$$

$$(1+p)\overrightarrow{AO} = \overrightarrow{AM} + p\overrightarrow{AN}; \ (3)$$

**From (1),(3) we get:**

$$\overrightarrow{AO} = \frac{1}{1+p} \cdot \frac{1}{1-\lambda}\overrightarrow{AB} - \frac{p}{1+p} \cdot \frac{\lambda}{1-\lambda}\overrightarrow{AC}; \ (4)$$

**From (2),(4) we have:**

$$\overrightarrow{AO} = \frac{1}{1+p} \cdot \frac{1}{1-\lambda} \cdot \frac{m+1}{m}\overrightarrow{AE} - \frac{p}{1+p} \cdot \frac{\lambda}{1-\lambda} \cdot \frac{n+1}{n}\overrightarrow{AF}$$

**$E, O, F$ are collinear if and only if**

$$\frac{1}{1+p} \cdot \frac{1}{1-\lambda} \cdot \frac{m+1}{m} - \frac{p}{1+p} \cdot \frac{\lambda}{1-\lambda} \cdot \frac{n+1}{n} = 1 \Leftrightarrow$$

$$n(m+1) - \lambda mp(n+1) = mn(p+1)(1-\lambda) \Leftrightarrow$$

$$n - mp\lambda = mnp - \lambda mn \Leftrightarrow n(1-mp) = m\lambda(p-n) \overset{mp=1}{\Longleftrightarrow} p = n.$$

**Pbl.8** **In acute $\Delta ABC$ the following relationship holds:**

$$\frac{sinA}{\sqrt{cosA}}+\frac{sinB}{\sqrt{cosB}}+\frac{sinC}{\sqrt{cosC}}>\pi$$

*Florică Anastase*

***Solution.***

$$We\ prove\ that:\ \frac{sinx}{\sqrt{cosx}}>x\ ,x\in\left(0,\frac{\pi}{2}\right)\Leftrightarrow\frac{sin^2x}{cosx}>x^2,x\in\left(0,\frac{\pi}{2}\right)$$

$$Fie\ f:\left(0,\frac{\pi}{2}\right)\to R,f(x)=\frac{sin^2x}{cosx}-x^2$$

$$f'(x)=\frac{sinx(1+cosx)}{cos^2x}-2x,x\in\left(0,\frac{\pi}{2}\right)$$

$$f''(x)=cosx+\frac{1}{cosx}+\frac{2sin^2x}{cos^3x}-2\geq\frac{2sin^2x}{cos^3x}>0$$

$$\Rightarrow f''(x)>0,x\in\left(0,\frac{\pi}{2}\right)\Rightarrow f'(x)>0,x\in\left(0,\frac{\pi}{2}\right)\Rightarrow f(x)>0,x\in\left(0,\frac{\pi}{2}\right)$$

$$\frac{sinA}{\sqrt{cosA}}+\frac{sinB}{\sqrt{cosB}}+\frac{sinC}{\sqrt{cosC}}>A+B+C$$

$$\frac{sinA}{\sqrt{cosA}}+\frac{sinB}{\sqrt{cosB}}+\frac{sinC}{\sqrt{cosC}}>\pi$$

$\boxed{Pbl.9}$ ***In acute $\Delta ABC$ the following relationship holds:***

$$\sum_{cyc}\left(2+\frac{\sqrt{h_bh_c}}{a}-\frac{2(s-a)^2}{bc}\right)\leq\sum_{cyc}(1+cscA)^{\frac{1}{1+cotA}}\cdot(1+secA)^{\frac{1}{1+tanA}}$$

*Florică Anastase*

***Solution.***

$$\text{Let: } f:(0,1)\to\mathbb{R},f(x)=log\left(\frac{1}{x}+1\right)$$

$$f'(x)=\frac{-1}{x(x+1)}<0,\forall x\in(0,1)\Rightarrow f-\text{decreasing.}$$

$$f''(x)=\frac{2x+1}{x^2(x+1)^2}>0,\forall x\in(0,1)\Rightarrow f-\text{convexe.}$$

$$log(1+sinx+cosx)=f\left(\frac{1}{sinx+cosx}\right)=f\left(\frac{sin^2x+cos^2x}{sinx+cosx}\right)$$

$$=f\left(\frac{sinx\cdot sinx+cosx\cdot cosx}{sinx+cosx}\right)\le\frac{sinxf(sinx)+cosxf(cosx)}{sinx+cosx}$$

$$=\frac{1}{1+cotx}log\left(\frac{1}{sinx}+1\right)+\frac{1}{1+tanx}log\left(\frac{1}{cosx}+1\right)$$

$$=log\left(\left(\frac{1}{sinx}+1\right)^{\frac{1}{1+cotx}}\cdot\left(\frac{1}{cosx}+1\right)^{\frac{1}{1+tanx}}\right)\Rightarrow$$

$$1+sinx+cosx\le\left(\frac{1}{sinx}+1\right)^{\frac{1}{1+cotx}}\cdot\left(\frac{1}{cosx}+1\right)^{\frac{1}{1+tanx}}$$

$$1+sinx+cosx\le(cscx+1)^{\frac{1}{1+cotx}}\cdot(secx+1)^{\frac{1}{1+tanx}}$$

$$\sum_{cyc}(1+cscA)^{\frac{1}{1+cotA}}\cdot(1+secA)^{\frac{1}{1+tanA}}\ge 3+\sum_{cyc}sinA+\sum_{cyc}cosA$$

$$=3+\frac{s}{R}+\left(1+\frac{r}{R}\right)=4+\left(\frac{1}{2}\sum_{cyc}\frac{h_b+h_c}{a}\right)+\left(2-2\sum_{cyc}\frac{(s-a)^2}{bc}\right)$$

$$\overset{Am-Gm}{\ge}6+\sum_{cyc}\left(\frac{\sqrt{h_b\cdot h_c}}{a}-2\cdot\frac{(s-a)^2}{bc}\right)$$

**Pbl. 10** **In $\Delta ABC$ the following relationship holds:**

$$\sum_{cyc}\frac{w_a^n}{h_a^n+m_a^n}\le\frac{3}{2}\cdot\left(\frac{R}{2r}\right)^{\frac{2n}{3}},n\in\mathbb{N}$$

*Florică Anastase*

***Solution.***

$$\begin{cases}\prod_{cyc}w_a=\dfrac{16Rr^2s^2}{s^2+r^2+2Rr}\\ \prod_{cyc}h_a=\dfrac{8F^3}{abc}\end{cases},h_a\le w_a\Rightarrow\prod_{cyc}\frac{w_a}{h_a}=\frac{8R^2}{s^2+r^2+2Rr}\ge 1,$$

$$\frac{8R^2}{s^2+r^2+2Rr}\overset{Gerretsen}{\le}\frac{8R^2}{16Rr-5r^2+r^2+2Rr}=\frac{8R^2}{18Rr-4r^2}\overset{Euler}{\le}\frac{R^2}{4r^2}=\left(\frac{R}{2r}\right)^2$$

Let $f:(0,\infty)\to(0,\infty), f(x)=\frac{e^{nx}}{1+e^{nx}}, n\in\mathbb{N}, f''(x)=\frac{n^2(1-e^{nx})e^{nx}}{(1+e^{nx})^3}<0\Rightarrow f$ –concave, thus

$$\sum_{cyc} f\left(log\left(\frac{w_a}{h_a}\right)\right)\overset{Jensen}{\leq} 3f\left(\frac{1}{3}\sum_{cyc} f\left(log\left(\frac{w_a}{h_a}\right)\right)\right)\Leftrightarrow$$

$$\sum_{cyc}\frac{\left(\frac{w_a}{h_a}\right)^n}{1+\left(\frac{w_a}{h_a}\right)^n}\leq 3\cdot\frac{\sqrt[3]{\prod_{cyc}\left(\frac{w_a}{h_a}\right)^n}}{1+\sqrt[3]{\prod_{cyc}\left(\frac{w_a}{h_a}\right)^n}}\Leftrightarrow\sum_{cyc}\frac{w_a^n}{h_a^n+w_a^n}\leq\frac{3}{2}\cdot\left(\frac{R}{2r}\right)^{\frac{n}{3}}\Leftrightarrow$$

$$w_a\leq m_a\Rightarrow\sum_{cyc}\frac{w_a^n}{h_a^n+m_a^n}\leq\frac{3}{2}\cdot\left(\frac{R}{2r}\right)^{\frac{2n}{3}}$$

$\boxed{Pbl.11}$ **In $\Delta ABC$ the following relationship holds:**

$$\left(\sum_{cyc}\frac{1}{m_am_b}\right)\left(4s+\sum_{cyc}\frac{bc}{a}\right)\geq\frac{54}{s}$$

***Florică Anastase***

***Solution.***

We prove that: $4m_bm_c\leq 2a^2+bc;(1)$

$$(1)\Leftrightarrow 16m_b^2m_c^2\leq(2a^2+bc)^2\Leftrightarrow 16\left(\frac{a^2+c^2}{2}-\frac{b^2}{4}\right)\left(\frac{a^2+b^2}{2}-\frac{c^2}{4}\right)\leq(2a^2+bc)^2$$

$$\Leftrightarrow(2a^2+2c^2-b^2)(2a^2+2b^2-c^2)\leq(2a^2+bc)^2\Leftrightarrow$$

$$(b-c)^2(a^2-b^2-c^2-2bc)\leq 0\Leftrightarrow(b-c)^2(a+b+c)\big(a-(b+c)\big)\leq 0$$

From $(1)\Rightarrow\frac{1}{4m_bm_c}\geq\frac{1}{2a^2+bc}\Rightarrow\frac{1}{m_bm_c}\geq\frac{4}{2a^2+bc}$

$(a,b,c),\left(\frac{1}{2a^2+bc},\frac{1}{2b^2+ca},\frac{1}{2c^2+ab}\right)$ reverse ordered, from Chebyshev's, it follows that:

$$\sum_{cyc}\frac{a}{2a^2+bc}\leq\frac{1}{3}\cdot\left(\sum_{cyc}a\right)\left(\sum_{cyc}\frac{1}{2a^2+bc}\right)\Leftrightarrow$$

$$\sum_{cyc}\frac{a}{2a^2+bc}\le\frac{2s}{3}\cdot\frac{1}{4}\left(\sum_{cyc}\frac{1}{m_am_b}\right)\Leftrightarrow\sum_{cyc}\frac{a}{2a^2+bc}\le\frac{s}{6}\cdot\left(\sum_{cyc}\frac{1}{m_am_b}\right)$$

$$\frac{1}{\sum_{cyc}\frac{a}{2a^2+bc}}\ge\frac{6}{s}\cdot\frac{1}{\sum_{cyc}\frac{1}{m_am_b}};(2)$$

$$\sum_{cyc}\frac{2a^2+bc}{a}=\sum_{cyc}\frac{1}{\frac{a}{2a^2+bc}}\overset{BCS}{\ge}\frac{9}{\sum_{cyc}\frac{a}{2a^2+bc}}\Leftrightarrow$$

$$\sum_{cyc}\left(2a+\frac{bc}{a}\right)\ge\frac{9}{\sum_{cyc}\frac{a}{2a^2+bc}}\Leftrightarrow 4s+\sum_{cyc}\frac{bc}{a}\ge\frac{9}{\sum_{cyc}\frac{a}{2a^2+bc}};(3)$$

**From $(2),(3)$, we get:**

$$4s+\sum_{cyc}\frac{bc}{a}\ge\frac{6}{s}\cdot\frac{9}{\sum_{cyc}\frac{1}{m_am_b}}\Leftrightarrow\left(\sum_{cyc}\frac{1}{m_am_b}\right)\left(4s+\sum_{cyc}\frac{bc}{a}\right)\ge\frac{54}{s}$$

**$\boxed{Pbl.12}$ In $\Delta ABC$ the following relationship holds:**

$$\sqrt[3]{\prod_{cyc}\left(1+\frac{1}{m_a}+\frac{1}{m_b}\right)}\ge\frac{\sum(a+2)sin\frac{A}{2}}{\sum asin\frac{A}{2}}>1+\frac{2}{s}$$

*Florică Anastase*

***Solution.***

$$\because\sum_{i=1}^{2}(a_i+b_i)\sum_{i=1}^{2}\frac{a_ib_i}{a_i+b_i}\le\left(\sum_{i=1}^{2}a_i\right)\left(\sum_{i=1}^{2}b_i\right)\ (Milne's\ ineq.n=2)$$

$$\sum_{cyc}(m_a+m_b)\sum_{cyc}\left(\frac{m_am_b}{m_a+m_b}\right)\le\left(\sum_{cyc}m_a\right)\left(\sum_{cyc}m_b\right)\Rightarrow$$

$$\sum_{cyc}\left(\frac{m_am_b}{m_a+m_b}\right)\ge\frac{(\sum m_a)^2}{2\sum m_a}\Rightarrow\sum_{cyc}\left(\frac{m_am_b}{m_a+m_b}\right)\le\frac{1}{2}\sum_{cyc}m_a\le\frac{1}{2}\sum_{cyc}\frac{b+c}{2}=\frac{2s}{2}=s$$

$$\because m_a\le\frac{b+c}{2},m_b\le\frac{c+a}{2},m_c\le\frac{a+b}{2}$$

$$\sum_{cyc}\frac{1}{x+\frac{m_am_b}{m_a+m_b}}\overset{BCS}{\geq}\frac{9}{3x+\sum\frac{m_am_b}{m_a+m_b}}\geq\frac{9}{3x+s}=\frac{3}{x+\frac{s}{3}}\Rightarrow$$

$$\int_0^1\sum_{cyc}\frac{1}{x+\frac{m_am_b}{m_a+m_b}}dx\geq\int_0^1\frac{3}{x+\frac{s}{3}}dx\Rightarrow$$

$$\sum_{cyc}log\left(1+\frac{m_a+m_b}{m_am_b}\right)\geq 3log\left(1+\frac{3}{s}\right)\Rightarrow\sqrt[3]{\prod_{cyc}\left(1+\frac{1}{m_a}+\frac{1}{m_b}\right)}\geq 1+\frac{3}{s};(1)$$

Now,

$$(a+b+c)\left(sin\frac{A}{2}+sin\frac{B}{2}+sin\frac{C}{2}\right)\overset{Chebyshev's}{\leq}3asin\frac{A}{2}+3bsin\frac{B}{2}+3csin\frac{C}{2}$$

$$\Leftrightarrow\sum_{cyc}(a-b)\left(sin\frac{A}{2}-sin\frac{B}{2}\right)\geq 0,(2)$$

On the other hand,

$$\sum_{cyc}(a+b-c)sin\frac{C}{2}>0,(3)$$

From (2),(3) it follows that:

$$\frac{1}{2}\cdot\frac{3}{s}\geq\frac{sin\frac{A}{2}+sin\frac{B}{2}+sin\frac{C}{2}}{asin\frac{A}{2}+bsin\frac{B}{2}+csin\frac{C}{2}}>\frac{1}{s};(4)$$

From (1),(4) it follows that:

$$\sqrt[3]{\prod_{cyc}\left(1+\frac{1}{m_a}+\frac{1}{m_b}\right)}\geq 1+\frac{3}{s}\geq 1+\frac{2\sum sin\frac{A}{2}}{\sum asin\frac{A}{2}}=\frac{\sum(a+2)sin\frac{A}{2}}{\sum asin\frac{A}{2}}>1+\frac{2}{s}$$

$Pbl.13$ **In $\Delta ABC$ the following relationship holds:**

$$\sum_{cyc}\frac{\sqrt{s_a}}{\sqrt{h_b}+\sqrt{h_c}}(w_a^2+m_bm_c)+\sum_{cyc}h_a^2\geq\frac{4rs^2}{R}$$

*Florică Anastase*

***Solution.***

$$\begin{cases} s_a \geq h_a \\ m_a \geq w_a \geq h_a \end{cases} \Rightarrow \sum_{cyc} \frac{\sqrt{s_a}}{\sqrt{h_b} + \sqrt{h_c}} (w_a^2 + m_b m_c) \geq \sum_{cyc} \frac{\sqrt{h_a}}{\sqrt{h_b} + \sqrt{h_c}} (h_a^2 + h_b h_c)$$

$$\sum_{cyc} h_a h_b = \sum_{cyc} \frac{2F}{a} \cdot \frac{2F}{b} = \frac{4F^2}{abc} \sum_{cyc} a = \frac{rs^2}{R}$$

**We must to prove that:**

$$\sum_{cyc} \frac{\sqrt{h_a}}{\sqrt{h_b} + \sqrt{h_c}} (h_a^2 + h_b h_c) + \sum_{cyc} h_a^2 \geq \sum_{cyc} h_a h_b \ ; (1)$$

**Let us denote:** $x = h_a, y = h_b, z = h_c$**. WLOG suppose:** $x \leq y \leq z$**, then:**

$$\frac{\sqrt{x}}{\sqrt{y} + \sqrt{z}} \leq \frac{\sqrt{y}}{\sqrt{z} + \sqrt{x}} \Leftrightarrow \sqrt{xz} + x \leq \sqrt{yz} + z \Leftrightarrow (x - y) + \sqrt{z}(\sqrt{x} - \sqrt{y}) \leq 0 \Leftrightarrow$$

$(\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y} + \sqrt{z}) \leq 0$**, which is clearly true. So,** $\frac{\sqrt{x}}{\sqrt{y}+\sqrt{z}} \leq \frac{\sqrt{y}}{\sqrt{z}+\sqrt{x}}$ **and analogous.**

$x^2 + yz \leq y^2 + zx \Leftrightarrow (x - y)(x + y - z) \leq 0$**, which is true. So, triplets:**

$(x^2 + yz;\ y^2 + zx; z^2 + xy)$ **and** $\left(\frac{\sqrt{x}}{\sqrt{y}+\sqrt{z}}; \frac{\sqrt{y}}{\sqrt{z}+\sqrt{x}}; \frac{\sqrt{z}}{\sqrt{x}+\sqrt{y}}\right)$ **have the same orientation.**

**From Chebyshev's Inequality, it follows that:**

$$\sum_{cyc} \frac{\sqrt{x}}{\sqrt{y} + \sqrt{z}} (x^2 + yz) \geq \frac{1}{3} \cdot \sum_{cyc} \frac{\sqrt{x}}{\sqrt{y} + \sqrt{z}} \cdot \sum_{cyc} (x^2 + yz) \overset{Nesbitt}{\geq} \frac{1}{3} \cdot \frac{3}{2} \cdot \sum_{cyc} (x^2 + yz) \ ; (2)$$

**From (1),(2) we must to prove that:**

$$\frac{3}{2} \cdot \sum_{cyc} (x^2 + yz) + \sum_{cyc} x^2 \geq \sum_{cyc} yz \Leftrightarrow \sum_{cyc} x^2 \geq \sum_{cyc} yz \ (true).$$

**Therefore,**

$$\sum_{cyc} \frac{\sqrt{s_a}}{\sqrt{h_b} + \sqrt{h_c}} (w_a^2 + m_b m_c) + \sum_{cyc} h_a^2 \geq \frac{4rs^2}{R}$$

$\boxed{Pbl.14}$ **In $\Delta ABC$ the following relationship holds:**

$$\left(1 + \frac{1}{a} \tan \frac{A}{2}\right)\left(1 + \frac{1}{b} \tan \frac{B}{2}\right)\left(1 + \frac{1}{c} \tan \frac{C}{2}\right) \geq \left(1 + \frac{9}{2} \cdot \frac{r}{s^2}\right)^3$$

*Florică Anastase*

**Solution.**

$$a\cot\frac{A}{2}+b\cot\frac{B}{2}+c\cot\frac{C}{2}\le\frac{2s^2}{3r};(1)$$

$$a\cot\frac{A}{2}=a\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}=\frac{as(s-a)}{F}\Rightarrow\sum_{cyc}\frac{as(s-a)}{F}\le\frac{2s^2}{3r}\Leftrightarrow$$

$$\sum_{cyc}\frac{a(b+c-a)}{2}\le\frac{2(a+b+c)^2}{3\cdot 4}\Leftrightarrow\sum_{cyc}ab\le\sum_{cyc}a^2;(true)$$

$$\sum_{cyc}\frac{1}{x+a\cot\frac{A}{2}}\overset{BCS}{\ge}\frac{9}{3x+\sum a\cot\frac{A}{2}}\overset{(1)}{\ge}\frac{9}{3x+\frac{2s^2}{3r}}=\frac{3}{x+\frac{2s^2}{9r}}\Leftrightarrow$$

$$\int_0^1\sum_{cyc}\frac{1}{x+a\cot\frac{A}{2}}dx\ge 3\int_0^1\frac{dx}{x+\frac{2s^2}{9r}}\Leftrightarrow$$

$$\sum_{cyc}\log\left(x+a\cot\frac{A}{2}\right)\Big|_0^1\ge 3\log\left(x+\frac{2s^2}{9r}\right)\Big|_0^1\Leftrightarrow$$

$$\sum_{cyc}\log\left(1+\frac{1}{a\cot\frac{A}{2}}\right)\ge 3\log\left(1+\frac{9r}{2s^2}\right)\Leftrightarrow$$

$$\log\left(\prod_{cyc}\left(1+\frac{1}{a}\tan\frac{A}{2}\right)\right)\ge\log\left(1+\frac{9r}{2s^2}\right)^3\Leftrightarrow$$

$$\prod_{cyc}\left(1+\frac{1}{a}\tan\frac{A}{2}\right)\ge\left(1+\frac{9r}{2s^2}\right)^3$$

$\boxed{Pbl.15}$ **In $\Delta ABC$ the following relationship holds:**

$$\left(1+\frac{s^2+r^2cot^2\frac{A}{2}}{cot\frac{A}{2}}\right)\left(1+\frac{s^2+r^2cot^2\frac{B}{2}}{cot\frac{B}{2}}\right)\left(1+\frac{s^2+r^2cot^2\frac{C}{2}}{cot\frac{C}{2}}\right)\ge\left(1+\frac{10rs}{3}\right)^3$$

*Florică Anastase*

**Solution.**

$$\because \sum_{cyc} \tan\frac{A}{2}\tan\frac{B}{2} 1;\ \ \tan\frac{B}{2}\tan\frac{C}{2} = \frac{r}{s}\cot\frac{A}{2}$$

Let be the function: $f(t) = \frac{t}{t^2+1}$ concave, from Jensen inequality:

$$f(x) + f(y) + f(z) \le 3f\left(\frac{x+y+z}{3}\right)$$

For $= \tan\frac{B}{2}\tan\frac{C}{2}; y = \tan\frac{C}{2}\tan\frac{A}{2}; z = \tan\frac{A}{2}\tan\frac{C}{2}$, we get:

$$\sum_{cyc} \frac{\cot\frac{A}{2}}{s^2 + r^2\cot^2\frac{A}{2}} \le \frac{9}{10rs}; (1) \Leftrightarrow \sum_{cyc} \frac{\cot\frac{A}{2}}{1 + \frac{r^2}{s^2}\cos^2\frac{A}{2}} \le \frac{9}{10}\cdot\frac{r}{s} \Leftrightarrow$$

$$\sum_{cyc} \frac{\tan\frac{B}{2}\tan\frac{C}{2}}{1 + \left(\tan\frac{B}{2}\tan\frac{C}{2}\right)^2} \overset{Jensen}{\le} 3\cdot\frac{\frac{1}{3}\sum\tan\frac{B}{2}\tan\frac{C}{2}}{\left(\frac{1}{3}\sum\tan\frac{B}{2}\tan\frac{C}{2}\right)^2 + 1} = \frac{9}{10}$$

Now,

$$\sum_{cyc} \frac{1}{x + \frac{\cot\frac{A}{2}}{s^2 + r^2\cot^2\frac{A}{2}}} \overset{Bergstrom}{\ge} \frac{9}{3x + \sum\frac{\cot\frac{A}{2}}{s^2 + r^2\cot^2\frac{A}{2}}} \ge \frac{9}{3x + \frac{9}{10rs}} = \frac{3}{x + \frac{3}{10rs}} \mid \int_0^1 \Leftrightarrow$$

$$\sum_{cyc} \log\left(x + \frac{\cot\frac{A}{2}}{s^2 + r^2\cot^2\frac{A}{2}}\right)\Bigg|_0^1 \ge 3\log\left(x + \frac{3}{10rs}\right)\Bigg|_0^1 \Leftrightarrow$$

$$\sum_{cyc} \log\left(1 + \frac{s^2 + r^2\cot^2\frac{A}{2}}{\cot\frac{A}{2}}\right) \ge 3\log\left(1 + \frac{10rs}{9}\right) \Leftrightarrow$$

$$\prod_{cyc}\left(1 + \frac{s^2 + r^2\cot^2\frac{A}{2}}{\cot\frac{A}{2}}\right) \ge \left(1 + \frac{10rs}{9}\right)^3; (2)$$

From (1),(2) it follows that:

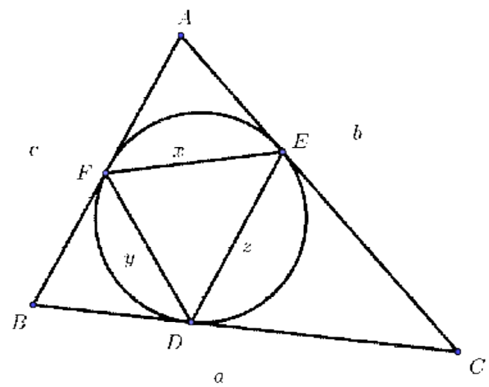
$$\left(1 + \frac{s^2 + r^2\cot^2\frac{A}{2}}{\cot\frac{A}{2}}\right)\left(1 + \frac{s^2 + r^2\cot^2\frac{B}{2}}{\cot\frac{B}{2}}\right)\left(1 + \frac{s^2 + r^2\cot^2\frac{C}{2}}{\cot\frac{C}{2}}\right) \ge \left(1 + \frac{10rs}{3}\right)^3$$

**$Pbl.16$** If $D,E,F$ –are contact points to incircle with $\Delta ABC$ such that $EF=x$, $FD=y, DE=z$ then prove that:

$$\left(\frac{a}{x}\right)^{n+1}+\left(\frac{b}{y}\right)^{n+1}+\left(\frac{c}{z}\right)^{n+1}\geq 3\cdot 2^{n+1},\quad n\in\mathbb{N}$$

*Florică Anastase*

*Solution.*




We have: $\mu(\angle FBD)=\frac{\pi}{2}-\frac{B}{2}, \mu(\angle EDC)=\frac{\pi}{2}-\frac{C}{2}$, so

$$\mu(\angle EDF)=\frac{\pi}{2}-\frac{A}{2}$$

$$EF=2r\cdot sin\left(\frac{\pi}{2}-\frac{A}{2}\right)$$

Let us denote: $\alpha=\frac{\pi}{2}-\frac{A}{2}, \beta=\frac{\pi}{2}-\frac{B}{2}, \gamma=\frac{\pi}{2}-\frac{C}{2}$ and

we have: $\alpha+\beta+\gamma=\pi;\ \alpha,\beta,\gamma\in\left(0,\frac{\pi}{2}\right)$

Hence,

$$\frac{a}{x}=\frac{2RsinA}{2rcos\frac{A}{2}}=\frac{2Rsin\frac{A}{2}cos\frac{A}{2}}{4Rsin\frac{A}{2}sin\frac{B}{2}sin\frac{C}{2}cos\frac{A}{2}}=\frac{1}{2ain\frac{B}{2}sin\frac{C}{2}}$$

$$\sum_{cyc}\frac{a}{x}=\sum_{cyc}\frac{1}{2sin\frac{B}{2}sin\frac{C}{2}}\geq\frac{3}{2}\cdot\sqrt{\frac{1}{sin^2\frac{A}{2}sin^2\frac{B}{2}sin^2\frac{C}{2}}}$$

$$\because sin\frac{A}{2}sin\frac{B}{2}sin\frac{C}{2}\leq\frac{1}{8}\Leftrightarrow 2r\leq R(Euler)\Rightarrow\frac{1}{sin\frac{A}{2}sin\frac{B}{2}sin\frac{C}{2}}\geq 8\Rightarrow$$

$$\sum_{cyc}\frac{a}{x}\geq 6;(1)$$

Now, we want to prove that:

$$\sum_{cyc}\frac{x}{a}\leq\frac{3}{2};(2)\Leftrightarrow\sum_{cyc}sin\frac{B}{2}sin\frac{C}{2}\leq\frac{3}{4}$$

**$Pbl.17$** In acute $\Delta ABC$ the following relationship holds:

$$\sum_{cyc}\frac{r_b+r_c}{a}\cdot\sum_{cyc}\left(\frac{a^2sin^3A}{b}+\frac{a^2cos^3A}{c}\right)\ge\frac{27r}{2}$$

*Florică Anastase*

***Solution.***

$$\sum_{cyc}\frac{r_b+r_c}{a}=\sum_{cyc}\frac{\frac{F}{s-b}+\frac{F}{s-c}}{a}=\sum_{cyc}\left(\frac{F}{a(s-b)}+\frac{F}{a(s-c)}\right)=F\cdot\sum_{cyc}\frac{s-c+s-b}{a(s-b)(s-c)}=$$

$$=F\cdot\sum_{cyc}\frac{a}{a(s-b)(s-c)}=F\cdot\sum_{cyc}\frac{1}{(s-b)(s-c)}=\frac{s}{r};(1)$$

**Now, we must to prove:**

$$\frac{s}{r}\cdot\sum_{cyc}\left(\frac{a^2sin^3A}{b}+\frac{a^2cos^3A}{c}\right)\ge\frac{27r}{2}\Leftrightarrow$$

$$\sum_{cyc}\left(\frac{a^2sin^3A}{b}+\frac{a^2cos^3A}{c}\right)\ge\frac{27r^2}{2s};(2)$$

**We have:**

$$\sum_{cyc}\left(\frac{a^2sin^3A}{b}+\frac{a^2cos^3A}{c}\right)=\sum_{cyc}\left(\frac{a^2sin^4A}{bsinA}+\frac{a^2cos^4A}{ccosA}\right)$$

$$\overset{Bergstrom}{\ge}\sum_{cyc}\frac{(asin^2A+acos^2A)^2}{bsinA+ccosA}=\sum_{cyc}\frac{a^2}{bsinA+ccosA}$$

$$\overset{BCS}{\ge}\sum_{cyc}\frac{a^2}{\sqrt{b^2+c^2}}\ge\sum_{cyc}\frac{a^2}{b+c}\overset{Bergstrom}{\ge}\frac{(a+b+c)^2}{2(a+b+c)}=s$$

$$=r^2\sum_{cyc}\frac{cot^2\frac{A}{2}}{s-a}\overset{Bergstrom}{\ge}r^2\cdot\frac{\left(\sum cot\frac{A}{2}\right)^2}{2s}=\frac{r^2}{2s}\cdot\left(\underbrace{\sum cot\frac{A}{2}}_{\ge3\sqrt{3}}\right)^2\ge\frac{27r^2}{2s}$$

**From $(1),(2)$ it follows that:**

$$\sum_{cyc}\frac{r_b+r_c}{a}\cdot\sum_{cyc}\left(\frac{a^2sin^3A}{b}+\frac{a^2cos^3A}{c}\right)\ge\frac{27r}{2}$$

**$Pbl.18$** **In acute $\Delta ABC, n \in \mathbb{N}, n \geq 2$ the following relationship holds:**

$$\sum_{cyc}\left(1-\sqrt[n]{\sin A}\right) \geq \sum_{cyc}\frac{1-\sin A\sin B}{2n+1-\sin A\sin B}$$

***Florică Anastase***

***Solution.***

$$\sum_{cyc}\left(1-\sqrt[n]{\sin A}\right) \geq \sum_{cyc}\frac{1-\sin A\sin B}{2n+1-\sin A\sin B}$$

$$3-\sum_{cyc}\sqrt[n]{\sin A} \geq \sum_{cyc}\frac{1-\sin A\sin B}{2n+1-\sin A\sin B}$$

$$3-\sum_{cyc}\frac{1-\sin A\sin B}{2n+1-\sin A\sin B} \geq \sum_{cyc}\sqrt[n]{\sin A}$$

$$\sum_{cyc}\left(1-\frac{1-\sin A\sin B}{2n+1-\sin A\sin B}\right) \geq \sum_{cyc}\sqrt[n]{\sin A}$$

$$\sum_{cyc}\frac{2n}{2n+1-\sin A\sin B} \geq \sum_{cyc}\sqrt[n]{\sin A}$$

$$\sum_{cyc}\frac{1}{2n+1-\sin A\sin B} \geq \frac{1}{2n}\sum_{cyc}\sqrt[n]{\sin A};\quad (1)$$

**Let be the function $f:[0,1]\to\mathbb{R}$,**

$$f(x) = 4n-(2n+1)\left(\sqrt[n]{x}+\sqrt[n]{\alpha}\right)+\alpha x\left(\sqrt[n]{x}+\sqrt[n]{\alpha}\right), \alpha\in[0,1]$$

$$f'(x) = -\frac{2n+1}{n\sqrt[n]{x^{n-1}}}+\alpha\left(\sqrt[n]{x}+\sqrt[n]{\alpha}\right)+\frac{\alpha}{n}\sqrt[n]{x}$$

$$f''(x) = \frac{2n+1}{n^2\cdot x\sqrt[n]{x^{n-1}}}+\frac{(n+1)\alpha}{n^2\sqrt[n]{x^{n-1}}}$$

$$f''(x) > 0 \Rightarrow f'_{[0,1]} - \text{increasing}; f'(1) = -2+\alpha\left(\sqrt[n]{\alpha}+1\right)-\frac{1-\alpha}{n}$$

$$\alpha\in[0,1], \alpha\left(\sqrt[n]{\alpha}+1\right)\leq 2 \Rightarrow f'(1)\leq 0 \Rightarrow f_{[0,1]} - \text{decreasing} \Rightarrow f(x)\geq f(1), \forall x\in[0,1]$$

$$f(1) = 4n-(2n+1)\left(\sqrt[n]{\alpha}+1\right)+\alpha\left(\sqrt[n]{\alpha}+1\right)\geq 0, \forall\alpha\in[0,1]$$

$$\text{Let } g(\alpha) = 4n-(2n+1)\left(\sqrt[n]{\alpha}+1\right)+\alpha\left(\sqrt[n]{\alpha}+1\right)\geq 0, \forall\alpha\in[0,1]$$

$$g'(\alpha)=\frac{-(2n+1)+\sqrt[n]{\alpha^{n-1}}\left((n+1)\sqrt[n]{\alpha}+1\right)}{n\sqrt[n]{\alpha^{n-1}}}\le 0,,\forall\alpha\in[0,1]\Rightarrow g-\text{decreasing}$$

$$f(1)=g(\alpha)\ge g(1)=0$$

**We have:**

$$f(\beta)=4n-(2n+1)(\sqrt[n]{\alpha}+\sqrt[n]{\beta})+\alpha\beta(\sqrt[n]{\alpha}+\sqrt[n]{\beta})\ge 0\Leftrightarrow$$

$$4n\ge(2n+1-\alpha\beta)(\sqrt[n]{\alpha}+\sqrt[n]{\beta})\Leftrightarrow\frac{1}{2n+1-\alpha\beta}\ge\frac{\sqrt[n]{\alpha}+\sqrt[n]{\beta}}{4n}$$

**Therefore,**

$$\frac{1}{2n+1-\sin A\sin B}\ge\frac{\sqrt[n]{\sin A}+\sqrt[n]{\sin B}}{4n}\ (and\ analogs);\quad(2)$$

**From $(1),(2)$ we get**

$$\sum_{cyc}(1-\sqrt[n]{\sin A})\ge\sum_{cyc}\frac{1-\sin A\sin B}{2n+1-\sin A\sin B}$$

**$\boxed{Pbl. 19}$ In $\Delta ABC, I$ –incenter, $R_a,R_b,R_c$ –circumradii of $\Delta IAB,\Delta IBC,\Delta IBC$. Prove that:**

$$\frac{a^2\cdot R_b^3R_c^3}{R_a}+\frac{b^2\cdot R_c^3R_a^3}{R_b}+\frac{c^2\cdot R_a^3R_b^3}{R_c}\ge\frac{16R^3F^2}{3}$$

*Florică Anastase*

***Solution.***

$$\because R_aR_bR_c=2R^2r;(1)$$

$$\mu(\angle BIC)=\pi-\frac{B+C}{2}=\frac{\pi}{2}-\frac{A}{2}$$

$$2R_a=\frac{a}{sin\left(\frac{\pi}{2}+\frac{A}{2}\right)}=\frac{2R\cdot sinA}{cos\frac{A}{2}}=4R\cdot sin\frac{A}{2}\Rightarrow R_a=2R\cdot sin\frac{A}{2}$$

$$R_aR_bR_c=8R^3\cdot sin\frac{A}{2}sin\frac{B}{2}sin\frac{C}{2}=2R^2r$$

$$\because\sum_{cyc}a\cdot R_b^2R_c^2=R^2\cdot abc;(2)$$

$$\sum_{cyc} a\cdot R_b^2R_c^2 = 32R^5\cdot\sum_{cyc} sinA\cdot sin^2\frac{B}{2}\cdot sin^2\frac{C}{2} =$$

$$= 64R^5\cdot sin\frac{A}{2}sin\frac{B}{2}sin\frac{C}{2}\left(\sum_{cyc} sin\frac{B}{2}sin\frac{C}{2}cos\frac{A}{2}\right)$$

**But,**

$$\sum_{cyc} sin\frac{B}{2}sin\frac{C}{2}cos\frac{A}{2} = sin\frac{A}{2}\left(sin\frac{B}{2}sin\frac{C}{2}+sin\frac{C}{2}sin\frac{B}{2}\right)+sin\frac{B}{2}sin\frac{C}{2}cos\frac{A}{2} =$$

$$= sin\frac{A}{2}sin\frac{B+C}{2}+sin\frac{B}{2}sin\frac{C}{2}cos\frac{A}{2} = cos\frac{A}{2}\left(cos\frac{B+C}{2}+sin\frac{B}{2}sin\frac{C}{2}\right) =$$

$$= cos\frac{A}{2}cos\frac{B}{2}cos\frac{C}{2}$$

**Hence,**

$$\sum_{cyc} a\cdot R_b^2R_c^2 = 8R^5\cdot sinAsinBsinC = 4R^3F = \frac{4R^3\cdot abc}{4R} = R^2\cdot abc$$

$$\frac{a^2\cdot R_b^3R_c^3}{R_a}+\frac{b^2\cdot R_c^3R_a^3}{R_b}+\frac{c^2\cdot R_a^3R_b^3}{R_c} = \frac{a^2\cdot R_b^4R_c^4}{R_aR_bR_c}+\frac{b^2\cdot R_c^4R_a^2}{R_aR_bR_c}+\frac{c^2\cdot R_a^4R_b^4}{R_aR_bR_c} =$$

$$= \frac{\left(a\cdot R_b^2R_c^2\right)^2}{R_aR_bR_c}+\frac{(b\cdot R_c^2R_a^2)^2}{R_aR_bR_c}+\frac{\left(c\cdot R_a^2R_b^2\right)^2}{R_aR_bR_c}\overset{Bergstrom}{\geq}$$

$$\geq \frac{\left(a\cdot R_b^2R_c^2+b\cdot R_c^2R_a^2+c\cdot R_a^2R_b^2\right)^2}{3R_aR_bR_c}\overset{(1),(2)}{=}\frac{(R^2\cdot abc)^2}{3\cdot 2R^2r} = \frac{R^2\cdot(abc)^2}{6r} =$$

$$= \frac{8R^4F^2}{3r}\overset{Euler}{\geq}\frac{16R^3F^2}{3}$$

$\boxed{Pbl.20}$ ***If $x,y,z>0$ then prove:***

$$\sum_{cyc}\frac{y}{x^3(1+y)}\geq\frac{3}{8}\cdot\left(\frac{15}{x^2+y^2+z^2}-1\right)$$

*Florică Anastase*

***Solution.***

$$\sum_{cyc}\frac{y}{x^3(1+y)}=\sum_{cyc}\frac{\frac{1}{x^3}}{\frac{1+y}{y}}=\sum_{cyc}\frac{\frac{1}{x^3}}{1+\frac{1}{y}}\overset{(H)}{\geq}\frac{\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^3}{3\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+3\right)}=$$

$$=\frac{1}{3}\cdot\frac{\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^3}{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+3}\overset{(1)}{\geq}\frac{1}{3}\cdot\frac{5\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^2-9}{8}=\frac{1}{24}\cdot\left[5\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^2-9\right]\overset{(2)}{\geq}$$

$$\overset{(2)}{\geq}\frac{1}{24}\cdot\left[5\cdot 3\left(\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}\right)-9\right]\overset{(B)}{\geq}$$

$$\overset{(B)}{\geq}\frac{1}{24}\cdot\left(5\cdot\frac{27}{xy+yz+zx}-9\right)\overset{(3)}{\geq}\frac{3}{8}\left(\frac{15}{x^2+y^2+z^2}-1\right)$$

$$From\ t=\frac{1}{x}+\frac{1}{y}+\frac{1}{z};(1)\Leftrightarrow\frac{t^3}{t+3}\geq\frac{5t^2-9}{8}\Leftrightarrow(t-3)^2(t+1)\geq 0,\forall t>0$$

$$(2)\Leftrightarrow\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^2\geq 3\left(\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}\right)\Leftrightarrow\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\geq\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}$$

$$(3)\Leftrightarrow x^2+y^2+z^2\geq xy+yz+zx$$

$$(H)\Leftrightarrow\frac{x^3}{a}+\frac{y^3}{b}+\frac{z^3}{c}\geq\frac{(x+y+z)^3}{3(a+b+c)};(Holder)$$

$$(B)\Leftrightarrow\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}\geq\frac{(x+y+z)^2}{a+b+c};(Bergstrom)$$

**Pbl. 21** ***If $x,y,z>0$ then prove:***

$$\sum_{cyc}\frac{x(y^3+z^3)}{(1+x)y^3z^3}\geq\frac{3}{4}\cdot\left(\frac{15}{x^2+y^2+z^2}-1\right)$$

***Florică Anastase***

***Solution.***

$$\sum_{cyc}\frac{x(y^3+z^3)}{(1+x)y^3z^3}=\sum_{cyc}\frac{\frac{y^3+z^3}{y^3z^3}}{\frac{1+x}{x}}=\sum_{cyc}\frac{\frac{1}{y^3}+\frac{1}{z^3}}{1+\frac{1}{x}}\overset{(H)}{\geq}\frac{2\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^3}{3\left(3+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)}\overset{(1)}{\geq}$$

$$\overset{(1)}{\geq}\frac{2}{3}\cdot\frac{5\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^2-9}{8}=\frac{1}{12}\cdot\left[5\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^2-9\right]\overset{(2)}{\geq}$$

$$\overset{(2)}{\geq}\frac{1}{12}\cdot\left[5\cdot 3\left(\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}\right)-9\right]\overset{(B)}{\geq}$$

$$\overset{(B)}{\geq}\frac{1}{12}\cdot\left(5\cdot\frac{27}{xy+yz+zx}-9\right)\overset{(3)}{\geq}\frac{3}{4}\left(\frac{15}{x^2+y^2+z^2}-1\right)$$

**Pbl. 22** ***If $x, y, z > 0, n \geq 0$ then prove:***

$$\sum_{cyc}\frac{x(ny^3+z^3)}{(1+x)y^3z^3}\geq\frac{3(n+1)}{8}\cdot\left(\frac{15}{x^2+y^2+z^2}-1\right)$$

***Florică Anastase***

***Solution.***

$$\sum_{cyc}\frac{x(ny^3+z^3)}{(1+x)y^3z^3}=\sum_{cyc}\frac{\frac{ny^3+z^3}{y^3z^3}}{\frac{1+x}{x}}=\sum_{cyc}\frac{\frac{1}{y^3}+\frac{n}{z^3}}{1+\frac{1}{x}}=\sum_{cyc}\frac{\frac{1}{y^3}}{1+\frac{1}{x}}+\sum_{cyc}\frac{\frac{n}{z^3}}{1+\frac{1}{x}}\overset{(H)}{\geq}$$

$$\overset{(H)}{\geq}\frac{\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^3}{3\left(3+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)}+\frac{n\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^3}{3\left(3+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)}=\frac{n+1}{3}\cdot\frac{\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^3}{3+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}\overset{(1)}{\geq}$$

$$\overset{(1)}{\geq}\frac{n+1}{3}\cdot\frac{5\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^2-9}{8}=\frac{n+1}{24}\cdot\left[5\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^2-9\right]\overset{(2)}{\geq}$$

$$\overset{(2)}{\geq}\frac{n+1}{24}\cdot\left[5\cdot 3\left(\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}\right)-9\right]\overset{(B)}{\geq}$$

$$\overset{(B)}{\geq}\frac{n+1}{24}\cdot\left(5\cdot\frac{27}{xy+yz+zx}-9\right)\overset{(3)}{\geq}\frac{3(n+1)}{8}\left(\frac{15}{x^2+y^2+z^2}-1\right)$$

**Pbl. 23** ***If $x, y, z > 0, n > 0$ then prove:***

$$\sum_{cyc}\frac{(x^3+y^3)z}{x^3y^3(1+nz)}\geq\frac{3}{4}\cdot\left(\frac{15}{x^2+y^2+z^2}-n^2\right)$$

***Florică Anastase***

***Solution.***

$$\sum_{cyc}\frac{(x^3+y^3)z}{x^3y^3(1+nz)}=\sum_{cyc}\frac{\frac{1}{x^3}+\frac{1}{y^3}}{\frac{1}{z}+n}\overset{(H)}{\geq}\frac{2}{3}\cdot\frac{\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^3}{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+3n}=$$

$$=\frac{2}{3}\cdot\frac{n^3\left(\frac{1}{nx}+\frac{1}{ny}+\frac{1}{nz}\right)^3}{n\left(\frac{1}{nx}+\frac{1}{ny}+\frac{1}{nz}+3\right)}=\frac{2n^2}{3}\cdot\frac{\left(\frac{1}{nx}+\frac{1}{ny}+\frac{1}{nz}\right)^3}{\frac{1}{nx}+\frac{1}{ny}+\frac{1}{nz}+3}\overset{(1)}{\geq}$$

$$\overset{(1)}{\geq}\frac{2n^2}{3}\cdot\frac{5\left(\frac{1}{nx}+\frac{1}{ny}+\frac{1}{nz}\right)^2-9}{8}=\frac{n^2}{12}\cdot\left[5\left(\frac{1}{nx}+\frac{1}{ny}+\frac{1}{nz}\right)^2-9\right]\overset{(2)}{\geq}$$

$$\overset{(2)}{\geq}\frac{1}{12}\cdot\left(5\cdot3\left(\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}\right)-9n^2\right)\overset{(B)}{\geq}\frac{1}{12}\cdot\left(5\cdot\frac{27}{xy+yz+zx}-9n^2\right)\overset{(3)}{\geq}$$

$$\overset{(3)}{\geq}\frac{1}{12}\cdot\left(5\cdot\frac{27}{x^2+y^2+z^2}-9n^2\right)=\frac{3}{4}\cdot\left(\frac{15}{x^2+y^2+z^2}-n^2\right)$$

**Pbl. 24** ***If $x,y,z>0, n,k>0$ then prove:***

$$\sum_{cyc}\frac{(kx^3+y^3)z}{x^3y^3(1+nz)}\geq\frac{3(k+1)}{8}\cdot\left(\frac{15}{x^2+y^2+z^2}-n^2\right)$$

***Florică Anastase***

***Solution.***

$$\sum_{cyc}\frac{(kx^3+y^3)z}{x^3y^3(1+nz)}=\sum_{cyc}\frac{\frac{1}{x^3}+\frac{k}{y^3}}{\frac{1}{z}+n}=\sum_{cyc}\frac{\frac{1}{x^3}}{\frac{1}{z}+n}+\sum_{cyc}\frac{\frac{k}{y^3}}{\frac{1}{z}+n}\overset{(H)}{\geq}$$

$$\overset{(H)}{\geq}\frac{1}{3}\cdot\frac{\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^3}{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+3n}+\frac{k}{3}\cdot\frac{\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^3}{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+3n}=\frac{k+1}{3}\cdot\frac{\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^3}{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+3n}=$$

$$=\frac{k+1}{3}\cdot\frac{n^3\left(\frac{1}{nx}+\frac{1}{ny}+\frac{1}{nz}\right)^3}{n\left(\frac{1}{nx}+\frac{1}{ny}+\frac{1}{nz}+3\right)}\overset{(1)}{\geq}\frac{n^2(k+1)}{3}\cdot\frac{5\left(\frac{1}{nx}+\frac{1}{ny}+\frac{1}{nz}\right)^2-9}{8}=$$

$$=\frac{n^2(k+1)}{24}\cdot\left[5\left(\frac{1}{nx}+\frac{1}{ny}+\frac{1}{nz}\right)^2-9\right]=\frac{k+1}{24}\cdot\left[5\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^2-9n^2\right]\overset{(2)}{\geq}$$

$$\overset{(2)}{\geq}\frac{k+1}{24}\cdot\left[5\cdot 3\left(\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}\right)-9n^2\right]\overset{(B)}{\geq}\frac{k+1}{24}\cdot\left(5\cdot\frac{27}{x^2+y^2+z^2}-9n^2\right)=$$

$$=\frac{3(k+1)}{8}\cdot\left(\frac{15}{x^2+y^2+z^2}-n^2\right)$$

**$\boxed{Pbl.25}$ Prove that:**

$$\frac{F_2}{(F_5-2)^2}+\frac{F_3}{(F_6-2)^2}+\cdots+\frac{F_n}{(F_{n+3}-2)^2}\leq\frac{L_{n+3}+L_{n+1}-5L_0}{5F_3^2(F_{n+2}-F_2)}$$

*Florică Anastase*

*Solution.*

$$S_n=\sum_{k=1}^{n}F_k=F_1+F_2+\cdots+F_n=F_{n+2}-F_2\Rightarrow$$

$$F_{i+3}-2F_2=F_{i+2}+F_{i+1}-2F_2=S_i+S_{i-1},\forall i=1,n$$

$F_5-2F_2=S_2+S_1=F_1+F_2+F_1=2F_1+F_2; F_6-2F_2=S_3+S_2=2F_1+2F_2+F_3$

$$F_{n+3}-2F_2=S_n+S_{n-1}=2F_1+2F_2+\cdots+2F_{n-1}+F_n$$

$$\frac{F_2}{(F_5-2)^2}+\frac{F_3}{(F_6-2)^2}+\cdots+\frac{F_n}{(F_{n+3}-2)^2}=$$

$$=\frac{F_2}{(2F_1+F_2)^2}+\frac{F_3}{(2F_1+2F_2+F_3)^2}+\cdots+\frac{F_n}{(2F_1+2F_2+\cdots+2F_{n-1}+F_n)^2}$$

$$\frac{F_2}{F_1(F_1+F_2)}+\frac{F_3}{(F_1+F_2)(F_1+F_2+F_3)}=\frac{F_2+F_3}{F_1(F_1+F_2+F_3)}$$

*Suppose:*

$$\frac{F_2}{F_1(F_1+F_2)}+\frac{F_3}{(F_1+F_2)(F_1+F_2+F_3)}+\cdots$$
$$+\frac{F_n}{(F_1+F_2+\cdots+F_{n-1})(F_1+F_2+\cdots+F_n)}=\frac{F_2+F_3+\cdots+F_n}{F_1(F_1+F_2+\cdots+F_n)}\Rightarrow$$

$$\frac{F_2}{F_1(F_1+F_2)}+\frac{F_3}{(F_1+F_2)(F_1+F_2+F_3)}+\cdots$$
$$+\frac{F_n}{(F_1+F_2+\cdots+F_{n-1})(F_1+F_2+\cdots+F_n)}+$$

$$+\frac{F_{n+1}}{(F_1+F_2+\cdots+F_n)(F_1+F_2+\cdots+F_{n+1})}=$$

$$=\frac{F_2+F_3+\cdots+F_n}{F_1(F_1+F_2+\cdots+F_n)}+\frac{F_{n+1}}{(F_1+F_2+\cdots+F_n)(F_1+F_2+\cdots+F_{n+1})}=$$

$$=\frac{(F_2+F_3+\cdots+F_n)^2+F_1(F_2+F_3+\cdots+F_n)+F_{n+1}(F_2+F_3+\cdots+F_n)+F_1F_{n+1}}{(F_1+F_2+\cdots+F_n)F_1(F_1+F_2+\cdots+F_{n+1})}$$

$$=\frac{F_2+F_3+\cdots+F_{n+1}}{F_1(F_1+F_2+\cdots+F_{n+1})}$$

***From*** $(x+y)^2\ge 4xy\Rightarrow \frac{1}{xy}\ge\frac{4}{(x+y)^2}$ ***we have:***

$$\frac{F_2}{F_1(F_1+F_2)}\ge\frac{4F_2}{(2F_1+F_2)^2}$$

$$\frac{F_3}{(F_1+F_2)(F_1+F_2+F_3)}\ge\frac{4F_3}{(2F_1+2F_2+F_3)^2}$$

$$\frac{F_n}{(F_1+F_2+\cdots+F_{n-1})(F_1+F_2+\cdots+F_n)}\ge\frac{4F_n}{(2F_1+2F_2+\cdots+2F_{n-1}+F_n)^2}$$

***Adding up relationships, we have:***

$$\frac{4F_2}{(2F_1+F_2)^2}+\frac{4F_3}{(2F_1+2F_2+F_3)^2}+\cdots+\frac{4F_n}{(2F_1+2F_2+\cdots+2F_{n-1}+F_n)^2}\le$$

$$\le\frac{F_2}{F_1(F_1+F_2)}+\frac{F_3}{(F_1+F_2)(F_1+F_2+F_3)}+\cdots$$

$$+\frac{F_n}{(F_1+F_2+\cdots+F_{n-1})(F_1+F_2+\cdots+F_n)}=\frac{F_2+F_3+\cdots+F_n}{F_1(F_1+F_2+\cdots+F_n)}\Rightarrow$$

$$\frac{F_2}{(2F_1+F_2)^2}+\frac{F_3}{(2F_1+2F_2+F_3)^2}+\cdots+\frac{F_n}{(2F_1+2F_2+\cdots+2F_{n-1}+F_n)^2}$$

$$\le\frac{F_2+F_3+\cdots+F_n}{4F_1(F_1+F_2+\cdots+F_n)}$$

$$\frac{F_2}{(2F_1+F_2)^2}+\frac{F_3}{(2F_1+2F_2+F_3)^2}+\cdots+\frac{F_n}{(2F_1+2F_2+\cdots+2F_{n-1}+F_n)^2}$$

$$\le\frac{F_{n+2}-F_3}{F_3^2(F_{n+2}-F_2)}$$

$$\frac{F_2}{(F_5-2)^2}+\frac{F_3}{(F_6-2)^2}+\cdots+\frac{F_n}{(F_{n+3}-2)^2}\le\frac{F_{n+2}-F_3}{F_3^2(F_{n+2}-F_2)}$$

From: $5F_n = L_{n+1} + L_{n-1} \Rightarrow F_{n+2} - F_3 = \frac{L_{n+3}-L_{n+1}-5L_0}{5} \Rightarrow \frac{F_{n+2}-F_3}{F_3^2(F_{n+2}-F_2)} = \frac{L_{n+3}+L_{n+1}-5L_0}{5F_3^2(F_{n+2}-F_2)}$

$$\frac{F_2}{(F_5-2)^2} + \frac{F_3}{(F_6-2)^2} + \cdots + \frac{F_n}{(F_{n+3}-2)^2} \le \frac{L_{n+3}+L_{n+1}-5L_0}{5F_3^2(F_{n+2}-F_2)}$$

$\boxed{Pbl.26}$ **If $(a_n)_{n\ge1}$ is in arithmetic progression with $a_1 > 0, r > 0$ then prove:**

$$\frac{1}{\sqrt{a_1F_1}} + \frac{1}{\sqrt{a_2F_2}} + \cdots + \frac{1}{\sqrt{a_nF_n}} \ge \frac{2n^2}{n + a_nF_{n+2} - r(F_{n+3} - F_3 - F_2) - a_1F_2}$$

*Florică Anastase*

*Solution.*

From A-G-M we have: $\sqrt{x} \le \frac{x+1}{2} \Rightarrow \frac{1}{\sqrt{x}} \ge \frac{2}{x+1}, \forall x \in \mathbb{R}_+^* = (0,\infty)$ then:

$$\frac{1}{\sqrt{a_1F_1}} + \frac{1}{\sqrt{a_2F_2}} + \cdots + \frac{1}{\sqrt{a_nF_n}} \ge \frac{2}{1+a_1F_1} + \frac{2}{1+a_2F_2} + \cdots + \frac{2}{1+a_nF_n} \overset{Bergström}{\ge}$$

$$\ge \frac{2n^2}{n + a_1F_1 + a_2F_2 + \cdots + a_nF_n}$$

From $F_{n+2} = F_{n+1} + F_n;\ F_1 = F_2 = 1$ we have $F_{k+2} = F_{k+1} + F_k, \forall k = \overline{1,n} \Rightarrow$

$$F_{k+2} - F_{k+1} = F_k, \forall k = \overline{1,n} \Rightarrow a_k \cdot F_k = (a_1 + (k-1)r)F_k = (a_1 - r)F_k + krF_k$$

But: $k \cdot F_k = k(F_{k+2} - F_{k+1}) = (k+2)F_{k+2} - (k+1)F_{k+1} - 2(F_{k+2} - F_{k+1}) - F_{k+1} =$

$$= (k+2)F_{k+2} - (k+1)F_{k+1} - 2(F_{k+2} - F_{k+1}) - (F_{k+3} - F_{k+2}) \Rightarrow$$

$$\sum_{k=1}^{n} k \cdot F_k = \sum_{k=1}^{n}[(k+2)F_{k+2} - (k+1)F_{k+1}] - 2\sum_{k=1}^{n}(F_{k+2} - F_{k+1}) - \sum_{k=1}^{n}(F_{k+3} - F_{k+2})$$

$$= (n+2)F_{n+2} - 2F_2 - 2(F_{n+2} - F_2) - (F_{n+3} - F_3) = nF_{n+2} - F_{n+3} + F_3$$

$$\sum_{k=1}^{n} a_k \cdot F_k = (a_1 - r)\sum_{k=1}^{n} F_k + r\sum_{k=1}^{n} k \cdot F_k =$$

$$= (a_1 - r)(F_{n+2} - F_2) + r(nF_{n+2} - F_{n+3} + F_3) =$$

$$= a_nF_{n+2} - r(F_{n+3} - F_3 - F_2) - a_1F_2.$$

So, $\frac{1}{\sqrt{a_1F_1}} + \frac{1}{\sqrt{a_2F_2}} + \cdots + \frac{1}{\sqrt{a_nF_n}} \ge \frac{2n^2}{n+a_nF_{n+2}-r(F_{n+3}-F_3-F_2)-a_1F_2}$

$\boxed{Pbl.27}$ **Prove that:**

$$\frac{1}{\sqrt{3F_1}}+\frac{1}{\sqrt{4F_2}}+\cdots+\frac{1}{\sqrt{(n+2)F_n}}\geq\frac{2n^2}{(1+F_{n+1})n+F_n}, \forall n\geq 2$$

*Florică Anastase*

***Solution.***

***From A-G-M we have:*** $\sqrt{x}\leq\frac{x+1}{2}\Rightarrow\frac{1}{\sqrt{x}}\geq\frac{2}{x+1}, \forall x\in\mathbb{R}_+^*=(0,\infty)$ ***then:***

$$\frac{1}{\sqrt{3F_1}}+\frac{1}{\sqrt{4F_2}}+\cdots+\frac{1}{\sqrt{(n+2)F_n}}\geq\frac{2}{1+3F_1}+\frac{2}{1+4F_2}+\cdots+\frac{2}{1+(n+2)F_n}\overset{Bergström}{\geq}$$

$$\geq\frac{2n^2}{n+3F_1+4F_2+\cdots+(n+2)F_n}$$

***For*** $a_1=3, r=1$ ***we get:*** $3F_1+4F_2+\cdots+(n+2)F_n=(1+F_{n+1})n+F_n$***, then***

$$\frac{1}{\sqrt{3F_1}}+\frac{1}{\sqrt{4F_2}}+\cdots+\frac{1}{\sqrt{(n+2)F_n}}\geq\frac{2n^2}{(1+F_{n+1})n+F_n}, \forall n\geq 2$$

$\boxed{Pbl. 28}$ **Prove that:**

$$\frac{1}{\sqrt{2L_1}}+\frac{1}{\sqrt{3L_2}}+\cdots+\frac{1}{\sqrt{nL_n}}\geq\frac{2n^2}{(1+L_{n+2})n-L_n+1}, \forall n\geq 2$$

*Florică Anastase*

***Solution.***

***From A-G-M we have:*** $\sqrt{x}\leq\frac{x+1}{2}\Rightarrow\frac{1}{\sqrt{x}}\geq\frac{2}{x+1}, \forall x\in\mathbb{R}_+^*=(0,\infty)$ ***then:***

$$\frac{1}{\sqrt{2L_1}}+\frac{1}{\sqrt{3L_2}}+\cdots+\frac{1}{\sqrt{nL_n}}\geq\frac{2}{1+2L_1}+\frac{2}{1+3L_2}+\cdots+\frac{2}{nL_n}\overset{Bergström}{\geq}$$

$$\geq\frac{2n^2}{n+2L_1+3L_2+\cdots+nL_n}$$

***For*** $a_1=2, r=1$ ***we get:*** $2L_1+3L_2+\cdots+nL_n=\sum_{k=1}^{n}(k+1)\cdot L_k=$

$$=(n+1)L_{n+2}-(L_{n+3}-L_3-L_2)-2L_2=nL_{n+2}-L_{n+1}+1$$

***Then***

$$\frac{1}{\sqrt{2L_1}}+\frac{1}{\sqrt{3L_2}}+\cdots+\frac{1}{\sqrt{nL_n}}\geq\frac{2n^2}{(1+L_{n+2})n-L_n+1}, \forall n\geq 2$$

$\boxed{Pbl.29}$ **Prove that:**

$$\sqrt{\frac{F_1^2+F_2^2}{F_1}}+\sqrt{\frac{F_1^2+F_2^2+F_3^2}{F_2}}+\cdots+\sqrt{\frac{F_1^2+F_2^2+\cdots+F_{n+1}^2}{F_n}}\geq\frac{2n^2F_{n+2}}{(n+1)F_{n+2}-F_1}$$

*Florică Anastase*

***Solution.***

**We have:** $F_{i+2}=F_{i+1}+F_i, \forall i\geq 0\Rightarrow F_{i+2}-F_i=F_{i+1}\ \forall i\geq 0\Rightarrow$

$$F_{i+2}\cdot F_{i+1}-F_{i+1}\cdot F_i=F_{i+1}^2\Rightarrow F_2^2+F_3^2+\cdots+F_{k+1}^2=F_{k+2}\cdot F_{k+1}-F_1F_2\Rightarrow$$

$$F_1^2+F_2^2+F_3^2+\cdots+F_{k+1}^2=F_{k+2}\cdot F_{k+1}\Rightarrow\frac{1}{F_1^2+F_2^2+F_3^2+\cdots+F_{k+1}^2}=\frac{1}{F_{k+2}\cdot F_{k+1}}$$

$$\Rightarrow\frac{F_k}{F_1^2+F_2^2+F_3^2+\cdots+F_{k+1}^2}=\frac{F_{k+2}-F_{k+1}}{F_{k+2}\cdot F_{k+1}}=\frac{1}{F_{k+1}}-\frac{1}{F_{k+2}}\Rightarrow$$

$$\sum_{k=1}^{n}\frac{F_k}{F_1^2+F_2^2+F_3^2+\cdots+F_{k+1}^2}=1-\frac{1}{F_{n+2}}$$

**From A-G-M we have:** $\sqrt{x}\leq\frac{x+1}{2}\Rightarrow\frac{1}{\sqrt{x}}\geq\frac{2}{x+1},\forall x\in\mathbb{R}_+^*=(0,\infty)$ **then:**

$$\sqrt{\frac{F_1^2+F_2^2}{F_1}}+\sqrt{\frac{F_1^2+F_2^2+F_3^2}{F_2}}+\cdots+\sqrt{\frac{F_1^2+F_2^2+\cdots+F_{n+1}^2}{F_n}}\geq$$

$$\geq\frac{2}{1+\frac{F_1}{F_1^2+F_2^2}}+\frac{2}{1+\frac{F_2}{F_1^2+F_2^2+F_3^2}}+\cdots+\frac{2}{1+\frac{F_n}{F_1^2+F_2^2+\cdots+F_{n+1}^2}}\overset{Bergström}{\geq}$$

$$\geq\frac{2n^2}{n+\frac{F_1}{F_1^2+F_2^2}+\frac{F_2}{F_1^2+F_2^2+F_3^2}+\cdots+\frac{F_n}{F_1^2+F_2^2+\cdots+F_{n+1}^2}}=$$

$$=\frac{2n^2}{n+1-\frac{1}{F_{n+2}}}=\frac{2n^2F_{n+2}}{(n+1)F_{n+2}-1}$$

$\boxed{Pbl.30}$ **Prove that:**

$$(n+1)^3F_1^2+(n+2)^3F_2^2+\cdots+(3n+1)^3F_{2n+1}^2\geq\frac{n(3F_{2n+3}-1)-F_{2n+2}+2}{2}$$

*Florică Anastase*

***Solution.***

$$(n+1)^3F_1^2+(n+2)^3F_2^2+\cdots+(3n+1)^3F_{2n+1}^2=$$

$$=\frac{[(n+1)F_1]^2}{\frac{1}{n+1}}+\frac{[(n+2)F_2]^2}{\frac{1}{n+2}}+\cdots+\frac{[(3n+1)F_{2n+1}]^2}{\frac{1}{3n+1}}\overset{Bergström}{\geq}$$

$$\geq\frac{[(n+1)F_1+(n+2)F_2+\cdots+(3n+1)F_{2n+1}]^2}{\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{3n+1}}$$

***But:*** $\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{3n+1}<2,\forall n\in\mathbb{N}$ ***from mathematical induction by*** $n\in\mathbb{N}$ ***and***

$$(n+1)F_1+(n+2)F_2+\cdots+(3n+1)F_{2n+1}=$$

$$=n(F_1+F_2+\cdots+F_{2n+1})+1\cdot F_1+2\cdot F_2+\cdots+(2n+1)\cdot F_{2n+1}=$$

$$=n(F_{2n+3}-F_2)+2nF_{2n+3}-F_{2n+2}+2=n(3F_{2n+3}-1)-F_{2n+2}+2$$

$$(n+1)^3F_1^2+(n+2)^3F_2^2+\cdots+(3n+1)^3F_{2n+1}^2\geq\frac{n(3F_{2n+3}-1)-F_{2n+2}+2}{2}$$

$\boxed{Pbl.31}$ **If $x,y,z>0,xyz=1,n\in(0,2]$ then prove:**

$$\sum_{cyc}\frac{(xy+z)(xz+y)}{(x+yz)(1+n(xy+z)(xz+y))}\leq\frac{2}{n}$$

*Florică Anastase*

***Solution.***

$$\sum_{cyc}\frac{(xy+z)(xz+y)}{(x+yz)(1+n(xy+z)(xz+y))}=\sum_{cyc}\frac{xyz(xy+z)(xz+y)}{x(x+yz)(yz+nyz(xy+z)(xz+y))}=$$

$$=\sum_{cyc}\frac{x(y^2+1)(z^2+1)}{(x^2+1)(yz+n(y^2+1)(z^2+1))}=\sum_{cyc}\frac{\frac{x}{1+x^2}}{\frac{y}{1+y^2}\cdot\frac{z}{1+z^2}+n}\quad(1)$$

$$Let: a=\frac{x}{1+x^2},b=\frac{y}{1+y^2},c=\frac{z}{1+z^2}\text{ and }a,b,c\in\left(0,\frac{1}{2}\right)$$

$$We\ must\ show:\ \frac{a}{bc+n}+\frac{b}{ca+n}+\frac{c}{ab+n}\leq\frac{2}{n}\quad(2)$$

$$We\ can\ consider\ 0\leq a\leq b\leq c\leq\frac{1}{2}\overset{(1),(2)}{\Leftrightarrow}$$

$$\frac{a}{bc+n}+\frac{b}{ca+n}+\frac{c}{ab+n}\le\frac{a}{ab+n}+\frac{b}{ab+n}+\frac{c}{ab+n}\overset{c<1}{\lneqq}\frac{a+b+1}{ab+n}\overset{?}{\lneqq}\frac{2}{n}\leftrightarrow$$

$n(a+b+1)\le 2ab+2n\leftrightarrow(2-n)ab+n(1-a)(1-b)\ge 0$ ***true for***

$$a,b,c\in\left(0,\frac{1}{2}\right),n\in(0,2]$$

$\boxed{Pbl.32}$ **If $a,b>0,a+b\in\left(\frac{1}{\pi},\frac{2}{\pi}\right)$, then:**

$$a^a\cdot b^b\cdot\left(1+cos\frac{1}{a}\right)^a\cdot\left(1+cos\frac{1}{b}\right)^b\le\left(a\left(1+cos\frac{1}{a}\right)+b\left(1+cos\frac{1}{b}\right)\right)^{a+b}$$

***Florică Anastase***

***Solution.***

$$Let: f:\left(\frac{\pi}{2},\pi\right)\to R,f(x)=log\left(\frac{x}{1+cosx}\right),f'(x)=\frac{1}{x}+\frac{sinx}{1+cosx},f''(x)$$
$$=\frac{x^2-cosx-1}{x^2(1+cosx)}$$

$$Let\ h:\left(\frac{\pi}{2},\pi\right)\to R,h(x)=x^2-cosx-1,h'(x)=2x+sinx>0\to h(x)>h\left(\frac{\pi}{2}\right)$$
$$=\pi+1>0\to f''(x)>0,\forall x\in\left(\frac{\pi}{2},\pi\right)\to f\ is\ convex.$$

$$From\ Jensen\ inequality\to f\left(\frac{1}{a+b}\right)=f\left(\frac{a}{a+b}\cdot\frac{1}{a}+\frac{b}{a+b}\cdot\frac{1}{b}\right)\le\frac{af\left(\frac{1}{a}\right)+bf\left(\frac{1}{b}\right)}{a+b}$$

$$\leftrightarrow$$

$$log\left(\frac{1}{(a+b)\cdot\left(1+cos\left(\frac{1}{a+b}\right)\right)}\right)\le\frac{a\cdot log\left(\frac{1}{a\left(1+cos\frac{1}{a}\right)}\right)+b\cdot log\left(\frac{1}{b\left(1+cos\frac{1}{b}\right)}\right)}{a+b}\leftrightarrow$$

$$a^a\cdot b^b\cdot\left(1+cos\frac{1}{a}\right)^a\cdot\left(1+cos\frac{1}{b}\right)^b\le(a+b)\cdot\left(\left(1+cos\left(\frac{1}{a+b}\right)\right)\right)^{a+b}\overset{cosx-convexe}{\lneqq}$$
$$\le\left(a\left(1+cos\frac{1}{a}\right)+b\left(1+cos\frac{1}{b}\right)\right)^{a+b}$$

$\boxed{Pbl.33}$

*If* $a, b, c, m, n > 0$ *then*:

$$\sum_{cyc}\frac{8a}{ma^2+nbc}\le(m+n)\left(\frac{1}{m^2}+\frac{1}{n^2}\right)\left(\sum_{cyc}\frac{a}{bc}\right)$$

*Florică Anastase*

***Solution.***

$$From: (ma^2+nbc)^2\ge 4mna^2bc\to\frac{a}{ma^2+nbc}\le\frac{ma^2+nbc}{4mnabc}\to$$

$$\sum_{cyc}\frac{a}{ma^2+nbc}\le\frac{1}{4mn}\sum_{cyc}\frac{ma^2+nbc}{abc}\le\frac{1}{4mn}\sum_{cyc}\frac{(m+n)a^2}{abc}=\frac{m+n}{4mn}\sum_{cyc}\frac{a}{bc}\quad(i)$$

$$\frac{m+n}{4mn}=\frac{m+n}{4}\cdot\frac{1}{mn}\overset{AGM}{\le}\frac{m+n}{4}\cdot\left(\frac{m+n}{2mn}\right)^2$$

$$=\frac{m+n}{4m^2n^2}\cdot\left(\frac{m+n}{2}\right)^2\overset{AGM}{\le}\frac{m+n}{8}\cdot\left(\frac{m^2+n^2}{m^2n^2}\right)\quad(ii)$$

***From (i),(ii) we have:***

$$\sum_{cyc}\frac{8a}{ma^2+nbc}\le(m+n)\left(\frac{1}{m^2}+\frac{1}{n^2}\right)\left(\sum_{cyc}\frac{a}{bc}\right)$$

$\boxed{Pbl.34}$ **If** $a_1, a_2, \dots, a_n > 0, n\in\mathbb{N}, n > 1$***.Then:***

$$\sum_{cyc}log_{1+a_1a_2}\left(1+a_2^{1+a_2}\right)\left(1+a_3^{1+a_3}\right)\ge 2n$$

*Florică Anastase*

***Solution.***

$$1+a_i^{1+a_i}=1+(1+a_i-1)^{1+a_i}\overset{Bernoulli}{\ge}1+a_i^2\Rightarrow$$

$$\left(1+a_i^{1+a_i}\right)\left(1+a_j^{1+a_j}\right)\ge\left(1+a_i^2\right)\left(1+a_j^2\right)\ge(1+a_ia_j)^2\Rightarrow$$

$$\sum_{cyc}log_{1+a_1a_2}\left(1+a_2^{1+a_2}\right)\left(1+a_3^{1+a_3}\right)\ge 2\sum_{cyc}log_{1+a_1a_2}(1+a_2a_3)\overset{Am-Gm}{\ge}$$

$$\geq 2n\sqrt[n]{\prod_{cyc} log_{1+a_1a_2}(1+a_2a_3)} \geq 2n$$

**Pbl. 35** If $a,b,c>0$, then

$$\left(1+a^{1+a^{1+a}}\right)\left(1+b^{1+b^{1+b}}\right)\left(1+c^{1+c^{1+c}}\right)\geq 8a^bb^cc^a$$

*Florică Anastase*

**Solution.**

$$1+a^{1+a}=1+(1+a-1)^{1+a}\overset{Bernoulli}{\geq} 1+a^2 \text{ and anlogous}$$

$$1+b^{1+b}\geq 1+b^2,\quad 1+c^{1+c}\geq 1+c^2$$

$$1+a^{1+a^{1+a}}\geq 1+a^{1+a^2}\overset{Am-Gm}{\geq} 1+a^{2a}\overset{Am-Gm}{\geq} 2\sqrt{a^{2a}}=2a^a \text{ and analogous}$$

$$1+b^{1+b^{1+b}}\geq 2b^b,\quad 1+c^{1+c^{1+c}}\geq 2c^c$$

$$\left(1+a^{1+a^{1+a}}\right)\left(1+b^{1+b^{1+b}}\right)\left(1+c^{1+c^{1+c}}\right)\geq 8a^ab^bc^c \ldots\ldots (1)$$

$$a^ab^bc^c\overset{?}{\geq} a^bb^cc^a \leftrightarrow (a-b)\log(a)+(b-c)\log(b)+(c-a)\log(c)\geq 0$$

$$\therefore \text{Let } 0<a\leq b\leq c\to a-b<b-c$$

$$\text{and } \log(a)\leq \log(b)\overset{Cebyshev}{\Rightarrow}$$

$$(a-b)\log(a)-(b-c)\log(b)\geq \frac{1}{2}(a-c)\log(ab)=\log\sqrt{ab}$$

$$(a-b)\log(a)+(b-c)\log(b)+(c-a)\log(c)\geq (a-c)\log\left(\frac{\sqrt{ab}}{c}\right)\geq 0 \ldots (2)$$

From (1) and (2) we have: $\left(1+a^{1+a^{1+a}}\right)\left(1+b^{1+b^{1+b}}\right)\left(1+c^{1+c^{1+c}}\right)\geq 8a^bb^cc^a$

**Pbl. 36** If $0<a_1\leq a_2\leq\ldots\leq a_n, n>0$, then prove:

$$\frac{a_1\cdot a_n}{n}\left(\sum_{k=1}^{n}\frac{1}{a_k}\right)^2\leq \sum_{k=1}^{n}\left(\frac{a_1+a_n}{a_k}-1\right)$$

*Florică Anastase*

**Solution.**

$$0 \geq \sum_{k=1}^{n} \frac{1}{a_k} \cdot (a_1 - a_k)(a_n - a_k) = \sum_{k=1}^{n} \frac{1}{a_k} \cdot (a_1 \cdot a_n - a_k \cdot a_n - a_1 \cdot a_k + a_k{}^2) =$$

$$= \sum_{k=1}^{n} \left(\frac{a_1 \cdot a_n}{a_k} - a_1 - a_n + a_k\right) = \sum_{k=1}^{n} \left(\frac{a_1 \cdot a_n}{a_k}\right) - n(a_1 + a_n) + \sum_{k=1}^{n} a_k \Rightarrow$$

$$a_1 \cdot a_n \cdot \sum_{k=1}^{n} \frac{1}{a_k} \leq \left(n(a_1 + a_n) - \sum_{k=1}^{n} a_k\right) \overset{Am-Gm}{\leq} \left(n(a_1 + a_n) - n\sqrt[n]{\prod a_k}\right)$$

$$\frac{a_1 \cdot a_n}{n} \cdot \sum_{k=1}^{n} \frac{1}{a_k} \leq a_1 + a_n - \sqrt[n]{\prod a_k} \overset{Am-Hm}{\leq} a_1 + a_n - \frac{n}{\sum_{k=1}^{n} \frac{1}{a_k}}$$

$$\frac{a_1 \cdot a_n}{n} \cdot \left(\sum_{k=1}^{n} \frac{1}{a_k}\right)^2 \leq (a_1 + a_n) \sum_{k=1}^{n} \frac{1}{a_k} - n$$

$$\frac{a_1 \cdot a_n}{n} \cdot \left(\sum_{k=1}^{n} \frac{1}{a_k}\right)^2 \leq \sum_{k=1}^{n} \left(\frac{a_1 + a_n}{a_k} - 1\right)$$

$\boxed{Pbl. 37}$ ***In acute $\Delta ABC$ the following relationship holds:***

$$\sum_{cyc} \left(\frac{a^2 sin^3 A}{b} + \frac{a^2 cos^3 A}{c}\right) \geq \frac{27r^2}{2s}$$

***Florică Anastase***

**Solution.**

$$\sum_{cyc} \left(\frac{a^2 sin^3 A}{b} + \frac{a^2 cos^3 A}{c}\right) = \sum_{cyc} \left(\frac{a^2 sin^4 A}{bsinA} + \frac{a^2 cos^4 A}{ccosA}\right)$$

$$\overset{Bergstrom}{\geq} \sum_{cyc} \frac{(asin^2 A + acos^2 A)^2}{bsinA + ccosA} = \sum_{cyc} \frac{a^2}{bsinA + ccosA}$$

$$\overset{BCS}{\geq} \sum_{cyc} \frac{a^2}{\sqrt{b^2 + c^2}} \geq \sum_{cyc} \frac{a^2}{b + c} \overset{Bergstrom}{\geq} \frac{(a + b + c)^2}{2(a + b + c)} = s$$

$$= r^2 \sum_{cyc} \frac{\cot^2 \frac{A}{2}}{s-a} \overset{Bergstrom}{\geq} r^2 \cdot \frac{\left(\sum \cot \frac{A}{2}\right)^2}{2s} = \frac{r^2}{2s} \cdot \left(\underbrace{\sum \cot \frac{A}{2}}_{\geq 3\sqrt{3}}\right)^2 \geq \frac{27r^2}{2s}$$

**Pbl. 38** ***If $a, b, c \in (0,1)$ or $a, b, c \in (1, \infty)$, $ab + bc + ca = abc$ then:***

$$2 \cdot \sqrt[4]{\left(\prod_{cyc} \tan^{-1} a\right)\left(\sum_{cyc} \tan^{-1} a\right)} \leq \tan^{-1}\left(\frac{\sqrt{\left(\sum_{cyc} a^2\right)\left(\sum_{cyc}(1-a)^2\right)}}{1-abc}\right)$$

***Florică Anastase***

***Solution.***

$$2\sqrt[4]{\tan^{-1}a \cdot \tan^{-1}b \cdot \tan^{-1}c(\tan^{-1}a + \tan^{-1}b + \tan^{-1}c)}$$

$$\overset{Am-Gm}{\leq} 2 \cdot \frac{2(\tan^{-1}a + \tan^{-1}b + \tan^{-1}c)}{4}$$

$$= \tan^{-1}\left(\frac{a+b+c-abc}{1-ab-bc-ca}\right) = \tan^{-1}\left(\frac{a+b+c-ab-bc-ca}{1-abc}\right)$$

$$= \tan^{-1}\left(\frac{a(1-b)+b(1-c)+c(1-a)}{1-abc}\right)$$

***Let be the function:*** $f(x) = \tan^{-1}x, f'(x) = \frac{1}{1+x^2} > 0 \forall x \in \mathbb{R} \Rightarrow f$ ***–increasing***

$$a(1-b)+b(1-c)+c(1-a) \overset{C.B.S}{\leq} \sqrt{(a^2+b^2+c^2)((1-a)^2+(1-b)^2+(1-c)^2)}$$

***If*** $a, b, c \in (0,1) \Rightarrow \begin{cases} 1-a > 0 \\ 1-b > 0 \\ 1-c > 0 \end{cases}$ ***and*** $1-abc > 0 \Rightarrow \frac{\sum a(1-b)}{1-abc} > 0$

***If*** $a, b, c \in (1, \infty)\,) \Rightarrow \begin{cases} 1-a < 0 \\ 1-b < 0 \\ 1-c < 0 \end{cases}$ ***and*** $1-abc < 0 \Rightarrow \frac{\sum a(1-b)}{1-abc} > 0$

**Pbl. 39**

$$If\ a, b, c \in \left(0, \frac{1}{2}\right), n \in \mathbb{N}, a^{n+1} + b^{n+1} + c^{n+1} = 1, then:$$

$$\frac{(1+a^n)^a(1+b^n)^b(1+c^n)^c}{(a+b^{n+1}+c^{n+1})(b+c^{n+1}+a^{n+1})(c+a^{n+1}+b^{n+1})}\le\left(\frac{1+a+b+c}{2a+2b+2c-1}\right)^{a+b+c}$$

*Florică Anastase*

***Solution***

$$Because\ a,b,c\in\left(0,\frac{1}{2}\right), n\in\mathbb{N},\to a^{n+1}<a^n<a<\frac{1}{2}\ and\ analogous\ \to a+b+c\in\left(0,\frac{3}{2}\right)$$

$$Hence\ a^n(a+b+c)<a(a+b+c)<\frac{3}{4}\ and\ analogous$$

$$Hence\ a+b^{n+1}+c^{n+1}\ge 1+a-\frac{a}{a+b+c}\ and\ analogous$$

$$(a+b^{n+1}+c^{n+1})(b+c^{n+1}+a^{n+1})(c+a^{n+1}+b^{n+1})$$

$$\ge\left(1+a-\frac{a}{a+b+c}\right)\left(1+b-\frac{b}{a+b+c}\right)\left(1+c-\frac{c}{a+b+c}\right)$$

$$\ge\left(2-\frac{1}{a+b+c}\right)^a\left(2-\frac{1}{a+b+c}\right)^b\left(2-\frac{1}{a+b+c}\right)^c$$

$$=\left(\frac{2(a+b+c)-1}{a+b+c}\right)^{a+b+c}\leftrightarrow$$

$$\frac{(2(a+b+c)-1)^{a+b+c}}{(a+b^{n+1}+c^{n+1})(b+c^{n+1}+a^{n+1})(c+a^{n+1}+b^{n+1})}\le(a+b+c)^{a+b+c}\leftrightarrow$$

$$\left(\frac{a+b+c+1}{a+b+c}\right)^{a+b+c}\cdot\frac{1}{(a+b^{n+1}+c^{n+1})(b+c^{n+1}+a^{n+1})(c+a^{n+1}+b^{n+1})}$$

$$\le\left(\frac{a+b+c+1}{2(a+b+c)-1}\right)^{a+b+c}\leftrightarrow$$

$$\left(\frac{a+a^{n+1}+b+b^{n+1}+c+c^{n+1}}{a+b+c}\right)^{a+b+c}$$

$$\cdot\frac{1}{(a+b^{n+1}+c^{n+1})(b+c^{n+1}+a^{n+1})(c+a^{n+1}+b^{n+1})}$$

$$\le\left(\frac{a+b+c+1}{2(a+b+c)-1}\right)^{a+b+c}\leftrightarrow$$

$$\frac{(1+a^n)^a(1+b^n)^b(1+c^n)^c}{(a+b^{n+1}+c^{n+1})(b+c^{n+1}+a^{n+1})(c+a^{n+1}+b^{n+1})} \le \left(\frac{1+a+b+c}{2a+2b+2c-1}\right)^{a+b+c}$$

**Pbl. 40** **If** $a_i > 1, i = \overline{1, n+1}, n \in \mathbb{N}$ **prove:**

$$\sum_{cyc} log^n_{a_1}(a_2 a_3 \cdot \ldots \cdot a_{n+1}) \ge n^{n+1}$$

***Florică Anastase***

***Solution:***

$$log^n_{a_1}(a_2 a_3 \cdot \ldots \cdot a_{n+1}) = \left(log_{a_1} a_2 + log_{a_1} a_3 + \cdots + log_{a_1} a_{n+1}\right)^n \overset{AM-GM}{\ge}$$

$$\ge n^n \cdot log_{a_1} a_2 \, log_{a_1} a_3 \cdot \ldots \cdot log_{a_1} a_{n+1}$$

$$\sum_{cyc} log^n_{a_1}(a_2 a_3 \cdot \ldots \cdot a_{n+1}) \ge n^n \sum_{cyc} log_{a_1} a_2 \, log_{a_1} a_3 \cdot \ldots \cdot log_{a_1} a_{n+1} \overset{AM-GM}{\ge}$$

$$\ge n^n \cdot n \cdot \sqrt[n]{log_{a_1} a_2 \, log_{a_2} a_1 \cdot \ldots \cdot log_{a_n} a_{n+1} \, log_{a_{n+1}} a_n} = n^{n+1}$$

**Pbl. 41**

***If*** $a, b, c \in (0,1)$ ***and*** $a + b + c = 1$, ***then:***

$$(b+c)^a \cdot (c+a)^b \cdot (a+b)^c \cdot (\sqrt{a} + \sqrt{b} + \sqrt{c})^2 \le 2$$

***Florică Anastase***

***Solution.***

$$Let: f: (0,1) \to R, f(x) = x \cdot log(1-x^2), f'(x) = log(1-x^2) - \frac{2x}{1-x^2}, f''(x)$$

$$= -\frac{2x \cdot (3-x^2)}{(1-x^2)^2} < 0, \forall x \in (0,1) \to f'' concave.$$

***From Jensen inequality:***

$$f\left(\frac{1}{\sqrt{a}+\sqrt{b}+\sqrt{c}}\right) = f\left(\frac{a+b+c}{\sqrt{a}+\sqrt{b}+\sqrt{c}}\right) \ge \frac{\sqrt{a} \cdot f(\sqrt{a}) + \sqrt{b} \cdot f(\sqrt{b}) + \sqrt{c} \cdot f(\sqrt{c})}{\sqrt{a}+\sqrt{b}+\sqrt{c}} \leftrightarrow$$

$$\frac{1}{\sqrt{a}+\sqrt{b}+\sqrt{c}}\cdot log\left(1-\frac{1}{(\sqrt{a}+\sqrt{b}+\sqrt{c})^2}\right)$$
$$\geq \frac{a\cdot log(1-a)+b\cdot log(1-b)+c\cdot log\,(1-c)}{\sqrt{a}+\sqrt{b}+\sqrt{c}} \leftrightarrow$$
$$\frac{(\sqrt{a}+\sqrt{b}+\sqrt{c})^2-(a+b+c)}{(\sqrt{a}+\sqrt{b}+\sqrt{c})^2}\geq log\big((1-a)^a\cdot(1-b)^b\cdot(1-c)^c\big)\leftrightarrow$$
$$(b+c)^a\cdot(c+a)^b\cdot(a+b)^c\cdot(\sqrt{a}+\sqrt{b}+\sqrt{c})^2\leq 2(\sqrt{ab}+\sqrt{bc}+\sqrt{ca})\leftrightarrow$$
$$(b+c)^a\cdot(c+a)^b\cdot(a+b)^c\cdot(\sqrt{a}+\sqrt{b}+\sqrt{c})^2\leq 2(a+b+c)\leftrightarrow$$
$$(b+c)^a\cdot(c+a)^b\cdot(a+b)^c\cdot(\sqrt{a}+\sqrt{b}+\sqrt{c})^2\leq 2$$

**Pbl. 42**

*If* $a_1, a_2, \dots, a_n > 0$, *then:*

$$\prod_{i=1}^{n}\left(1+a_i^{1+a_i^{1+a_i}}\right)\geq 2^n\left(\prod_{i=1}^{n}a_i\right)^{\frac{1}{n}\sum_{i=1}^{n}a_i}$$

***Florică Anastase***

***Solution.***

$$1+a_i^{1+a_i}=1+(1+a_i-1)^{1+a_i}\overset{Benoulli}{\geq}1+a_i^2\rightarrow$$

$$1+a_i^{1+a_i^{1+a_i}}\geq 1+a_i^{1+a_i^2}\overset{Am-Gm}{\geq}2a_i^{a_i}\rightarrow\prod_{i=1}^{n}\left(1+a_i^{1+a_i^{1+a_i}}\right)\geq 2^n\prod_{i=1}^{n}a_i^{a_i}\ \dots\dots(1^\circ)$$

$$We\ must\ show:\ \prod_{i=1}^{n}a_i^{a_i}\geq\left(\prod_{i=1}^{n}a_i\right)^{\frac{1}{n}\sum_{i=1}^{n}a_i}\leftrightarrow\sum_{i=1}^{n}a_i log(a_i)$$
$$\geq\frac{1}{n}\left(\sum_{i=1}^{n}a_i\right)\left(\sum_{i=1}^{n}log(a_i)\right)\ true,$$

***Cebyshev inequalities for sequences*** $(a_i)_{i\geq1}, (log\,(a_i))_{i\geq1}$**......**$(2^\circ)$

***From*** $(1^\circ), (2^\circ)$ ***we have:***

$$\prod_{i=1}^{n}\left(1+a_i^{1+a_i^{1+a_i}}\right)\geq 2^n\left(\prod_{i=1}^{n}a_i\right)^{\frac{1}{n}\sum_{i=1}^{n}a_i}$$

**Pbl. 43** **If $a,b,c>1$, then:**

$$\sum_{cyc}\log_{a+b}\left(1+b^{b+1}\right)\left(1+c^{c+1}\right)\geq 6(a+b)^{c-b}(b+c)^{a-c}(c+a)^{b-a}$$

*Florică Anastase*

***Solution:***

***From Bernoulli's inequality, we have:***

$$\begin{cases}a^{a+1}=(1+a-1)^{a+1}\geq a^2\\ b^{b+1}=(1+b-1)^{b+1}\geq b^2\\ c^{c+1}=(1+c-1)^{c+1}\geq c^2\end{cases}$$

$$\to\begin{cases}\left(1+a^{a+1}\right)\left(1+b^{b+1}\right)\geq(1+a^2)(1+b^2)\geq(a+b)^2\\ \left(1+b^{b+1}\right)\left(1+c^{c+1}\right)\geq(1+b^2)(1+c^2)\geq(b+c)^2\\ \left(1+c^{c+1}\right)\left(1+a^{a+1}\right)\geq(1+c^2)(1+a^2)\geq(c+a)^2\end{cases}\to$$

$$\sum_{cyc}\log_{a+b}\left(1+b^{b+1}\right)\left(1+c^{c+1}\right)\geq\sum_{cyc}\log_{a+b}(b+c)^2=$$

$$=2\sum_{cyc}\log_{a+b}(b+c)\overset{Am-Gm}{\geq}6\cdot\sqrt[3]{\prod_{cyc}\log_{a+b}(b+c)}=6\quad(i)$$

$\therefore\ x^xy^yz^z\geq x^zy^xz^y,\forall x,y,z>1\leftrightarrow(z-x)\ln x+(x-y)\ln y+(y-z)\ln z\leq 0$

$$If\ 1\leq x\leq y\leq z\to(\ln x\leq\ln y\leq\ln z,z-x\geq x-y)\overset{Chebyshev's}{\Rightarrow}$$

$$(z-x)\ln x+(x-y)\ln y\leq\frac{1}{2}(z-y)\ln(xy)=(z-y)\ln\sqrt{xy}$$

$$\to(z-x)\ln x+(x-y)\ln y+(y-z)\ln z\leq(z-y)\ln\sqrt{xy}+(y-z)\ln z$$

$$=(z-y)\ln\frac{\sqrt{xy}}{z}\leq 0\ \therefore$$

$$From: x=b+c,y=c+a,z=a+b\to(a+b)^{c-b}(b+c)^{a-c}(c+a)^{b-a}\leq 1\quad(ii)$$

$$From\ (i),(ii) \to \sum_{cyc} log_{a+b}(1+b^{b+1})(1+c^{c+1}) \ge 6(a+b)^{c-b}(b+c)^{a-c}(c+a)^{b-a}$$

**Pbl. 44** ***If*** $a,b,c \in (0,1), n \in \mathbb{N}, n \ge 2$ ***then prove:***

$$\sum_{cyc}\left(1-\sqrt[n]{sina}\right) \ge \sum_{cyc}\frac{1-sinasinb}{2n+1-sinasinb}$$

***Florică Anastase***

***Solution.***

$$\sum_{cyc}\left(1-\sqrt[n]{sina}\right) \ge \sum_{cyc}\frac{1-sinasinb}{2n+1-sinasinb}$$

$$3-\sum_{cyc}\sqrt[n]{sina} \ge \sum_{cyc}\frac{1-sinasinb}{2n+1-sinasinb}$$

$$3-\sum_{cyc}\frac{1-sinasinb}{2n+1-sinasinb} \ge \sum_{cyc}\sqrt[n]{sina}$$

$$\sum_{cyc}\left(1-\frac{1-sinasinb}{2n+1-sinasinb}\right) \ge \sum_{cyc}\sqrt[n]{sina}$$

$$\sum_{cyc}\frac{2n}{2n+1-sinasinb} \ge \sum_{cyc}\sqrt[n]{sina}$$

$$\sum_{cyc}\frac{1}{2n+1-sinasinb} \ge \frac{1}{2n}\sum_{cyc}\sqrt[n]{sina};\quad (1)$$

***Let be the function*** $f:[0,1] \to \mathbb{R}$,

$$f(x) = 4n-(2n+1)\left(\sqrt[n]{x}+\sqrt[n]{\alpha}\right)+\alpha x\left(\sqrt[n]{x}+\sqrt[n]{\alpha}\right), \alpha \in [0,1]$$

$$f'(x) = -\frac{2n+1}{n\sqrt[n]{x^{n-1}}}+\alpha\left(\sqrt[n]{x}+\sqrt[n]{\alpha}\right)+\frac{\alpha}{n}\sqrt[n]{x}$$

$$f''(x) = \frac{2n+1}{n^2\cdot x\sqrt[n]{x^{n-1}}}+\frac{(n+1)\alpha}{n^2\sqrt[n]{x^{n-1}}}$$

$$f''(x) > 0 \Rightarrow f'_{[0,1]} - increasing; f'(1) = -2+\alpha\left(\sqrt[n]{\alpha}+1\right)-\frac{1-\alpha}{n}$$

$\alpha \in [0,1], \alpha\left(\sqrt[n]{\alpha}+1\right) \le 2 \Rightarrow f'(1) \le 0 \Rightarrow f_{[0,1]}$ –decreasing $\Rightarrow f(x) \ge f(1), \forall x \in [0,1]$

$$f(1) = 4n - (2n+1)\left(\sqrt[n]{\alpha}+1\right) + \alpha\left(\sqrt[n]{\alpha}+1\right) \ge 0, \forall \alpha \in [0,1]$$

Let $g(\alpha) = 4n - (2n+1)\left(\sqrt[n]{\alpha}+1\right) + \alpha\left(\sqrt[n]{\alpha}+1\right) \ge 0, \forall \alpha \in [0,1]$

$$g'(\alpha) = \frac{-(2n+1) + \sqrt[n]{\alpha^{n-1}}\left((n+1)\sqrt[n]{\alpha}+1\right)}{n\sqrt[n]{\alpha^{n-1}}} \le 0,, \forall \alpha \in [0,1] \Rightarrow g - decreasing$$

$$f(1) = g(\alpha) \ge g(1) = 0$$

**We have:**

$$f(\beta) = 4n - (2n+1)\left(\sqrt[n]{\alpha}+\sqrt[n]{\beta}\right) + \alpha\beta\left(\sqrt[n]{\alpha}+\sqrt[n]{\beta}\right) \ge 0 \Leftrightarrow$$

$$4n \ge (2n+1-\alpha\beta)\left(\sqrt[n]{\alpha}+\sqrt[n]{\beta}\right) \Leftrightarrow \frac{1}{2n+1-\alpha\beta} \ge \frac{\sqrt[n]{\alpha}+\sqrt[n]{\beta}}{4n}$$

**Therefore,**

$$\frac{1}{2n+1-sinasinb} \ge \frac{\sqrt[n]{sina}+\sqrt[n]{sinb}}{4n} \ (and\ analogs); \ (2)$$

**From** $(1),(2)$ **we get**

$$\sum_{cyc}\left(1-\sqrt[n]{sina}\right) \ge \sum_{cyc}\frac{1-sinasinb}{2n+1-sinasinb}$$

$\boxed{Pbl.45}$ **If $a,b,c > 0$ then:**

$$\frac{4}{ac+b\sqrt{a}} + \frac{4}{ab+c\sqrt{b}} + \frac{4}{bc+a\sqrt{c}} \le \frac{1+\sqrt{a}}{bc} + \frac{1+\sqrt{b}}{ca} + \frac{1+\sqrt{c}}{ab}$$

***Florică Anastase***

***Solution.***

$$\begin{cases} (a\sqrt{b}+c)^2 \ge 4ac\sqrt{b} \\ (b\sqrt{c}+a)^2 \ge 4ab\sqrt{c} \\ (c\sqrt{a}+b)^2 \ge 4bc\sqrt{a} \end{cases} \to \begin{cases} \dfrac{4\sqrt{a}}{c\sqrt{a}+b} \le \dfrac{c\sqrt{a}+b}{bc} \\ \dfrac{4\sqrt{b}}{a\sqrt{b}+c} \le \dfrac{a\sqrt{b}+c}{ac} \\ \dfrac{4\sqrt{c}}{b\sqrt{c}+a} \le \dfrac{b\sqrt{c}+a}{ab} \end{cases} \to \begin{cases} \dfrac{4}{ac+b\sqrt{a}} \le \dfrac{c\sqrt{a}+b}{abc} \\ \dfrac{4}{ab+c\sqrt{b}} \le \dfrac{a\sqrt{b}+c}{abc} \\ \dfrac{4}{bc+a\sqrt{c}} \le \dfrac{b\sqrt{c}+a}{abc} \end{cases}$$

$$\frac{4}{ac+b\sqrt{a}} + \frac{4}{ab+c\sqrt{b}} + \frac{4}{bc+a\sqrt{c}} \le \frac{c\left(1+\sqrt{a}\right)+a\left(1+\sqrt{b}\right)+b\left(1+\sqrt{c}\right)}{abc} \leftrightarrow$$

$$\frac{4}{ac+b\sqrt{a}}+\frac{4}{ab+c\sqrt{b}}+\frac{4}{bc+a\sqrt{c}}\le\frac{1+\sqrt{a}}{ab}+\frac{1+\sqrt{b}}{bc}+\frac{1+\sqrt{c}}{ac}\quad (i)$$

$$Let\ 0<a\le b\le c\to 0<\frac{1}{bc}\le\frac{1}{ca}\le\frac{1}{ab}\to$$

$$\begin{bmatrix}1+\sqrt{a} & 1+\sqrt{b} & 1+\sqrt{c}\\ \frac{1}{bc} & \frac{1}{ca} & \frac{1}{ab}\end{bmatrix}\ge\begin{bmatrix}1+\sqrt{a} & 1+\sqrt{b} & 1+\sqrt{c}\\ \frac{1}{ab} & \frac{1}{bc} & \frac{1}{ac}\end{bmatrix}\to$$

$$\frac{1+\sqrt{a}}{ab}+\frac{1+\sqrt{b}}{bc}+\frac{1+\sqrt{c}}{ac}\le\frac{1+\sqrt{a}}{bc}+\frac{1+\sqrt{b}}{ac}+\frac{1+\sqrt{c}}{ab}$$

$$So:\ \frac{4}{ac+b\sqrt{a}}+\frac{4}{ab+c\sqrt{b}}+\frac{4}{bc+a\sqrt{c}}\le\frac{1+\sqrt{a}}{bc}+\frac{1+\sqrt{b}}{ca}+\frac{1+\sqrt{c}}{ab}$$

**Pbl. 46** ***If*** $0<a,b,c<1$ ***then:***

$$\prod_{cyc}\frac{(1+ab)(1+ac)}{1+a\sqrt{bc}}\ge\left(1+\sqrt[3]{a^2b^2c^2}\right)^3$$

***Florică Anastase***

***Solution. For*** $x,y,z>0$ ***we have:***

$$(1+x)(1+y)\ge\left(1+\sqrt{xy}\right)^2\Leftrightarrow 1+x+y+xy\ge 1+2\sqrt{xy}+xy\Leftrightarrow$$

$$x+y-2\sqrt{xy}\ge 0\Leftrightarrow\left(\sqrt{x}-\sqrt{y}\right)^2\ge 0\ \textit{true.}$$

$$(1+x)(1+y)(1+z)\ge\left(1^3+\sqrt[3]{x^3}\right)\left(1^3+\sqrt[3]{y^3}\right)\left(1^3+\sqrt[3]{z^3}\right)\overset{Holder}{\ge}$$

$$\ge\left(1\cdot 1\cdot 1+\sqrt[3]{xyz}\right)^3=\left(1+\sqrt[3]{xyz}\right)^3;(*)$$

***Now,***

$$\frac{(1+ab)(1+ac)}{1+a\sqrt{bc}}\ge\frac{\left(1+\sqrt{ab\cdot ac}\right)^2}{1+a\sqrt{bc}}=1+a\sqrt{bc}$$

$$\prod_{cyc}\frac{(1+ab)(1+ac)}{1+a\sqrt{bc}}\ge\prod_{cyc}\left(1+a\sqrt{bc}\right)\overset{by(*)}{\ge}\left(1+\sqrt[3]{a^2b^2c^2}\right)^3$$

**Pbl. 47** ***If*** $a_i,b_i\in(0,1);p,q\in\mathbb{N}^*$ ***then prove:***

$$\sum_{i=1}^{n} log_{a_i} \sqrt[n]{\frac{2a_i^{2p} \cdot b_i^{2q}}{a_i^{2p} + b_i^{2q}}} + \sum_{i=1}^{n} log_{b_i} \sqrt[n]{\frac{2a_i^{2q} \cdot b_i^{2p}}{a_i^{2q} + b_i^{2p}}} \geq \left(\sqrt{p} + \sqrt{q}\right)^2$$

*Florică Anastase*

**Solution.**

$$log_{a_i} \frac{2a_i^{2p} \cdot b_i^{2q}}{a_i^{2p} + b_i^{2q}} \overset{HM-GM}{\geq} log_{a_i} \sqrt{a_i^{2p} \cdot b_i^{2q}} = log_{a_i}\left(a_i^p \cdot b_i^q\right) = p + q\, log_{a_i} b_i$$

$$log_{b_i} \frac{2a_i^{2q} \cdot b_i^{2p}}{a_i^{2q} + b_i^{2p}} \overset{HM-GM}{\geq} log_{b_i} \sqrt{a_i^{2q} \cdot b_i^{2p}} = log_{b_i}\left(a_i^q \cdot b_i^p\right) = p + q\, log_{b_i} a_i$$

*Adding, we get:*

$$log_{a_i} \frac{2a_i^{2p} \cdot b_i^{2q}}{a_i^{2p} + b_i^{2q}} + log_{b_i} \frac{2a_i^{2q} \cdot b_i^{2p}}{a_i^{2q} + b_i^{2p}} \geq 2p + q\left(log_{a_i} b_i + log_{b_i} a_i\right) \overset{AM-GM}{\geq} 2p + 2q$$

$$\sum_{i=1}^{n} \left(log_{a_i} \frac{2a_i^{2p} \cdot b_i^{2q}}{a_i^{2p} + b_i^{2q}} + log_{b_i} \frac{2a_i^{2q} \cdot b_i^{2p}}{a_i^{2q} + b_i^{2p}}\right) \geq 2np + 2nq \Leftrightarrow$$

$$\sum_{i=1}^{n} log_{a_i} \sqrt[n]{\frac{2a_i^{2p} \cdot b_i^{2q}}{a_i^{2p} + b_i^{2q}}} + \sum_{i=1}^{n} log_{b_i} \sqrt[n]{\frac{2a_i^{2q} \cdot b_i^{2p}}{a_i^{2q} + b_i^{2p}}} \geq 2p + 2q \geq \left(\sqrt{p} + \sqrt{q}\right)^2$$

**Pbl. 48** *Solve in $\mathbb{Q}$ the system:*

$$\begin{cases} 11(x^4 - y^4) + 4xy(x^2 + y^2) + x = 0 \\ 2(x^4 - y^4) - 22xy(x^2 + y^2) + y = 0 \end{cases}$$

*Florică Anastase*

**Solution.**

$$\begin{cases} 11(x^4 - y^4) + 4xy(x^2 + y^2) + x = 0 \\ 2(x^4 - y^4) - 22xy(x^2 + y^2) + y = 0 \end{cases}$$

$$\begin{cases} \dfrac{x}{x^2 + y^2} = -11(x^2 - y^2) - 4xy \\ \dfrac{y}{x^2 + y^2} = -2(x^2 - y^2) + 22xy \end{cases}$$

*Let be* $z \in \mathbb{C}, z = x + yi$ *with* $\frac{1}{z} = \frac{x-yi}{x^2+y^2}$ *and* $z^2 = x^2 - y^2 + 2xyi$

*From system we get:*

$$\frac{x-yi}{x^2+y^2}=-11(x^2-y^2)-4xy+2(x^2-y^2)-22xyi=(-11+2i)(x^2-y^2+2xyi)$$

$$\Leftrightarrow\frac{1}{z}=(-11+2i)z^2\Leftrightarrow z^3=\frac{1}{(1-2i)^3}\Leftrightarrow z=\frac{1+2i}{5}$$

*Therefore,*

$$x=\frac{1}{5}\ \text{și}\ y=\frac{2}{5}$$

**Pbl. 49** **If** $0<x<y<z$ **and** $a,b,c>0$

$$\Omega_1=\left(1+a\sqrt{\frac{z}{y}}\right)\left(1+b\sqrt{\frac{z}{z-y}}\right)+\left(1+b\sqrt{\frac{y}{x}}\right)\left(1+c\sqrt{\frac{y}{y-x}}\right)+\left(1+c\sqrt{\frac{z}{x}}\right)\left(1+a\sqrt{\frac{z}{z-x}}\right)$$

$$\Omega_2=\left(1+\sqrt[3]{2\sqrt{2}abc}\right)^2$$

**Prove that:** $\Omega_1>3\Omega_2$

*Florică Anastase*

***Solution.*** ***From BCS inequality we have:***

$$(m^2+n^2)(p^2+q^2)\geq(mp+nq)^2$$

$$\left(1+a\sqrt{\frac{z}{y}}\right)\left(1+b\sqrt{\frac{z}{z-y}}\right)\geq\left(1+\sqrt{ab}\cdot\sqrt[4]{\frac{z}{y}\cdot\frac{z}{z-y}}\right)^2\overset{(1)}{\geq}\left(1+\sqrt{2ab}\right)^2$$

$$(1)\sqrt[4]{\frac{z}{y}\cdot\frac{z}{z-y}}\geq\sqrt{2}\Leftrightarrow\frac{z^2}{y(z-y)}\geq 4\Leftrightarrow(b-2a)^2\geq 0$$

*Similarly:*

$$\left(1+b\sqrt{\frac{y}{x}}\right)\left(1+c\sqrt{\frac{y}{y-x}}\right)\geq\left(1+\sqrt{2bc}\right)^2$$

$$\left(1+c\sqrt{\frac{z}{x}}\right)\left(1+a\sqrt{\frac{z}{z-x}}\right)\geq\left(1+\sqrt{2ca}\right)^2$$

*Therefore,*

$$\Omega_1 = \left(1 + a\sqrt{\frac{z}{y}}\right)\left(1 + b\sqrt{\frac{z}{z-y}}\right) + \left(1 + b\sqrt{\frac{y}{x}}\right)\left(1 + c\sqrt{\frac{y}{y-x}}\right) + \left(1 + c\sqrt{\frac{z}{x}}\right)\left(1 + a\sqrt{\frac{z}{z-x}}\right)$$

$$\geq \left(1 + \sqrt{2ab}\right)^2 + \left(1 + \sqrt{2bc}\right)^2 + \left(1 + \sqrt{2ca}\right)^2 \overset{Bergstrom}{\geq}$$

$$\overset{Bergstrom}{\geq} \frac{\left(3 + \sqrt{2ab} + \sqrt{2bc} + \sqrt{2ca}\right)^2}{3} \overset{AM-GM}{\geq}$$

$$\overset{AM-GM}{\geq} \frac{\left(3 + 3\sqrt[3]{2\sqrt{2}abc}\right)^2}{3} = 3\left(1 + \sqrt[3]{2\sqrt{2}abc}\right)^2 = \Omega_2$$

**$\boxed{Pbl.50}$ If $a, b, c \in (1,2), f: (2,3) \to (0,\infty)$ continuous with $f'(x) < 0$ and $f''(x) < 0, \forall x \in (2,3)$ then:**

$$\sum_{cyc} f\left(\frac{(a+1)(b+1)}{1+\sqrt{ab}}\right) \geq 2 \cdot \sqrt[4]{\prod_{cyc} f(a+1) \cdot \sum_{cyc} f(a+1)}$$

***Florică Anastase***

***Solution:***

$$a, b, c \in (1,2) \to (1+a), (1+b), (1+c) \in (2,3)$$

$$\frac{(a+1)(b+1)}{1+\sqrt{ab}} \leq \frac{a+b+2}{2} \leftrightarrow \left(\sqrt{a} - \sqrt{b}\right)^2\left(\sqrt{ab} - 1\right) \geq 0, \forall a, b$$

$$\in (1,2) \text{ and analogous} \dots\dots (1)$$

$$\frac{(a+1)(b+1)}{1+\sqrt{ab}} \geq \frac{(1+\sqrt{ab})^2}{1+\sqrt{ab}} = 1 + \sqrt{ab} > 2$$

$$\frac{(a+1)(b+1)}{1+\sqrt{ab}} < 3 \leftrightarrow \left(\sqrt{ab} - 1\right)\left(\sqrt{ab} - 2\right) < 0 \text{ true.}$$

$$\sqrt{xy} + \sqrt{yz} + \sqrt{zx} \geq 2\sqrt[4]{xyz(x+y+z)}, \forall x, y, z > 0 \dots\dots (2)$$

$$\text{Let } z = \max\{x, y, z\} \text{ and } a = \frac{x}{z}, b = \frac{y}{z}, a, b \in [0,1] \overset{(2)}{\Leftrightarrow} (a + b + ab)^2$$

$$\geq 4ab\sqrt{a^2 + b^2 + 1}$$

$$\text{But } (a + b + ab)^2 = a^2b^2 + (a+b)^2 + 2ab(a+b)$$

$$\geq 2ab(a + b + 2), \text{then we have:}$$

$$(a+b+2)^2 \geq 4(a+b+1), true\ from\ (a+b+2)^2 = (a+b)^2+4+4(a+b)$$
$$\geq 4(a+b+1) \geq 4(a^2+b^2+1), \forall a,b \in [0,1]$$

**From (1),(2) we have:**

$$f\left(\frac{(a+1)(b+1)}{1+\sqrt{ab}}\right) \geq f\left(\frac{(a+1)+(b+1)}{2}\right) \geq \frac{f(a+1)+f(b+1)}{2}$$
$$\geq \sqrt{f(a+1)\cdot f(b+1)}$$
$$\sum_{cyc} f\left(\frac{(a+1)(b+1)}{1+\sqrt{ab}}\right) \geq \sum_{cyc} \sqrt{f(a+1)\cdot f(b+1)} \geq$$
$$\geq 2\cdot \sqrt[4]{\left(\prod_{cyc} f(a+1)\right)\cdot\left(\sum_{cyc} f(a+1)\right)}$$

**Pbl. 51** **If $a,b,c \in (0,1)$ and $a+b+c=1$, then:**

$$(b+c)^a\cdot(c+a)^b\cdot(a+b)^c\cdot(\sqrt{a}+\sqrt{b}+\sqrt{c})^2 \leq 2$$

*Florică Anastase*

***Solution:***

$$Let: f:(0,1)\to R, f(x) = x\cdot log(1-x^2), f'(x) = log(1-x^2) - \frac{2x}{1-x^2}, f''(x)$$
$$= -\frac{2x\cdot(3-x^2)}{(1-x^2)^2} < 0, \forall x \in (0,1) \to f''concave.$$

**From Jensen inequality:**

$$f\left(\frac{1}{\sqrt{a}+\sqrt{b}+\sqrt{c}}\right) = f\left(\frac{a+b+c}{\sqrt{a}+\sqrt{b}+\sqrt{c}}\right) \geq \frac{\sqrt{a}\cdot f(\sqrt{a})+\sqrt{b}\cdot f(\sqrt{b})+\sqrt{c}\cdot f(\sqrt{c})}{\sqrt{a}+\sqrt{b}+\sqrt{c}} \leftrightarrow$$
$$\frac{1}{\sqrt{a}+\sqrt{b}+\sqrt{c}}\cdot log\left(1-\frac{1}{(\sqrt{a}+\sqrt{b}+\sqrt{c})^2}\right)$$
$$\geq \frac{a\cdot log(1-a)+b\cdot log(1-b)+c\cdot log(1-c)}{\sqrt{a}+\sqrt{b}+\sqrt{c}} \leftrightarrow$$
$$\frac{(\sqrt{a}+\sqrt{b}+\sqrt{c})^2-(a+b+c)}{(\sqrt{a}+\sqrt{b}+\sqrt{c})^2} \geq log\left((1-a)^a\cdot(1-b)^b\cdot(1-c)^c\right) \leftrightarrow$$
$$(b+c)^a\cdot(c+a)^b\cdot(a+b)^c\cdot(\sqrt{a}+\sqrt{b}+\sqrt{c})^2 \leq 2(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}) \leftrightarrow$$

$$(b+c)^a\cdot(c+a)^b\cdot(a+b)^c\cdot(\sqrt{a}+\sqrt{b}+\sqrt{c})^2\le 2(a+b+c)\leftrightarrow$$

$$(b+c)^a\cdot(c+a)^b\cdot(a+b)^c\cdot(\sqrt{a}+\sqrt{b}+\sqrt{c})^2\le 2$$

**Pbl. 52** If $a_1,a_2,\dots,a_n>0$ such that $a_1^3+a_2^3+\dots+a_n^3=1$, then:

$$\prod_{k=1}^{n}\left(\frac{1}{1+a_k^2}\right)^{2a_k^2}\le\left(\frac{\sum_{k=1}^n a_k}{1+\sum_{k=1}^n a_k}\right)^{1+\sum_{k=1}^n a_k}$$

*Florică Anastase*

*Soluție:*

$$\prod_{k=1}^{n}\left(\frac{1}{1+a_k^2}\right)^{2a_k^2}\overset{GM-AM}{\le}\prod_{k=1}^{n}\left(\frac{1}{1+a_k^2}\right)^{a_k(1+a_k^2)}\overset{?}{\le}\left(\frac{\sum_{k=1}^n a_k}{1+\sum_{k=1}^n a_k}\right)^{1+\sum_{k=1}^n a_k}$$

$$f:(0,\infty)\to R,\ f(x)=ln\left(\frac{1}{1+x}\right)^{1+x}=-(1+x)\,ln(1+x),\ f'(x)=-(1+ln(1+x)),$$

$$f''(x)=-\frac{1}{1+x}<0,\forall x>0\to f\ concave\ \overset{Jensen}{\Rightarrow}$$

$$f\left(\frac{1}{\sum_{k=1}^n a_k}\right)=f\left(\frac{\sum_{k=1}^n a_k^3}{\sum_{k=1}^n a_k}\right)\ge\frac{\sum_{k=1}^n a_k f(a_k^2)}{\sum_{k=1}^n a_k}\leftrightarrow$$

$$ln\left(\frac{1}{1+\frac{1}{\sum_{k=1}^n a_k}}\right)^{1+\frac{1}{\sum_{k=1}^n a_k}}\ge\frac{\sum_{k=1}^n a_k ln\left(\frac{1}{1+a_k^2}\right)^{1+a_k^2}}{\sum_{k=1}^n a_k}\leftrightarrow$$

$$ln\left(\frac{1}{1+\frac{1}{\sum_{k=1}^n a_k}}\right)^{1+\frac{1}{\sum_{k=1}^n a_k}}\ge\frac{\sum_{k=1}^n ln\left(\frac{1}{1+a_k^2}\right)^{a_k(1+a_k^2)}}{\sum_{k=1}^n a_k}\leftrightarrow$$

$$ln\left(\frac{\sum_{k=1}^n a_k}{1+\sum_{k=1}^n a_k}\right)^{\frac{1+\sum_{k=1}^n a_k}{\sum_{k=1}^n a_k}}\ge\frac{ln\left(\prod_{k=1}^n\left(\left(\frac{1}{1+a_k^2}\right)^{a_k(1+a_k^2)}\right)\right)}{\sum_{k=1}^n a_k}\leftrightarrow$$

$$ln\left(\frac{\sum_{k=1}^{n}a_k}{1+\sum_{k=1}^{n}a_k}\right)^{1+\sum_{k=1}^{n}a_k}\geq ln\left(\prod_{k=1}^{n}\left(\frac{1}{1+a_k^2}\right)^{a_k(1+a_k^2)}\right)\leftrightarrow$$

$$\left(\frac{\sum_{k=1}^{n}a_k}{1+\sum_{k=1}^{n}a_k}\right)^{1+\sum_{k=1}^{n}a_k}\geq\prod_{k=1}^{n}\left(\frac{1}{1+a_k^2}\right)^{a_k(1+a_k^2)}\geq\prod_{k=1}^{n}\left(\frac{1}{1+a_k^2}\right)^{2a_k^2}$$

**Pbl. 53** **If $a,b,c\in\left(0,\frac{1}{2}\right)$, $k>0$, $S_k:=a^k+b^k+c^k$. If $S_{n+1}=1$, then**

$$(2S_1-1)^{S_1}(1+a^n)^a(1+b^n)^b(1+c^n)^c\leq$$
$$\leq(1+S_1)^{S_1}(a+b^{n+1}+c^{n+1})(a^{n+1}+b+c^{n+1})(a^{n+1}+b^{n+1}+c)$$

*Florică Anastase*

***Solution.***

$$(a+b^{n+1}+c^{n+1})(a^{n+1}+b+c^{n+1})(a^{n+1}+b^{n+1}+c)=$$

$$=(1+a-a^{n+1})(1+b-b^{n+1})(1+c-c^{n+1})\overset{Bernoulli}{\geq}(2-a^n)^a(2-b^n)^b(2-c^n)^c,\ pt.\ a,b,c\in\left(0,\frac{1}{2}\right)$$

***We must to prove:***

$$(2S_1-1)^{S_1}(1+a^n)^a(1+b^n)^b(1+c^n)^c\leq(1+S_1)^{S_1}(2-a^n)^a(2-b^n)^b(2-c^n)^c$$

$$\left(\frac{2S_1-1}{S_1+1}\right)^{S_1}\leq\frac{(2-a^n)^a(2-b^n)^b(2-c^n)^c}{(1+a^n)^a(1+b^n)^b(1+c^n)^c}$$

$$S_1\,ln\left(\frac{2S_1-1}{S_1+1}\right)\leq ln\left(\frac{(2-a^n)^a(2-b^n)^b(2-c^n)^c}{(1+a^n)^a(1+b^n)^b(1+c^n)^c}\right)$$

$$ln\left(\frac{2-\frac{1}{S_1}}{1+\frac{1}{S_1}}\right)\leq\frac{1}{S_1}ln\left(\frac{2-a^n}{1+a^n}\right)^a\left(\frac{2-b^n}{1+b^n}\right)^b\left(\frac{2-c^n}{1+c^n}\right)^c\quad(*)$$

$$Let\ \ f:(0,2)\to R,\ f(x)=ln\left(\frac{2-x}{1-x}\right)=ln(2-x)-ln(1+x)$$

$$f'(x)=-\frac{3}{(2-x)(1+x)},\ f''(x)=\frac{3(1-2x)}{(2-x)^2(1+x)^2}\geq0,\ x\in\left(0,\frac{1}{2}\right)\to f\ convex$$

$$f\left(\frac{1}{S_1}\right)=f\left(\frac{S_{n+1}}{S_1}\right)\overset{Jensen}{\leq}\frac{af(a^n)+bf(b^n)+cf(c^n)}{S_1}\leftrightarrow(*)$$

**Pbl. 54** For $a, b, c \in (0,1) \cup (1,\infty), a+b+c=abc$. Prove that

$$a^{a^2}\cdot b^{b^2}\cdot c^{c^2}\cdot (a+b+c)^{a^2+b^2+c^2} \ge (a^2+b^2+c^2)^{a^2+b^2+c^2}$$

*Florică Anastase*

***Solution.***

$$a^{a^2}\cdot b^{b^2}\cdot c^{c^2}\cdot (a+b+c)^{a^2+b^2+c^2} \ge (a^2+b^2+c^2)^{a^2+b^2+c^2}$$

$$a^{a^2}\cdot b^{b^2}\cdot c^{c^2}\cdot (abc)^{a^2+b^2+c^2} \ge (a^2+b^2+c^2)^{a^2+b^2+c^2}$$

$$\left(\frac{abc}{a^2+b^2+c^2}\right)^{a^2+b^2+c^2} \ge \frac{1}{a^{a^2}\cdot b^{b^2}\cdot c^{c^2}}$$

$$ln\left(\frac{abc}{a^2+b^2+c^2}\right)^{a^2+b^2+c^2} \ge ln\frac{1}{a^{a^2}\cdot b^{b^2}\cdot c^{c^2}}$$

$$(a^2+b^2+c^2)\,ln\left(\frac{abc}{a^2+b^2+c^2}\right) \ge a^2\,ln\left(\frac{1}{a}\right)+b^2\,ln\left(\frac{1}{b}\right)+c^2\,ln\left(\frac{1}{c}\right)$$

$$\frac{a^2+b^2+c^2}{abc}ln\left(\frac{abc}{a^2+b^2+c^2}\right) \ge \frac{a}{bc}ln\left(\frac{1}{a}\right)+\frac{b}{ca}ln\left(\frac{1}{b}\right)+\frac{c}{ab}ln\left(\frac{1}{c}\right)\ (*)$$

$$Let\ \ f:(0,1)\cup(1,\infty)\to R, f(x)=x\,ln\left(\frac{1}{x}\right)=-xlnx\ c$$

$$f'(x)=-(lnx+1), f''(x)=-\frac{1}{x}<0\to f\ concave\ on\ (0,1)\cup(1,\infty)$$

$$\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}=1\to$$

$$f\left(\frac{c}{ab}+\frac{a}{bc}+\frac{b}{ca}\right)\ge\frac{1}{ab}f(c)+\frac{1}{bc}f(b)+\frac{1}{ca}f(a)\leftrightarrow$$

$$f\left(\frac{a^2+b^2+c^2}{abc}\right)\ge\frac{af(a)+bf(b)+cf(c)}{abc}\leftrightarrow$$

$$\frac{a^2+b^2+c^2}{abc}ln\left(\frac{abc}{a^2+b^2+c^2}\right)\ge\frac{af(a)+bf(b)+cf(c)}{abc}\leftrightarrow$$

$$(a^2+b^2+c^2)\,ln\left(\frac{abc}{a^2+b^2+c^2}\right)\ge\frac{a}{bc}ln\left(\frac{1}{a}\right)+\frac{b}{ca}ln\left(\frac{1}{b}\right)+\frac{c}{ab}ln\left(\frac{1}{c}\right)\leftrightarrow(*)$$

**Pbl. 55** If $x\in\left(\pi,\frac{3\pi}{2}\right)$, then:

$$\frac{\left(\frac{1+\sin(\sin x)}{\sin x}\right)^{\frac{1}{1+\cot x}}\cdot\left(\frac{1+\sin(\cos x)}{\cos x}\right)^{\frac{1}{1+\tan x}}}{(1+\sin(\sin x))\sin x+(1+\sin(\cos x))\cos x}\leq 1$$

*Florică Anastase*

***Solution:***

$$Let\ f:\left(\pi,\frac{3\pi}{2}\right)\to R, f(x)=\log\left(\frac{x}{1+\sin x}\right), f'(x)=\frac{1}{x}-\frac{\cos x}{1+\sin x}, f''(x)=\frac{x^2-\sin x-1}{x^2(1+\sin x)}$$

$$Let\ h(x)=x^2-\sin x-1, h'(x)=2x-\cos x, h''(x)=2+\sin x>0,\ \forall x\in\left(\pi,\frac{3\pi}{2}\right)$$

$$\to h'(x)>h'(\pi)=2\pi+1>0\to h(x)>h(\pi)=\pi^2-1\to f''(x)>0\to f\text{ is convex for } x\in\left(\pi,\frac{3\pi}{2}\right)$$

$$\text{With Jensen inequality, we obtain: } f\left(\frac{\sin^2x+\cos^2x}{\sin x+\cos x}\right)\leq\frac{\sin x f(\sin x)+\cos x f(\cos x)}{\sin x+\cos x}\leftrightarrow$$

$$\log\left(\frac{1}{(\sin x+\cos x)\left(1+\sin\left(\frac{1}{\sin x+\cos x}\right)\right)}\right)\leq\frac{\log\left(\left(\frac{\sin x}{1+\sin(\sin x)}\right)^{\sin x}\left(\frac{\cos x}{1+\sin(\cos x)}\right)^{\cos x}\right)}{\sin x+\cos x}$$

$$\left(\frac{1+\sin(\sin x)}{\sin x}\right)^{\frac{\sin x}{\sin x+\cos x}}\left(\frac{1+\sin(\cos x)}{\cos x}\right)^{\frac{\cos x}{\sin x+\cos x}}\leq$$

$$\leq(\sin x+\cos x)\left(1+\sin\left(\frac{1}{\sin x+\cos x}\right)\right)\overset{\sin x\text{ is convex for } x\in\left(\pi,\frac{3\pi}{2}\right)}{\leq}$$

$$\leq(\sin x+\cos x)\left(1+\frac{\sin x\cdot\sin(\sin x)+\cos x\cdot\sin(\cos x)}{\sin x+\cos x}\right)\leftrightarrow$$

$$\left(\frac{1+\sin(\sin x)}{\sin x}\right)^{\frac{1}{1+\cot x}}\left(\frac{1+\sin(\cos x)}{\cos x}\right)^{\frac{1}{1+\tan x}}$$
$$\le (1+\sin(\sin x))\sin x+(1+\sin(\cos x))\cos x$$

**Pbl.56** **Pentru $a,b,c\in(1,\infty)$, să se arate că:**

$$\left(\log_{ab^2c^2} a\right)^2+\left(\log_{a^2bc^2} b\right)^2+\left(\log_{a^2b^2c} c\right)^2\ge\frac{3}{25}$$

*Florică Anastase*

***Soluție:***

***QM≥AM:*** $\sqrt{\frac{x^2+y^2+z^2}{3}}\ge\frac{x+y+z}{3}$ ***we get:***

$$\left(\log_{ab^2c^2} a\right)^2+\left(\log_{a^2bc^2} b\right)^2+\left(\log_{a^2b^2c} c\right)^2\ge\frac{\left(\log_{ab^2c^2} a+\log_{a^2bc^2} b+\log_{a^2b^2c} c\right)^2}{3}\ \textbf{(1)}$$

***We must show:***

$\log_{ab^2c^2} a+\log_{a^2bc^2} b+\log_{a^2b^2c} c\ge\frac{3}{5}$ ***(2), but for*** $t\in(0,1)\cup(1,\infty)$ ***we have***

$$\log_{ab^2c^2} a+\log_{a^2bc^2} b+\log_{a^2b^2c} c=$$
$$=\frac{\log_t a}{\log_t a+2(\log_t b+\log_t c)}+\frac{\log_t b}{\log_t b+2(\log_t a+\log_t c)}$$
$$+\frac{\log_t c}{\log_t c+2(\log_t a+\log_t b)}$$

**Let:** $\log_t a=x,\ \log_t b=y,\ \log_t c=z;\ x,y,z\in(0,\infty)$,

$$\frac{x}{x+2(y+z)}+\frac{y}{y+2(x+z)}+\frac{z}{z+2(x+y)}$$
$$=\frac{x^2}{x^2+2(xy+xz)}+\frac{y^2}{y^2+2(xy+yz)}+\frac{z^2}{z^2+2(xz+yz)}$$
$$\ge\frac{(x+y+z)^2}{x^2+y^2+z^2+4(xy+xz+yz)}$$

***Inequality (2) becomes:***

$$\frac{(x+y+z)^2}{x^2+y^2+z^2+4(xy+xz+yz)}\ge\frac{3}{5}\Leftrightarrow$$

$$(x-y)^2+(y-z)^2+(z-x)^2\ge 0$$

**Pbl. 57** If $a_i>1,\ i=\overline{1,n+1},\ n\in\mathbb{N}$ prove:

$$\sum_{cyc} log_{a_1}^n(a_2a_3\cdot\ldots\cdot a_{n+1})\cdot log_{a_1a_2^2a_3^2\cdot\ldots\cdot a_{n*1}^2} a_1\ge\frac{2n^n}{3}$$

*Florică Anastase*

*Solution.*

$$\sum_{cyc} log_{a_1}^n(a_2a_3\cdot\ldots\cdot a_{n+1})\cdot log_{a_1a_2^2a_3^2\cdot\ldots\cdot a_{n*1}^2} a_1\overset{Cebâşev}{\ge}$$

$$\ge\frac{1}{n}\left(\sum_{czc} log_{a_1}^n(a_2a_3\cdot\ldots\cdot a_{n+1})\right)\left(\sum_{cyc} log_{a_1a_2^2a_3^2\cdot\ldots\cdot a_{n*1}^2} a_1\right)\quad(1)$$

$$log_{a_1}^n(a_2a_3\cdot\ldots\cdot a_{n+1})=\left(log_{a_1}a_2+log_{a_1}a_3+\cdots+log_{a_1}a_{n+1}\right)^n\overset{AM-GM}{\ge}$$

$$\ge n^n\cdot log_{a_1}a_2\, log_{a_1}a_3\cdot\ldots\cdot log_{a_1}a_{n+1}$$

$$\sum_{cyc} log_{a_1}^n(a_2a_3\cdot\ldots\cdot a_{n+1})\ge n^n\sum_{cyc} log_{a_1}a_2\, log_{a_1}a_3\cdot\ldots\cdot log_{a_1}a_{n+1}\overset{AM-GM}{\ge}$$

$$\ge n^n\cdot n\cdot\sqrt[n]{log_{a_1}a_2\, log_{a_2}a_1\cdot\ldots\cdot log_{a_n}a_{n+1}\, log_{a_{n+1}}a_n}=n^{n+1}\quad(2)$$

$$\sum_{cyc} log_{a_1a_2^2a_3^2\cdot\ldots\cdot a_{n*1}^2} a_1=\sum_{cyc}\frac{lna_1}{lna_1+2(lna_2+lna_3+\cdots+lna_{n+1}}=$$

$$=\sum_{cyc}\frac{ln^2a_1}{ln^2a_1+2(lna_2lna_3+\cdots+lna_nlna_{n+1})}\ge$$

$$\ge\frac{(lna_1+lna_2+\cdots+lna_{n+1})^2}{ln^2a_1+ln^2a_2+\cdots+ln^2a_{n+1}+4(lna_1lna_2+lna_2lna_3+\cdots+lna_{n+1}lna_1)}\ge\frac{2}{3}$$

$$\Leftrightarrow$$

$$ln^2a_1+ln^2a_2+\cdots+ln^2a_{n+1}\ge lna_1lna_2+lna_2lna_3+\cdots+lna_nlna_{n+1}\quad(3)$$

*From (1),(2),(3) we get the proposed problem.*

$\boxed{Pbl.58}$ **If $a_i > 1$, $i = \overline{1, n+1}$, $n \in \mathbb{N}$ prove:**

$$\sum_{cyc} log_{a_1}^n(a_2a_3 \cdot \ldots \cdot a_{n+1}) \geq n^{n+1}$$

*Florică Anastase*

***Solution:***

$$log_{a_1}^n(a_2a_3 \cdot \ldots \cdot a_{n+1}) = \left(log_{a_1} a_2 + log_{a_1} a_3 + \cdots + log_{a_1} a_{n+1}\right)^n \overset{AM-GM}{\geq}$$

$$\geq n^n \cdot log_{a_1} a_2 \, log_{a_1} a_3 \cdot \ldots \cdot log_{a_1} a_{n+1}$$

$$\sum_{cyc} log_{a_1}^n(a_2a_3 \cdot \ldots \cdot a_{n+1}) \geq n^n \sum_{cyc} log_{a_1} a_2 \, log_{a_1} a_3 \cdot \ldots \cdot log_{a_1} a_{n+1} \overset{AM-GM}{\geq}$$

$$\geq n^n \cdot n \cdot \sqrt[n]{log_{a_1} a_2 \, log_{a_2} a_1 \cdot \ldots \cdot log_{a_n} a_{n+1} \, log_{a_{n+1}} a_n} = n^{n+1}$$

$\boxed{Pbl.59}$ ***If $x, y, z \in \mathbb{R} - \left\{(2k+1)\frac{\pi}{2} \mid k \in \mathbb{Z}\right\}$ then prove:***

$$2\prod_{cyc} cosx \cdot \sum_{cyc} sinxsin(y-z)tanx + \sum_{cyc} sinxsin(y-z)sin(y+z-x) = 0$$

*Florică Anastase*

***Solution.***

$$\sum_{cyc} sinxsin(y-z)tanx = \sum_{cyc} \frac{sin(y-z)}{cosx} - \sum_{cyc} cosxsin(y-z)$$

***But***

$$\sum_{cyc} cosxsin(y-z) = 0$$

***Hence,***

$$\sum_{cyc} \frac{sin(y-z)}{cosx} = \frac{1}{\prod cosx} \cdot \sum_{cyc} cosycoszsin(y-z) =$$

$$= \frac{1}{2\prod cosx} \cdot \left[\sum_{cyc} cos(y+z)sin(y-z) + \frac{1}{2}\sum_{cyc} sin2(y-z)\right]; \quad (1)$$

***Now,***

$$\frac{1}{2}\sum_{cyc} sin2(y-z) = -2sin(x-y)sin(y-z)sin(z-x)$$

***Then,***

$$\sum_{cyc} sinxsin(y-z)tanx + \frac{sin(x-y)sin(y-z)(sin(z-x)}{\prod cosx} = 0;\ \ (2)$$

***On the other hand, we have:***

$$sinxsin(y-z)sin(y+z-x) = sin(y-z)\cdot\frac{2sinxsin(y+z-x)}{2} =$$

$$= sin(y-z)\cdot\frac{cos(y+z-2x)-cos(y+z)}{2} =$$

$$= \frac{1}{2}sin(y-z)cos(y+z-2x) - \frac{1}{2}sin(y-z)cos(y+z) =$$

$$= \frac{1}{4}[sin2(y-x)+sin2(x-z)] - \frac{1}{4}(sin2y - sin2z)$$

***Hence,***

$$\sum_{cyc} sinxsin(y-z)sin(y+z-x) = -\frac{1}{2}\sum_{cyc} sin2(y-z) =$$

$$= -\frac{sin2(y-z)+sin2(z-x)+sin2(x-y)}{2} =$$

$$= -sin(y-x)cos(x+y-2z) - sin(x-y)cos(x-y) =$$

$$= sin(x-y)[cos(x+y-2z)-cos(x-y)] =$$

$$= 2sin(x-y)sin(y-z)sin(z-x);\ \ (3)$$

***From*** $(1),(2),(3)$ ***we get:***

$$2\prod_{cyc} cosx\cdot\sum_{cyc} sinxsin(y-z)tanx + \sum_{cyc} sinxsin(y-z)sin(y+z-x) = 0$$

$\boxed{Pbl.60}$ **If $x,y,z>0$, then:**

$$4\cdot\sqrt[4]{\left(\prod_{cyc} x\right)\cdot\left(\sum_{cyc} x\right)} + \sum_{cyc} x \le \sum_{cyc}\left(\sqrt{\frac{x^2+y^2}{2}} + 2\sqrt{xy}\right) \le 3\cdot\sum_{cyc} x$$

*Florică Anastase*

***Solution:***

$$u=\sqrt{\frac{x^2+y^2}{2}},v=\sqrt{xy}\rightarrow\begin{cases}2u^2=x^2+y^2\\ v^2=xy\end{cases}\rightarrow\begin{cases}x^2+y^2+2xy=2u^2+2v^2\\ (x+y)^2=2(u^2+v^2)\end{cases}$$

$$0\le(u-v)^2\rightarrow(u+v)^2\le 2(u^2+v^2)\rightarrow(u+v)^2\le(x+y)^2$$

$$u+v\le x+y\rightarrow\begin{cases}\sqrt{\frac{x^2+y^2}{2}}+\sqrt{xy}\le x+y\\ \sqrt{xy}\le\frac{x+y}{2}\end{cases}\rightarrow\sqrt{\frac{x^2+y^2}{2}}+2\sqrt{xy}\le\frac{3}{2}(x+y)\rightarrow$$

$$\sum_{cyc}\left(\sqrt{\frac{x^2+y^2}{2}}+2\sqrt{xy}\right)\le 3(x+y+z)\ \ldots\ldots(1)$$

$$\sqrt{xy}+\sqrt{yz}+\sqrt{zx}\ge 2\cdot\sqrt[4]{xyz(x+y+z)},\forall x,y,z>0$$

$$Let\ z=max\{x,y,z\}\ and\ a=\frac{x}{z},b=\frac{y}{z},a,b\in[0,1]\overset{(2)}{\Rightarrow}(a+b+ab)^2$$

$$\ge 4ab\sqrt{a^2+b^2+1}$$

$$But\ (a+b+ab)^2=a^2b^2+(a+b)^2+2ab(a+b)$$

$$\ge 2ab(a+b+2),then\ we\ have:$$

$$(a+b+2)^2\ge 4(a+b+1),true\ from\ (a+b+2)^2=(a+b)^2+4+4(a+b)$$

$$\ge 4(a+b+1)\ge 4(a^2+b^2+1),\forall a,b\in[0,1]$$

$$\sum_{cyc}\left(\sqrt{\frac{x^2+y^2}{2}}+2\sqrt{xy}\right)\ge\sum_{cyc}\left(\frac{x+y}{2}+2\sqrt{xy}\right)$$

$$\ge x+y+z+4\cdot\sqrt[4]{xyz(x+y+z)}\ \ldots\ldots(2)$$

$\boxed{Pbl.61}$ ***If*** $0<a<b<1$, ***then:***

$$\int_0^{\frac{\pi}{4}}\frac{a+bsinx}{b+asinx}dx\cdot\int_0^{\frac{\pi}{4}}\frac{1}{b+asinx}dx\ge\frac{\pi}{4}\left(\frac{1}{b}-\frac{\sqrt{2}}{2b+a\sqrt{2}}\right)$$

***Florică Anastase***

***Solution.***

$$Let: f,g:\left[0,\frac{\pi}{4}\right]\to R, f(x)=\frac{a+bsinx}{b+asinx}, g(x)=\frac{1}{b+asinx}\ derivable\ with$$

$$f'(x)=\frac{(b^2-a^2)cosx}{(b+asinx)^2}>0, g'(x)=-\frac{acosx}{(b+asinx)^2}<0$$

$$\to f\ is\ increasing\ and\ g\ decreasing$$

$$\overset{Chebychev's}{\Rightarrow}\int_0^{\frac{\pi}{4}}\frac{a+bsinx}{b+asinx}dx\cdot\int_0^{\frac{\pi}{4}}\frac{1}{b+asinx}dx\geq\frac{\pi}{4}\int_0^{\frac{\pi}{4}}\frac{a+bsinx}{(b+asinx)^2}dx\quad (i)$$

$$I=\int_0^{\frac{\pi}{4}}\frac{a+bsinx}{(b+asinx)^2}dx=\frac{b}{a}\int_0^{\frac{\pi}{4}}\frac{\frac{a^2}{b}-b+(b+asinx)}{(b+asinx)^2}dx$$

$$=\frac{a^2-b^2}{a}\int_0^{\frac{\pi}{4}}\frac{dx}{(b+asinx)^2}+\frac{b}{a}\int_0^{\frac{\pi}{4}}\frac{dx}{b+asinx}\quad (ii)$$

$$Let\ t=\frac{cosx}{b+asinx}\to dt=-\frac{b}{a}\left(\frac{1}{b+asinx}+\frac{a^2-b^2}{b(b+asinx)^2}\right)dx\to$$

$$t=-\frac{b}{a}\int_0^{\frac{\pi}{4}}\frac{dx}{b+asinx}-\frac{a^2-b^2}{b}\int_0^{\frac{\pi}{4}}\frac{dx}{(b+asinx)^2}\quad (iii)$$

$$From\ (ii),(iii) we\ get: I=\left.\frac{-cosx}{b+asinx}\right|_0^{\frac{\pi}{4}}=\frac{1}{b}-\frac{\sqrt{2}}{2b+a\sqrt{2}}$$

$$So:\int_0^{\frac{\pi}{4}}\frac{a+bsinx}{b+asinx}dx\cdot\int_0^{\frac{\pi}{4}}\frac{1}{b+asinx}dx\geq\frac{\pi}{4}\left(\frac{1}{b}-\frac{\sqrt{2}}{2b+a\sqrt{2}}\right)$$

**Pbl. 62** **For $n,p\in\mathbb{N}, p\geq 2, n\geq p$ find:**

$$\Omega=\lim_{n\to\infty}\left(-n+\sum_{i=1}^{n}\sqrt[p]{1+ln\left(1+\frac{i^{p-1}}{n^p}\right)}\right)$$

*Florică Anastase*

***Solution.***

$$From: \lim_{x\to 0}\frac{(1+ln(1+x))^{\frac{1}{p}}-1}{x}=\frac{1}{p}\Leftrightarrow \forall n\in\mathbb{N},\exists\zeta_n>0 \text{ such that:}$$

$$\frac{1}{p}-\zeta_n\le\frac{\left(1+ln\left(1+\frac{i^{p-1}}{n^p}\right)\right)^{\frac{1}{p}}-1}{\frac{i^{p-1}}{n^p}}\le\frac{1}{p}+\zeta_n$$

$$\left(\frac{1}{p}-\zeta_n\right)\sum_{i=1}^{n}\frac{i^{p-1}}{n^p}\le\sum_{i=1}^{n}\sqrt[p]{1+ln\left(1+\frac{i^{p-1}}{n^p}\right)}-n\le\left(\frac{1}{p}+\zeta_n\right)\sum_{i=1}^{n}\frac{i^{p-1}}{n^p}$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i^{p-1}}{n^p}=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^{p-1}=\int_0^1 x^{p-1}dx=\frac{1}{p}$$

$$So, \lim_{n\to\infty}\left(-n+\sum_{i=1}^{n}\sqrt[p]{1+ln\left(1+\frac{i^{p-1}}{n^p}\right)}\right)=\frac{1}{p^2}$$

**Pbl. 63** Let be $(a_n)_{n\ge1},(f_n(x))_{n\ge1}; n\in\mathbb{N}, n\ge 7, x>1$

$$a_n=\left(\prod_{k=1}^{n}\binom{n}{k}\right)^2, f_n(x)=\int_x^{x^2}\frac{1}{log\sqrt[n]{t}}dt$$

Then find:

$$\Omega_1=\lim_{x\to\infty}f_n(x) \text{ and } \Omega_2=\lim_{n\to\infty}\left(\frac{1}{a_n}\lim_{x\to1}f_n(x)\right)$$

*Florică Anastase*

*Solution.*

Let $g:(1,\infty)\to\mathbb{R}, g(t)=\frac{1}{logt}$ and $G:(1,\infty)\to\mathbb{R}, G'(t)=g(t)$

How $f_n(x)=G(x^2)-G(x)$ then $f_n$ – differentiable

$$f'_n(x)=2x\cdot g(x)-g(x)=\frac{2x}{logx^2}-\frac{1}{logx}=\frac{x-1}{logx}>0\Rightarrow f_n(x)\uparrow x\in(1,\infty)$$

How $\frac{1}{logt}>\frac{1}{logx^2}, \forall t\in(x,x^2), x>1$ we have

$$f_n(x)=\int_x^{x^2}\frac{1}{log\sqrt[n]{t}}dt=n\int_x^{x^2}\frac{1}{logt}dt\geq n\int_x^{x^2}\frac{1}{logx^2}dt=\frac{n(x^2-x)}{2logx},\forall x>1$$

$$\Rightarrow\Omega_1=\lim_{x\to\infty}f_n(x)=\lim_{x\to\infty}\frac{n(x^2-x)}{2logx}\overset{L'H}{=}n\lim_{x\to\infty}\frac{2x-1}{2\cdot\frac{1}{x}}=n\lim_{x\to\infty}\frac{2x^2-x}{2}=+\infty$$

$$\frac{1}{tlogt}\leq\frac{1}{logt}\leq\frac{1}{t-1}+\frac{1}{3-t},\forall t\in(1,3)\quad(*)$$

$$\frac{1}{tlogt}\leq\frac{1}{logt},\forall t\in(1,3)\Leftrightarrow\frac{1}{logt}\left(1-\frac{1}{t}\right)\geq 0,\forall t\in(1,3)\Leftrightarrow$$

$$\frac{t-1}{tlogt}\geq 0,\forall t\in(1,3)(true)$$

***Now,***

$$\frac{1}{logt}\leq\frac{1}{t-1}+\frac{1}{3-t},\forall t\in(1,3)\Leftrightarrow\frac{1}{logt}\leq\frac{2}{-t^2+4t-3},\forall t\in(1,3)\Leftrightarrow$$

$$logt+\frac{t^2-4t+3}{2}\geq 0,\forall t\in(1,3)$$

*Let* $h(t)=\ logt+\frac{t^2-4t+3}{2},t\in(1,3);h'(t)=\frac{(t-1)^2}{2}\geq 0,\forall t\in(1,3)\Rightarrow h(t)\uparrow t\in(1,3)$

***From*** $(*)$ ***we have:***

$$\int_x^{x^2}\frac{1}{tlogt}dt\leq\int_x^{x^2}\frac{1}{logt}dt\leq\int_x^{x^2}\left(\frac{1}{t-1}+\frac{1}{3-t}\right)dt$$

$$n\int_x^{x^2}\frac{1}{tlogt}dt\leq f_n(x)\leq n\int_x^{x^2}\left(\frac{1}{t-1}+\frac{1}{3-t}\right)dt$$

$$\int_x^{x^2}\frac{1}{tlogt}dt=log(logt)|_x^{x^2}=log2,\forall x\in(1,\sqrt{3})\Rightarrow\lim_{x\to 1}\int_x^{x^2}\frac{1}{tlogt}dt=log2$$

$$\int_x^{x^2}\left(\frac{1}{t-1}+\frac{1}{3-t}\right)dt=log\left|\frac{t-1}{3-t}\right|\Bigg|_x^{x^2}=log\left(\frac{(x+1)(3-x)}{3-x^2}\right),\forall x\in(1,\sqrt{3})$$

$$\lim_{x\to 1}\int_x^{x^2}\left(\frac{1}{t-1}+\frac{1}{3-t}\right)dt = \log 2$$

***So, we have:***

$$\lim_{x\to 1} f_n(x) = n\log 2$$

***Let be:***

$$b_n = \prod_{k=0}^{n}\binom{n}{k} = \prod_{k=1}^{n-1}\binom{n}{k} = \prod_{k=1}^{n-1}\frac{n!}{k!\,(n-k)!} = \frac{(n!)^{n-1}}{[1!\cdot 2!\cdot \ldots \cdot (n-1)!]^2}$$

$$2 < e_n \cong \frac{n+1}{\sqrt[n+1]{(n+1)!}} < 3; n \ge 6 \ \ (*)$$

$$\frac{b_{n+1}}{b_n} = \frac{(n+1)^n}{n!} = \frac{(n+1)^{n+1}}{(n+1)!} \overset{(*)}{>} 2^{n+1} \Rightarrow b_{n+1} > 2^{n+1}\cdot b_n$$

$$\frac{b_{n+1}}{b_n} = \frac{(n+1)^n}{n!} = \frac{(n+1)^{n+1}}{(n+1)!} \overset{(*)}{<} 3^{n+1} \Rightarrow b_{n+1} < 2^{n+1}\cdot b_n$$

***Suppose:*** $\begin{cases} b_n > 2^{\frac{n^2}{2}} \\ b_{n+1} > 2^{n+1}\cdot b_n \end{cases} \Rightarrow b_{n+1} > 2^{n+1}\cdot 2^{\frac{n^2}{2}} = 2^{\frac{n^2+2n+2}{2}} > 2^{\frac{(n+1)^2}{2}}$

***Suppose:*** $\begin{cases} b_n < 3^{n^2} \\ b_{n+1} < 3^{n+1}\cdot b_n \end{cases} \Rightarrow b_{n+1} < 3^{n+1}\cdot 3^{n^2} = 3^{n^2+n+1} < 3^{(n+1)^2}$

***Therefore,***

$$2^{n^2} < \left(\prod_{k=1}^{n}\binom{n}{k}\right)^2 < 3^{n^2}, \forall n \ge 7$$

$$\frac{n\log 2}{3^{n^2}} < \frac{n\log 2}{a_n} < \frac{n\log 2}{2^{n^2}}$$

$$0 \le \frac{n}{2^{n^2}} = \frac{n}{2^n\cdot 2^n\cdot \ldots \cdot 2^n} < \frac{n}{2^n} = \frac{n}{(1+1)^n} =$$

$$= \frac{n}{1+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}} = \frac{n}{1+n+\frac{n(n-1)}{2}+\cdots} < \frac{n}{\frac{n(n-1)}{2}} = 0$$

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{a_n}\lim_{x\to 1} f_n(x)\right) = \lim_{n\to\infty}\left(\frac{nlog2}{a_n}\right) = 0$$

**Pbl. 64** Let $(a_n)_{n\geq 1}, a_1 = e, a_n = e^n a_{n-1}^n$ and $(b_n)_{n\geq 1}$ such that:

$$\left(1+\frac{1}{n}\right)^{n+b_n} = \prod_{k=1}^{n}\left(1+\frac{1}{loga_k}\right)$$

Find: $\Omega = \lim_{n\to\infty} b_n$

*Florică Anastase*

**Solution.**

$$a_n = e^n a_{n-1}^n \Leftrightarrow loga_n = n + nloga_{n-1} = n(1+loga_{n-1})$$

Let: $x_n = loga_n;\ x_1 = 1 \Rightarrow x_n = n(1+x_{n-1}), x_1 = 1$

$$1 + x_k = k(1+x_{k-1}) + 1 \Rightarrow \frac{1+x_k}{k!} - \frac{1+x_{k-1}}{(k-1)!} = \frac{1}{k!} \Rightarrow$$

$$\frac{1+x_n}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} = E_n$$

$$\prod_{k=1}^{n}\left(1+\frac{1}{loga_k}\right) = \prod_{k=1}^{n}\left(1+\frac{1}{x_k}\right) = \prod_{k=1}^{n}\left(\frac{1}{k+1}\cdot\frac{x_{k+1}}{x_k}\right) = \frac{x_{n+1}}{(n+1)!} = \frac{1+x_n}{n!}$$

$$\lim_{n\to\infty}\prod_{k=1}^{n}\left(1+\frac{1}{loga_k}\right) = \lim_{n\to\infty}\frac{1+x_n}{n!} = e$$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\left(\frac{log\left(1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}\right)}{log\left(1+\frac{1}{n}\right)} - n\right)$$

$$= \lim_{n\to\infty}\left(\frac{log\left(1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}\right)-1}{log\left(1+\frac{1}{n}\right)} + \frac{1}{log\left(1+\frac{1}{n}\right)} - n\right);\ \ (1)$$

$$\lim_{n\to\infty}\left(\frac{log\left(1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}\right)-1}{log\left(1+\frac{1}{n}\right)}\right) \overset{L.C-S}{\cong} \lim_{n\to\infty}\frac{log\left(1+\frac{\frac{1}{(n+1)!}}{1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}}\right)}{log\left(1-\frac{1}{(n+1)^2}\right)}$$

$$= \lim_{n\to\infty}\frac{\log\left(1+\frac{\frac{1}{(n+1)!}}{1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}}\right)^{n^2}}{\log\left(1-\frac{1}{(n+1)^2}\right)^{n^2}} = 0;\ \ (2)$$

$$\lim_{n\to\infty}\left(\frac{1}{\log\left(1+\frac{1}{n}\right)}-n\right) = \lim_{x\to 0}\left(\frac{1}{\log(1+x)}-\frac{1}{x}\right) = \frac{1}{2};\ (3)$$

***From (1)+(2)+(3) we have:***

$$\Omega = \lim_{n\to\infty} b_n = \frac{1}{2}$$

$\boxed{Pbl.65}$ ***Let $(a_n)_{n\ge 1}$ be a sequence of real numbers such that $a_1 = a > 0$***

***$na_{n+1} = (n+1)(a\cdot a_n + n\cdot a^{n+1}), n \ge 1$ then find:***

$$\Omega = \lim_{n\to\infty}\left(\sqrt[n^2]{\prod_{k=1}^{n} a_k}\right)^{\frac{n^2}{a_n}+\sum_{k=1}^{n}\left(\frac{k^2}{a_k}\right)}$$

***Florică Anastase***

***Solution.***

$$a_1 = a > 0$$

$$n = 1:\quad a_2 = 2(a\cdot a_1 + a^2) = 4a^2 = 2^2a^2$$

$$n = 2:\ \ 2a_3 = 3(a\cdot a_2 + 2a^3) = 9a^3 = 3^2a^3$$

***Let be:*** $a_n = n^2a^n \Rightarrow na_{n+1} = (n+1)(a\cdot n^2a^n + na^{n+1}) = n(n+1)^2a^{n+1}\ |:n \Rightarrow$

$$a_{n+1} = (n+1)^2a^{n+1}$$

$$\prod_{k=1}^{n} a_k = 1^2\cdot 2^2 3^2\cdot\ldots\cdot n^2\cdot a\cdot a^2\cdot a^3\cdot\ldots\cdot a^n = (n!)^2\cdot a^{\frac{n(n+1)}{2}}$$

$$\frac{n^2}{a_n}+\sum_{k=1}^{n}\left(\frac{k^2}{a_k}\right) = \frac{n^2}{n^2a^n}+\frac{1^2}{a}+\frac{2^2}{2^2a^2}+\frac{3^2}{3^2a^3}+\cdots+\frac{n^2}{n^2a^n} =$$

$$=\frac{1}{a^n}+\frac{1}{a}+\frac{1}{a^2}+\cdots+\frac{1}{a^n}=\frac{1}{a^n}+\frac{1}{a}\cdot\frac{\left(\frac{1}{a}\right)^n-1}{\frac{1}{a}-1}=\frac{1}{a^n}+\left(\frac{1}{a^{n+1}}-\frac{1}{a}\right)\cdot\frac{a}{1-a}=$$

$$=\frac{1}{a^n}\cdot\frac{2-a}{1-a}-\frac{1}{1-a}$$

$$\log\Omega=\lim_{n\to\infty}\log\left(\sqrt[n^2]{\prod_{k=1}^{n}a_k}\right)^{\frac{n^2}{a_n}+\sum_{k=1}^{n}\left(\frac{k^2}{a_k}\right)}=\lim_{n\to\infty}\log\left(\prod_{k=1}^{n}a_k\right)^{\frac{\frac{1}{a^n}\cdot\frac{2-a}{1-a}-\frac{1}{1-a}}{n^2}}=$$

$$=\lim_{n\to\infty}\log\left((n!)^2\cdot a^{\frac{n(n+1)}{2}}\right)^{\frac{\frac{1}{a^n}\cdot\frac{2-a}{1-a}-\frac{1}{1-a}}{n^2}}=$$

$$=\lim_{n\to\infty}\frac{\frac{1}{a^n}\cdot\frac{2-a}{1-a}-\frac{1}{1-a}}{n^2}\cdot\left(2\log(n!)+\frac{n(n+1)}{2}\log a\right)$$

**Case 1.** $a>1\Rightarrow\frac{1}{a^n}\cdot\frac{2-a}{1-a}\to 0$

$$\log\Omega=\frac{1}{a-1}\lim_{n\to\infty}\left(\frac{2\log n!}{n^2}+\underbrace{\frac{n(n+1)}{2n^2}\log a}_{\frac{1}{2}\log a}\right)=\frac{1}{2(a-1)}\log a+\frac{2}{a-1}\lim_{n\to\infty}\frac{\log n!}{n^2}=$$

$$\overset{LC-S}{=}\frac{1}{2(a-1)}\log a+\frac{2}{a-1}\lim_{n\to\infty}\frac{\log(n+1)!-\log n!}{(n+1)^2-n^2}=$$

$$=\frac{1}{2(a-1)}\log a+\frac{2}{a-1}\lim_{n\to\infty}\frac{\log(n+1)}{2n+1}=$$

$$=\frac{1}{2(a-1)}\log a\Rightarrow\Omega=e^{\frac{1}{(1-a)\log a^2}}$$

**Case 2.** $0<a<1\Rightarrow a^n\to 0$

$$\Omega=\lim_{n\to\infty}\left(\sqrt[n^2]{n!\,a^{\frac{n(n+1)}{2}}}\right)^{\frac{1}{a^n}+\frac{1}{a}\cdot\frac{\left(\frac{1}{a}\right)^n-1}{\frac{1}{a}-1}}=\lim_{n\to\infty}\left[\left(n!\,a^{\frac{n(n+1)}{2}}\right)^{\frac{1}{n^2a^n}}\cdot\left(n!\,a^{\frac{n(n+1)}{2}}\right)^{\frac{\left(\frac{1}{a}\right)^n-1}{n^2(1-a)}}\right]=$$

$$= \lim_{n\to\infty}\left[\underbrace{\left(n!\,a^{\frac{n(n+1)}{2}}\right)^{\frac{1}{n^2a^n}}}_{L}\cdot\underbrace{\left(n!\,a^{\frac{n(n+1)}{2}}\right)^{\frac{1-a^n}{n^2(1-a)a^n}}}_{L^{\frac{1}{1-a}}}\right] = L^{1+\frac{1}{1-a}}$$

$$L = \lim_{n\to\infty}\left(n!\,a^{\frac{n(n+1)}{2}}\right)^{\frac{1}{n^2a^n}} \overset{\frac{1}{a}=b>1}{\Longrightarrow} logL = \frac{b^n}{n^2}log\left(n! - \frac{n(n+1)}{2}logb\right) \Rightarrow$$

$$\lim_{n\to\infty} logL = \lim_{n\to\infty}\frac{\frac{logn!}{n^2} - \frac{logb}{2}}{\frac{1}{b^n}} = -\infty \Rightarrow L = \lim_{n\to\infty}\left(n!\,a^{\frac{n(n+1)}{2}}\right)^{\frac{1}{n^2a^n}} = 0 \Rightarrow \Omega = 0$$

**Pbl. 66**

Let be $(I_n)_{n\geq1}, I_n = \int_1^{a^2}\frac{dx}{x(1+\sqrt{x})^n}; a \in \mathbb{R}, a \geq 2;$

$\Omega(a) = \lim_{n\to\infty}(1+I_n)\cdot\sum_{k=1}^{n}\frac{a^k-2^k}{k\cdot(2a)^k}$. Then prove:

$$\frac{a-2}{4a} \leq \Omega(a) \leq \frac{a-1}{a+1}$$

*Florică Anastase*

**Solution.**

$$I_n = \int_1^{a^2}\frac{dx}{x(1+\sqrt{x})^n} \overset{t=\sqrt{x}}{=} \int_1^{a}\frac{2tdt}{t^2(1+t)^n} = 2\int_1^{a}\frac{dt}{t(1+t)^n} \overset{1+t=u}{=} 2\int_2^{a+1}\frac{du}{(u-1)u^n} =$$

$$= 2\int_2^{a+1}\left(\frac{1}{u-1} - \frac{1}{u} - \frac{1}{u^2} - \cdots -\right)du = 2\int_2^{a+1}\left(\frac{1}{u-1} - \frac{1}{u}\cdot\frac{\frac{1}{u^n}-1}{\frac{1}{u}-1}\right)du =$$

$$= 2\int_2^{a+1}\left(\frac{1}{u-1} + \frac{1}{u}\cdot\frac{u}{1-u}\right)du = 2\int_2^{a+1}\left(\frac{1}{u-1} + \frac{1}{1-u}\right)du = 0 \Rightarrow I_n \to 0$$

$$\sum_{k=1}^{n}\frac{a^k-2^k}{k\cdot(2a)^k} = \sum_{k=1}^{n}\left(\frac{1}{k\cdot2^k} - \frac{1}{k\cdot a^k}\right) = \sum_{k=1}^{n}\left(\frac{\frac{1}{2^k}}{k} - \frac{\frac{1}{a^k}}{k}\right)$$

*Let be the sum:*

$$1+x+x^2+\dots+x^{n-1}=\frac{-1}{x-1}=\frac{1}{1-x};(x\in(0,1),x^n\to 0)\Big|\int \Leftrightarrow$$

$$x+\frac{x^2}{2}+\frac{x^3}{3}+\dots+\frac{x^n}{n}=-log(1-x)\Rightarrow\sum_{k=1}^{n}\frac{x^k}{k}=-log(1-x)$$

*Therefore,*

$$\sum_{k=1}^{n}\frac{a^k-2^k}{k\cdot(2a)^k}=\sum_{k=1}^{n}\left(\frac{\frac{1}{2^k}}{k}-\frac{\frac{1}{a^k}}{k}\right)=-log\frac{1}{2}+log\left(1-\frac{1}{a}\right)=log\frac{2(a-1)}{a}$$

So, $\Omega(a)=log\frac{2(a-1)}{a}$

*We must show:* $\frac{a-2}{4a}\le\Omega(a)\le\frac{a-1}{a+1}$

*Case 1) Let be the function* $f(a)=log\frac{2(a-1)}{a}-\frac{a-2}{4a};a\in[2,\infty)$

$$f'(a)=\frac{a+1}{2a^2(a-1)}>0,\forall a>2\Rightarrow f(a)\nearrow[2,\infty)\Rightarrow f(a)\ge f(2)=0,\forall a\in[2,\infty)$$

$$\Rightarrow log\frac{2(a-1)}{a}\ge\frac{a-2}{4a},\forall a\in[2,\infty);\ \ (1)$$

*Case 2) Let be the function* $g(a)=log\frac{2(a-1)}{a}-\frac{a-1}{a+1},a\in[2,\infty)$

$$g'(a)=\frac{-a^2+4a+1}{a(a-1)(a+1)^2};g(a)=0\Leftrightarrow a_1=2+\sqrt{5};(a>2);\ g(a_1)<0$$

| $a$ | $2$ $\qquad$ $a_1$ $\qquad$ $\infty$ |
|---|---|
| $g'(a)$ | $+ + + + +0 - - - - - - - - - - -$ |
| $g(a)$ | $-\frac{1}{3}$ $\nearrow\nearrow\nearrow$ $g(a_1)$ $\searrow\searrow\searrow$ $(log2-1)$ |

*We get,* $g(a)<0,,\forall a\in[2,\infty);(2)$

*From (1),(2) we get:*

$$\frac{a-2}{4a}\le\Omega(a)\le\frac{a-1}{a+1}$$

**Pbl. 67** *If* $S_n=\sum_{k=1}^{n}3^{k-1}\cdot sin^3\frac{\pi}{3^{k+1}}$ *and* $I=\pi\int_{\frac{1}{\sqrt{3}}}^{1}\frac{x}{tan^{-1}x}dx$ *then find:*

*Florică Anastase*

***Solution.***

$$I = \pi\int_{\frac{1}{\sqrt{3}}}^{1}\frac{x}{\tan^{-1}x}dx$$

**Let be the function** $f:\left[\frac{1}{\sqrt{3}},1\right]\to\mathbb{R}, f(x)=\frac{x}{\tan^{-1}x}; f'(x)>0; f''(x)>0, \forall x\in\left[\frac{1}{\sqrt{3}},1\right]\Rightarrow$

$$\left(1-\frac{1}{\sqrt{3}}\right)f\left(\frac{1}{\sqrt{3}}\right) < \int_{\frac{1}{\sqrt{3}}}^{1}f(x)\,dx < \frac{1}{2}\left(1-\frac{1}{\sqrt{3}}\right)\left[f\left(\frac{1}{\sqrt{3}}\right)+f(1)\right]\ (Hermite\ Hadamard)$$

$$\Leftrightarrow 2\left(\sqrt{3}-1\right) < \pi\int_{\frac{1}{\sqrt{3}}}^{1}\frac{x}{\tan^{-1}x}dx < \frac{3+\sqrt{3}}{3}$$

$$\Rightarrow [I] = \left[\pi\int_{\frac{1}{\sqrt{3}}}^{1}\frac{x}{\tan^{-1}x}dx\right] = 1$$

$$(\therefore \sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha)$$

**For** $\alpha = \frac{\pi}{3^2},\frac{\pi}{3^3}\dots,\frac{\pi}{3^{n+1}}$ **we get:**

$$\begin{cases} \sin\frac{\pi}{3} = 3\sin\frac{\pi}{3^2} - 4\sin^3\frac{\pi}{3^2}\Big|\cdot 3^0 \\ \sin\frac{\pi}{3^2} = 3\sin\frac{\pi}{3^3} - 4\sin^3\frac{\pi}{3^3}\Big|\cdot 3^1 \\ \vdots \\ \sin\frac{\pi}{3^n} = 3\sin\frac{\pi}{3^{n+1}} - 4\sin^3\frac{\pi}{3^{n+1}}\Big|\cdot 3^{n-1} \end{cases}$$

**Therefore,**

$$S_n = \sum_{k=1}^{n}3^{k-1}\cdot\sin^3\frac{\pi}{3^{k+1}} = \sin^3\frac{\pi}{3^2} + 3\cdot\sin^3\frac{\pi}{3^3} + \dots + 3^{n-1}\cdot\sin^3\frac{\pi}{3^{n+1}} =$$

$$= \sin^3\frac{\pi}{3\cdot 3} + 3\cdot\sin^3\frac{\pi}{3\cdot 3^2} + \dots + 3^{n-1}\cdot\sin^3\frac{\pi}{3\cdot 3^n} =$$

$$=\frac{1}{4}\left(3^n\sin\frac{\pi}{3^{n+1}}-\sin\frac{\pi}{3}\right)=\frac{1}{4}\left(3^n\sin\frac{\pi}{3^{n+1}}-\frac{\sqrt{3}}{2}\right)$$

$$\Omega=\lim_{n\to\infty}([I]\cdot S_n)^n=\lim_{n\to\infty}\left(\frac{1}{4}\left(3^n\sin\frac{\pi}{3^{n+1}}-\frac{\sqrt{3}}{2}\right)\right)^n=\lim_{n\to\infty}\left(\frac{1}{4}\left(\frac{\sin\frac{\pi}{3^{n+1}}}{\frac{\pi}{3^n}}\cdot\pi-\frac{\sqrt{3}}{2}\right)\right)^n$$

$$=\lim_{n\to\infty}\left(\frac{2\pi-\sqrt{3}}{8}\right)^n=0$$

**Pbl. 68** ***Let $(x_n)_{n\geq1}, (y_n)_{n\geq1}$ be sequences of positive real numbers such that:***

$$x_1>1, x_{n+1}=\frac{1+(n-1)x_n^n}{nx_n^{n-1}};\ y_1>0, y_{n+1}=\frac{(n+1)n^ny_n}{y_n^n+n^n(n-1)}$$

***Then find:*** $\lim_{n\to\infty}\left(\frac{x_n+y_n}{y_n}\right)^{\frac{\sqrt{n}}{x_n}}$

***Florică Anastase***

***Solution.***

$$x_{n+1}=\frac{1+(n-1)x_n^n}{nx_n^{n-1}}\overset{AM-GM}{\geq}\frac{\sqrt[n]{x_n^{n(n-1)}}}{x_n^{n-1}}=1\Rightarrow x_n\geq1, \forall n\in\mathbb{N};\ \ (1)$$

$$x_{n+1}-x_n=\frac{1+(n-1)x_n^n}{nx_n^{n-1}}-x_n=\frac{1+(n-1)x_n^n-nx_n^n}{nx_n^{n-1}}=\frac{1-x_n^n}{nx_n^{n-1}}\overset{(1)}{\leq}0, \forall n\in\mathbb{N}\Rightarrow$$

$(x_n)_{n\geq1}$ ***–decreasing; (2) and*** $x_n\in(1,x_1)$.

***From (1),(2) we get*** $(x_n)_{n\geq1}$ ***–convergent; (3)***

$$y_{n+1}=\frac{(n+1)n^ny_n}{y_n^n+n^n(n-1)}\Leftrightarrow\frac{y_{n+1}}{n+1}=\frac{n^ny_n}{y_n^n+n^n(n-1)}\Leftrightarrow$$

$$\frac{n+1}{y_{n+1}}=\frac{y_n^n+n^n(n-1)}{n^ny_n}=\frac{y_n^{n-1}}{n^n}+\frac{n-1}{y_n}=\frac{1}{n\left(\frac{n}{y_n}\right)^{n-1}}+\frac{n-1}{n}\cdot\frac{n}{y_n}\overset{\frac{n}{y_n}=x_n}{=}$$

$$=\frac{1}{nx_n^{n-1}}+\frac{(n-1)x_n}{n}=\frac{1+(n-1)x_n^n}{nx_n^{n-1}}=x_{n+1}$$

***So,***

$$\frac{n}{y_n} = x_n, \forall n \in \mathbb{N} \Rightarrow y_n = \frac{n}{x_n} \overset{(3)}{\Longrightarrow} \lim_{n\to\infty} y_n = +\infty$$

$$\lim_{n\to\infty}\left(\frac{x_n+y_n}{y_n}\right)^{\frac{\sqrt{n}}{x_n}} = \lim_{n\to\infty}\left(1+\frac{x_n}{y_n}\right)^{\frac{\sqrt{n}}{x_n}} = \lim_{n\to\infty}\left[\left(1+\frac{x_n}{y_n}\right)^{\frac{y_n}{x_n}}\right]^{\frac{\sqrt{n}}{y_n}} = e^{\lim_{n\to\infty}\left(\frac{n}{y_n}\cdot\frac{1}{\sqrt{n}}\right)} =$$

$$= e^{\lim_{n\to\infty}\frac{x_n}{\sqrt{n}}} \overset{(3)}{=} 1$$

**Pbl.69** Let $(a_n)_{n\geq 1}, a_n \in (0,\infty)$ be sequence of real numbers such that $a_1 = \sqrt{a}, a > 0, a_{n+1}^2 = n \cdot a_n + 1$ then find:

$$\Omega = \lim_{n\to\infty}\frac{a_n}{n^3}\int_0^1 \sqrt[n]{\frac{x^{2n}+1}{x^n+1}}\,dx, n \in \mathbb{N}, n \geq 2$$

*Florică Anastase*

**Solution 1.**

$$a_1 = \sqrt{a}, a > 0;\ a_{n+1}^2 = n \cdot a_n + 1$$

Let: $a_{n+1}^2 = x_{n+1} > 0 \Rightarrow x_1 = a > 0, a_{n+1} = \sqrt{x_{n+1}}$;

$$x_{n+1} = n \cdot \sqrt{x_n} + 1$$

How $x_{n+1} = n \cdot \sqrt{x_n} + 1 > 1 \Rightarrow x_n > 1, \forall n \geq 2, n \in \mathbb{N}$ then

$$n \cdot \sqrt{x_n} < x_{n+1} < (n+1)\sqrt{x_n}, \forall n \geq 2 \Leftrightarrow$$

$$\log n + \frac{1}{2}\log x_n < \log x_{n+1} < \log(n+1) + \frac{1}{2}\log x_n \Leftrightarrow$$

$$2^{n+1}\log n + 2^n \log x_n < 2^{n+1}\log x_{n+1} < 2^{n+1}\log(n+1) + 2^n \log x_n$$

$$2^{n+1}\log n < 2^{n+1}\log x_{n+1} - 2^n \log x_n < 2^{n+1}\log(n+1)$$

Let: $y_n = 2^n \log x_n \Rightarrow 2^{n+1}\log n < y_{n+1} - y_n < 2^{n+1}\log(n+1)$ and summing, we get:

$$\sum_{k=3}^{n} 2^k \log(k-1) < y_n - y_2 < \sum_{k=3}^{n} 2^k \log k$$

$$\frac{y_2}{2^n} + \frac{1}{2^n}\sum_{k=3}^{n} 2^k \log(k-1) - 2\log n < \log\left(\frac{x_n}{n^2}\right) < \frac{y_2}{2^n} + \frac{1}{2^n}\sum_{k=3}^{n} 2^k \log k - 2\log n$$

$$\lim_{n\to\infty}\left(\frac{1}{2^n}\sum_{k=3}^{n}2^k log(k-1)-2log\, n\right)=\lim_{n\to\infty}\frac{1}{2^n}\left(\sum_{k=3}^{n}2^k log\, k-2^{n+1}log\, n\right)\overset{L.C-Stolz}{=}$$

$$=\lim_{n\to\infty}\frac{2^{n+1}log\, n-2^{n+2}log(n+1)+2^{n+1}log\, n}{2^{n+1}-2^n}=4\lim_{n\to\infty}log\left(\frac{n+1}{n}\right)=0;(1)$$

***Analogously,***

$$\lim_{n\to\infty}\left(\frac{1}{2^n}\sum_{k=3}^{n}2^k log\, k-2log\, n\right)=0;(2)$$

***From (1),(2) we get:*** $\lim_{n\to\infty}log\left(\frac{x_n}{n^2}\right)=0\Rightarrow\lim_{n\to\infty}\left(\frac{x_n}{n^2}\right)=\lim_{n\to\infty}\left(\frac{a_n^2}{n^2}\right)=1\Rightarrow\lim_{n\to\infty}\left(\frac{a_n}{n}\right)=1$

***Now,***

$$\frac{x^{2n}+1}{x^n+1}>\sqrt{2}-1\Leftrightarrow x^{2n}-(\sqrt{2}-1)x^n+2-\sqrt{2}>0; t=x^n>0\Rightarrow$$

$$t^2-(\sqrt{2}-1)t+2-\sqrt{2}>0,\Delta_t=-5+2\sqrt{2}<0\Rightarrow$$

$$\sqrt{2}-1<\int_0^1\sqrt[n]{\frac{x^{2n}+1}{x^n+1}}dx;\ \ (3)$$

$$\sqrt[n]{\frac{x^{2n}+1}{x^n+1}}=\sqrt[n]{(x^{2n}+1)\cdot\underbrace{1\cdot 1\dots .1}_{(n-2)}\cdot\frac{1}{x^n+1}}\overset{AM-GM}{\leq}\frac{(x^{2n}+1)+n-2+\frac{1}{x^n+1}}{n}\Leftrightarrow$$

$$\sqrt[n]{\frac{x^{2n}+1}{x^n+1}}\leq\frac{1}{n}\left(x^{2n}+n-1+\frac{1}{x^n+1}\right)$$

$$\int_0^1\sqrt[n]{\frac{x^{2n}+1}{x^n+1}}dx\leq\frac{1}{n}\int_0^1 x^{2n}dx+\frac{n-1}{n}+\frac{1}{n}\underbrace{\int_0^1\frac{1}{x^n+1}}_{\leq 1}\leq\frac{1}{n(2n+1)}+1;(4)$$

***From (3),(4) we have:***

$$\sqrt{2}-1<\int_0^1\sqrt[n]{\frac{x^{2n}+1}{x^n+1}}dx<\frac{1}{n(2n+1)}+1\Leftrightarrow$$

$$(\sqrt{2}-1)\frac{a_n}{n}\cdot\frac{1}{n^2}<\frac{a_n}{n^3}\int_0^1\sqrt[n]{\frac{x^{2n}+1}{x^n+1}}dx<\frac{a_n}{n}\cdot\frac{1}{n^2}\left(\frac{1}{n(2n+1)}+1\right)$$

$$So, \Omega = \lim_{n\to\infty} \frac{a_n}{n^3} \int_0^1 \sqrt[n]{\frac{x^{2n}+1}{x^n+1}}dx = 0$$

***Solution 2.***

$$a_1 = \sqrt{a}, a > 0;\ a_{n+1}^2 = n \cdot a_n + 1$$

$$\textit{Let: } a_{n+1}^2 = x_{n+1} > 0 \Rightarrow x_1 = a > 0, a_{n+1} = \sqrt{x_{n+1}};$$

$$x_{n+1} = n \cdot \sqrt{x_n} + 1$$

$$\textit{How } x_{n+1} = n \cdot \sqrt{x_n} + 1 > 1 \Rightarrow x_n > 1, \forall n \ge 2, n \in \mathbb{N} \textit{ then}$$

$$n \cdot \sqrt{x_n} < x_{n+1} < (n+1)\sqrt{x_n}, \forall n \ge 2 \Leftrightarrow$$

$$\log n + \frac{1}{2}\log x_n < \log x_{n+1} < \log(n+1) + \frac{1}{2}\log x_n \Leftrightarrow$$

$$2^{n+1}\log n + 2^n \log x_n < 2^{n+1}\log x_{n+1} < 2^{n+1}\log(n+1) + 2^n \log x_n$$

$$2^{n+1}\log n < 2^{n+1}\log x_{n+1} - 2^n \log x_n < 2^{n+1}\log(n+1)$$

*Let:* $y_n = 2^n \log x_n \Rightarrow 2^{n+1}\log n < y_{n+1} - y_n < 2^{n+1}\log(n+1)$ *and summing, we get:*

$$\sum_{k=3}^{n} 2^k \log(k-1) < y_n - y_2 < \sum_{k=3}^{n} 2^k \log k$$

$$\frac{y_2}{2^n} + \frac{1}{2^n}\sum_{k=3}^{n} 2^k \log(k-1) - 2\log n < \log\left(\frac{x_n}{n^2}\right) < \frac{y_2}{2^n} + \frac{1}{2^n}\sum_{k=3}^{n} 2^k \log k - 2\log n$$

$$\lim_{n\to\infty}\left(\frac{1}{2^n}\sum_{k=3}^{n} 2^k \log(k-1) - 2\log n\right) = \lim_{n\to\infty}\frac{1}{2^n}\left(\sum_{k=3}^{n} 2^k \log k - 2^{n+1}\log n\right) \overset{L.C-Stolz}{=}$$

$$= \lim_{n\to\infty}\frac{2^{n+1}\log n - 2^{n+2}\log(n+1) + 2^{n+1}\log n}{2^{n+1} - 2^n} = 4\lim_{n\to\infty}\log\left(\frac{n+1}{n}\right) = 0;\ (1)$$

*Analogously,*

$$\lim_{n\to\infty}\left(\frac{1}{2^n}\sum_{k=3}^{n} 2^k \log k - 2\log n\right) = 0;\ (2)$$

*From (1),(2) we get:* $\lim_{n\to\infty}\log\left(\frac{x_n}{n^2}\right) = 0 \Rightarrow \lim_{n\to\infty}\left(\frac{x_n}{n^2}\right) = \lim_{n\to\infty}\left(\frac{a_n^2}{n^2}\right) = 1 \Rightarrow \lim_{n\to\infty}\left(\frac{a_n}{n}\right) = 1$

*Now, let be the function:*

$$f: [0,1] \to \mathbb{R}, f(x) = \frac{x^{2n}+1}{x^n+1}; f'(x) = \frac{nx^{n-1}(x^{2n}+2x^n-1)}{(x^n+1)^2}$$

| $x$ | $0$ | $\sqrt[n]{\sqrt{2}-1}$ | $1$ |
|---|---|---|---|
| $f'(x)$ | $- - - - - - - - -$ | $0$ | $+ + + + + + +$ |
| $f(x)$ | $1 \searrow$ | $2(\sqrt{2}-1)$ | $\nearrow 1$ |

***We have:*** $2(\sqrt{2}-1) \le f(x) \le 1; \forall x \in [0,1] \Rightarrow$

$$\sqrt[n]{2(\sqrt{2}-1)} < \sqrt[n]{\frac{x^{2n}+1}{x^n+1}} < 1; \forall x \in [0,1] \Rightarrow$$

$$\frac{a_n}{n}\cdot\frac{1}{n^2}\sqrt[n]{2\left(\sqrt{2}-1\right)} < \frac{a_n}{n^3}\cdot\sqrt[n]{\frac{x^{2n}+1}{x^n+1}} < \frac{a_n}{n}\cdot\frac{1}{n^2}; \forall x \in [0,1] \Rightarrow$$

$$\frac{a_n}{n}\cdot\frac{1}{n^2}\sqrt[n]{\frac{2}{\sqrt{2}+1}} < \frac{a_n}{n^3}\int_0^1\sqrt[n]{\frac{x^{2n}+1}{x^n+1}} < \frac{a_n}{n}\cdot\frac{1}{n^2}; \forall x \in [0,1]$$

$$So, \Omega = \lim_{n\to\infty}\frac{a_n}{n^3}\int_0^1\sqrt[n]{\frac{x^{2n}+1}{x^n+1}}dx = 0$$

**$\boxed{Pbl.70}$ *If $(b_n)_{n\ge 2}, b_n = \frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}} - \frac{n^2}{\sqrt[n]{n!}}$ be Bătinețu sequence. Find:***

$$\Omega = \lim_{n\to\infty}\left(1+\frac{b_n}{n}\right)^{\frac{1}{n^{n-2}}\cdot\sum_{k=0}^{n}\left(\frac{n^k}{k+1}\binom{n}{k}\right)}$$

***Florică Anastase***

***Solution:***

$$b_n = \frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}} - \frac{n^2}{\sqrt[n]{n!}} = \frac{n^2}{\sqrt[n]{n!}}\cdot(u_n - 1),$$

$$u_n = \frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}} - \frac{\sqrt[n]{n!}}{n^2}, \forall n \ge 2$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\left(\frac{n+1}{n}\cdot\frac{n+1}{\sqrt[n]{(n+1)!}}\cdot\frac{\sqrt[n]{n!}}{n}\right) = 1\cdot e\cdot\frac{1}{e}$$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\left(\frac{n}{\sqrt[n]{n!}}\cdot\frac{u_n-1}{\log u_n}\cdot\log u_n^n\right) =$$

$$= e\cdot 1\cdot \lim_{n\to\infty}\left(\log\left(\left(\frac{n+1}{n}\right)^{2n}\cdot\frac{n!\cdot\sqrt[n+1]{(n+1)!}}{(n+1)!}\right)\right)$$

$$= e\cdot\log\left(\lim_{n\to\infty} e_n^2\cdot\frac{n+1}{\sqrt[n+1]{(n+1)!}}\right) = e\cdot\log\left(e^2\cdot\frac{1}{e}\right) = e$$

$$Let: a_n = \frac{1}{n^{n-1}}\cdot\sum_{k=0}^{n}\left(\frac{n^k}{k+1}\binom{n}{k}\right)$$

$$\int_0^1 (1+nx)^n dx = \int_0^1\left(\sum_{k=0}^{n}\binom{n}{k}n^k x^k\right)dx =$$

$$= \sum_{k=0}^{n}\binom{n}{k}n^k\int_0^1 x^k dx = \sum_{k=0}^{n}\frac{n^k}{k+1}\binom{n}{k}\ \dots (1)$$

$$\int_0^1 (1+nx)^n dx \underset{dt=ndx}{\overset{t=nx}{=}} \int_1^{n+1}\frac{t^n}{n}dt = \frac{(n+1)^{n+1}-1}{n(n+1)}\ \dots (2)$$

***From (1)+(2) we have:***

$$a_n = \frac{1}{n^{n-1}}\cdot\sum_{k=0}^{n}\left(\frac{n^k}{k+1}\binom{n}{k}\right) = \left(1+\frac{1}{n}\right)^n - \frac{1}{(n+1)n^n}$$

$$\Rightarrow \lim_{n\to\infty} a_n = n$$

$$\Omega = \lim_{n\to\infty}\left(1+\frac{b_n}{n}\right)^{\frac{1}{n^{n-2}}\cdot\sum_{k=0}^{n}\left(\frac{n^k}{k+1}\binom{n}{k}\right)}$$

$$= \lim_{n\to\infty}\left(1+\frac{b_n}{n}\right)^{na_n} = \lim_{n\to\infty}\left[\left(1+\frac{b_n}{n}\right)^{\frac{n}{b_n}}\right]^{a_n\cdot b_n} = e^{\lim\limits_{n\to\infty}(a_n\cdot b_n)} = e^{e^2}$$

**$\boxed{Pbl.71}$ If $n\in\mathbb{N}, n\geq 2$ and $x>0$, find:**

$$\Omega = \lim_{n\to\infty}\frac{1}{n}\int_{\frac{1}{n}}^{1}\frac{dx}{x\left(1+x\sqrt{x\sqrt[3]{x\sqrt[4]{x\cdot\ldots\cdot x\sqrt[n]{x}}}}\right)}$$

*Florică Anastase*

**Solution.**

$$x\sqrt{x\sqrt[3]{x\sqrt[4]{x\cdot\ldots\cdot x\sqrt[n]{x}}}}=x^{\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}}=x^{E_n},\ undo\ E_n:=\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}$$

$$I=\int_{\frac{1}{n}}^{1}\frac{dx}{x(1+x^{E_n})}$$

$$=\int_{\frac{1}{n}}^{1}\left(\frac{1}{x}-\frac{x^{E_n-1}}{1+x^{E_n}}\right)dx=\left(\log x-\frac{\log(1+x^{E_n})}{E_n}\right)\Bigg|_{1/n}^{1}=\frac{\log\left(1+\left(\frac{1}{n}\right)^{E_n}\right)}{E_n}$$

$$+\log\left(\frac{1}{n}\right)-\frac{\log 2}{E_n}$$

$$\Omega=\lim_{n\to\infty}\frac{\frac{\log\left(1+\left(\frac{1}{n}\right)^{E_n}\right)}{E_n}+\log\left(\frac{1}{n}\right)-\frac{\log 2}{E_n}}{n}=\lim_{n\to\infty}\frac{E_n}{n^{E_n-1}}\log\left(1+\frac{1}{n^{E_n}}\right)^{n^{E_n}}=0$$

$\boxed{Pbl.72}$ **Prove that:**

$$\int_0^e\frac{\ln(e+x)}{\sqrt{e^2+x^2}}dx\ge\ln(1+\sqrt{2})\ln(e\sqrt{2})$$

*Florică Anastase*

**Solution.**

$$We\ have:\int_0^e\frac{\ln(e+x)}{\sqrt{e^2+x^2}}dx=\int_0^e\frac{1}{e^2+x^2}\sqrt{e^2+x^2}\ln(e+x)\,dx$$

***Applying Chebychev's weighted inequality***

$$f,g:[a,b]\to R\ same\ monotonically,p:[a,b]\to R\ integrable,then:$$

$$\left(\int_a^b p(x)dx\right)\left(\int_a^b p(x)f(x)g(x)dx\right)\ge\left(\int_a^b p(x)f(x)dx\right)\left(\int_a^b p(x)g(x)dx\right)$$

$$Let\ p(x)=\frac{1}{e^2+x^2}, f(x)=\sqrt{e^2+x^2}, g(x)=ln(e+x)\ on\ [0,e]$$

$$Then: \left(\int_0^e \frac{1}{e^2+x^2}\right)\left(\int_0^e \frac{ln\,(e+x}{\sqrt{e^2+x^2}}dx\right)\geq\left(\int_0^e \frac{\sqrt{e^2+x^2}}{e^2+x^2}dx\right)\left(\int_0^e \frac{ln\,(e+x)}{e^2+x^2}dx\right)\Leftrightarrow$$

$$\frac{\pi}{4e}\int_0^e \frac{ln\,(e+x)}{\sqrt{e^2+x^2}}dx\geq ln\,(1+\sqrt{2})\int_0^e \frac{ln\,(e+x)}{e^2+x^2}dx \qquad (1)$$

$$\int_0^e \frac{ln\,(e+x)}{e^2+x^2}dx \overset{x=et,dx=edt}{=} \int_0^1 \frac{lne(1+t)}{e^2(1+t^2)}edt=\frac{1}{e}\int_0^1 \frac{dt}{1+t^2}+\frac{1}{e}\int_0^e \frac{ln\,(1+t)}{1+t^2}dt=$$

$$=\frac{\pi}{4e}+\frac{1}{e}\int_0^e \frac{ln\,(1+t)}{1+t^2}dt \overset{t=tgu,dt=\frac{du}{cos^2u}}{=} \frac{\pi}{4e}+\frac{1}{e}\int_0^{\frac{\pi}{4}} ln(1+tgu)\,du=$$

$$=\frac{\pi}{4e}+\frac{1}{e}\int_0^{\frac{\pi}{4}} ln\left(\frac{sinu+cosu}{cosu}\right)du=\frac{\pi}{4e}+\frac{1}{e}\int_0^e ln\left(\frac{\sqrt{2}cos\left(\frac{\pi}{4}-u\right)}{cosu}\right)du=$$

$$=\frac{\pi}{4e}+\frac{\pi}{8e}ln2+\underbrace{\frac{1}{e}\int_0^{\frac{\pi}{4}} ln\,(cos\left(\frac{\pi}{4}-u\right)du-\frac{1}{e}\int_0^e ln(cosu)\,du}_{0}=$$

$$=\frac{\pi}{4e}+\frac{\pi}{8e}ln2=\frac{\pi}{4e}ln\left(e\sqrt{2}\right) \qquad (2)$$

***From (1),(2) we obtain the proposed problem.***

$\boxed{Pbl.73}$ **Prove that:**

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{cotx}{sinx+cosx}dx\geq\frac{3}{\pi}\sqrt{\frac{\pi}{3}log3}$$

***Florică Anastase***

***Solution:***

$$I=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{cotx}{sinx+cosx}dx=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{sinx}\cdot\frac{1}{1+tanx}dx \overset{Cebâşev}{\geq} (*)$$

$$Fie\ f,g:\left[\frac{\pi}{6},\frac{\pi}{3}\right]\to R, f(x)=\frac{1}{sinx}, g(x)=\frac{1}{1+tanx}\ decreasing$$

$$(*)\geq\frac{6}{\pi}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{1}{sinx}dx\cdot\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{1}{1+tanx}dx$$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{1}{1+tanx}dx\overset{tanx=t;dx=\frac{dt}{1+t^2}}{\cong}\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}}\frac{dt}{(1+t^2)(1+t)}=$$

$$=\frac{1}{2}\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}}\frac{1}{1+t}dt+\frac{1}{2}\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}}\frac{1}{1+t^2}dt-\frac{1}{2}\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}}\frac{t}{1+t^2}dt=\frac{1}{4}\left(\frac{\pi}{3}+log3\right)$$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{1}{sinx}dx=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{1+tan\frac{x}{2}}{2tan\frac{x}{2}}dx=log\left(tan\frac{x}{2}\right)\Bigg|_{\frac{\pi}{6}}^{\frac{\pi}{3}}=log\frac{1+\sqrt{3}}{3-\sqrt{3}}>0$$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{cotx}{sinx+cosx}dx\geq\frac{3}{\pi}\cdot\frac{1}{2}\left(\frac{\pi}{3}+log3\right)\cdot log\frac{1+\sqrt{3}}{3-\sqrt{3}}\overset{AM-GM}{\geq}\frac{3}{\pi}\sqrt{\frac{\pi}{3}log3}$$

**Pbl.74** **If:** $\omega_n=1-\frac{\binom{n}{1}}{3}+\frac{\binom{n}{2}}{5}-\dots+\frac{(-1)^n\binom{n}{n}}{2n+1}$. **Find:**

$$\Omega=\lim_{n\to\infty}\left(1+\frac{\sqrt[n]{\omega_n}}{n!}\right)^{\frac{n!}{e^n}}$$

*Florică Anastase*

**Solution.**

$$(1-x^2)^n=\binom{n}{0}-\binom{n}{1}x^2+\binom{n}{2}x^4-\dots+(-1)^n\binom{n}{n}x^{2n}$$

$$I_n=\int_0^1(1-x^2)^n\cdot x'dx=(1-x^2)^n\cdot x\Big|_0^1+2n\int_0^1(1-x^2)^{n-1}\cdot x^2dx=$$

$$=-2n\int_0^1(1-x^2-1)(1-x^2)^{n-1}dx=-2n\int_0^1(1-x^2)^ndx+2n\int_0^1(1-x^2)^{n-1}dx=$$

$$=-2nI_n+2nI_{n-1}\Rightarrow I_n=\frac{2^{2n}\cdot(n!)^2}{(2n+1)!}$$

$$\lim_{n\to\infty}\sqrt[n]{\omega_n}=\lim_{n\to\infty}\sqrt[n]{\frac{2^{2n}\cdot(n!)^2}{(2n+1)!}}\overset{Criteriul\ C-D'Alembert}{=}\lim_{n\to\infty}\frac{2^{2(n+1)}((n+1)!)^2}{(2n+3)!}\cdot\frac{(2n+1)!}{2^{2n}(n!)^2}=1$$

$$\Rightarrow$$

$$\Omega=\lim_{n\to\infty}\left(1+\frac{\sqrt[n]{\omega_n}}{n!}\right)^{\frac{n!}{e^n}}=e^{\lim_{n\to\infty}\frac{\sqrt[n]{\omega_n}}{n!}\cdot\frac{n!}{e^n}}=e^0=1$$

**Pbl. 75** **Find:**

$$\Omega=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{\sum_{k=1}^{n}(-1)^{k-1}\frac{1}{k}C_n^k}$$

***Florică Anastase***

***Solution.***

$$Let\ P(k): E_k=\sum_{k=1}^{n}(-1)^{k-1}\frac{1}{k}C_n^k, then: E_1=1, E_2=2-\frac{1}{2}=1+\frac{1}{2}$$

$$We\ have\ to\ prove: E_{k+1}-E_k=\frac{1}{k+1}$$

$$E_{k+1}-E_k=\sum_{i=1}^{n}\frac{(-1)^{i-1}}{i}\left(C_{k+1}^i-C_k^i\right)+(-1)^{k-1}\frac{1}{k+1}=$$

$$=\sum_{i=1}^{k}\frac{(-1)^{i-1}}{i}C_k^{i-1}+(-1)^{k-1}\frac{1}{k+1}=$$

$$=\sum_{i=1}^{k}\frac{(-1)^{i-1}k!}{i!\,(k-i-1)!}+\frac{(-1)^{k-1}}{k+1}=\frac{1}{k+1}\sum_{i=1}^{k}(-1)^{i-1}C_{k+1}^i+\frac{(-1)^{i-1}}{k+1}=$$

$$=-\frac{1}{k+1}\left(\sum_{i=0}^{k+1}\left(-C_{k+1}^1+C_{k+1}^2+\cdots+(-1)^kC_{k+1}^k\right)\right)+\frac{(-1)^{k-1}}{k+1}=$$

$$=-\frac{1}{k+1}\left((1-1)^{k+1}-1-(-1)^{k+1}\right)+\frac{(-1)^{k-1}}{k+1}=$$

$$=\frac{1}{k+1}+\frac{(-1)^k}{k+1}+\frac{(-1)^{k-1}}{k+1}=\frac{1}{k+1}\Rightarrow$$

$$E_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}, n \geq 1$$

$$\Omega = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^{\sum_{k=1}^{n}(-1)^{k-1}\frac{1}{k}C_n^k} = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}} =$$

$$= e^{\lim_{n\to\infty}\frac{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}}{n}} \overset{L.C-S}{=} 1$$

**Pbl. 76**

*If $n, k \in \mathbb{N}, n \geq k, a_n = (n+1)\sum_{k=0}^{n}\int_0^1 (1-x)^{n-k}x^k dx$, find:*

$$\Omega = \lim_{n\to\infty} n(a_n - 2)$$

*Florică Anastase*

***Solution.***

$$I = \int_0^1 (tx + 1 - x)^n dx = \int_0^1 ((t-1)x + 1)^n dx = \left(\frac{((t-1)x+1)^{n+1}}{(n+1)(t-1)}\right)\Big|_0^1$$

$$= \frac{1}{n+1}(t^n + t^{n-1} + \cdots + t + 1) \quad (i)$$

$$I = \int_0^1 (tx + 1 - x)^n dx = \int_0^1 ((1-x) + tx)^n dx = \sum_{k=0}^{n}\binom{n}{0} t^k \int_0^1 (1-x)^{n-k}x^k dx \quad (ii)$$

***From (i),(ii) we have:***

$$\sum_{k=0}^{n}\binom{n}{0} t^k \int_0^1 (1-x)^{n-k}x^k dx = \frac{1}{n+1}(t^n + t^{n-1} + \cdots + t + 1) \text{ and identifyng the coefficients}$$

$$\sum_{k=0}^{n}\int_0^1 (1-x)^{n-k}x^k dx = \frac{1}{n+1}\sum_{k=0}^{n}\frac{1}{\binom{n}{k}} \to a_n = \sum_{k=0}^{n}\frac{1}{\binom{n}{k}}$$

$$Let\ n \geq 6 \to \binom{n}{k} \geq \binom{n}{3}, 3 \leq k \leq n-3 \to$$

$$\begin{cases} a_n > \left(\frac{1}{\binom{n}{0}}+\frac{1}{\binom{n}{n}}\right)+\left(\frac{1}{\binom{n}{1}}+\frac{1}{\binom{n}{n-1}}\right) > 2+\frac{2}{n} \\ a_n < \left(\frac{1}{\binom{n}{0}}+\frac{1}{\binom{n}{n}}\right)+\left(\frac{1}{\binom{n}{1}}+\frac{1}{\binom{n}{n-1}}\right)+\left(\frac{1}{\binom{n}{2}}+\frac{1}{\binom{n}{n-2}}\right)+\frac{n-5}{\binom{n}{3}} \end{cases} \to$$

$$2+\frac{2}{n} < a_n < 2+\frac{2}{n}+\frac{4}{n(n-1)}+\frac{6(n-5)}{(n-1)(n-2)} \to \Omega = \lim_{n\to\infty} n(a_n-2) = 2$$

**Pbl.77**

*If $a, b > 0$, then $\exists c \in \left[0, \frac{\pi}{4}\right]$ such that:*

$$\int_0^{\frac{\pi}{4}} \frac{dx}{(x+1)(a^2\cos^2 x + b^2\sin^2 x)} = \frac{1}{ab(\pi+4)}\left(\pi\tan^{-1}\frac{b\tan c}{a} + 4\tan^{-1}\frac{b}{a}\right)$$

*Florică Anastase*

***Solution:***

***Theorem(Bonnet-Weierstrass):***

***If $f:[a,b]\to R$ decreasing function of $C^1$ class and $g:[a,b]\to R$ continuous function, then $\exists c \in [a,b]$ such that:***

$$\int_a^b f(x)g(x)dx = f(a)\int_a^c g(x)dx + f(b)\int_c^b g(x)dx$$

***Proof.***

***Let $h:[a,b]\to R, h(x) = f(x)-f(b)$ decreasing and $h(x)\ge 0, \forall x \in [a,b]$.***

***From theorem 2 of means $\exists c \in [a,b]$ such that:***

$$\int_a^b g(x)h(x)dx = h(a)\int_a^c g(x)dx$$

$$\int_a^b g(x)\big(f(x)-f(b)\big)dx = \big(f(a)-f(b)\big)\int_a^c g(x)dx$$

$$\int_a^b f(x)g(x)dx = f(b)\int_a^b g(x)dx + \big(f(b)-f(a)\big)\int_a^c g(x)dx =$$

$$= f(a)\int_a^c g(x)dx + f(b)\int_c^b g(x)dx$$

*q.e.d.*

**Let** $f, g: \left[0, \frac{\pi}{4}\right] \to R, g(x) = \frac{1}{a^2cos^2x+b^2sin^2x}, f(x) = \frac{1}{x+1}, f'(x) = -\frac{1}{(x+1)^2} < 0$

$$then f\ is\ decreasing$$

$$G(x) = \int \frac{dx}{a^2cos^2x + b^2sin^2x} =$$

$$\int \frac{1}{a^2 + b^2tan^2x} \cdot \frac{dx}{cos^2x} = \frac{1}{b^2}\int \frac{dt}{\left(\frac{a}{b}\right)^2 + t^2} = \frac{1}{ab}tan^{-1}\frac{btanx}{a} + C$$

$$Then\ \exists c \in \left[0, \frac{\pi}{4}\right]\ for\ witch:$$

$$\int_0^{\frac{\pi}{4}} \frac{dx}{(x+1)(a^2cos^2x + b^2sin^2x)} = f(0)\big(G(c) - G(0)\big) + f\left(\frac{\pi}{4}\right)\big(G(b) - G(c)\big)$$

$$= \frac{1}{ab}tan^{-1}\frac{btanc}{a} + \frac{1}{\frac{\pi}{4}+1} \cdot \frac{1}{ab}\left(tan^{-1}\frac{b}{a} - tan^{-1}\frac{btanc}{a}\right)$$

$$= \frac{1}{ab(\pi+4)}\left(\pi tan^{-1}\frac{btanc}{a} + 4tan^{-1}\frac{b}{a}\right)$$

**Pbl. 78**

$$Pentru\ m, n, p \in \mathbb{N}, m, n, p \geq 2\ și\ a_i > 0, a_i \neq 1, i = \overline{1, p}.\ Să\ se\ calculeze:$$

$$\lim_{x\to 0}\left(\frac{a_1^{\sqrt[n]{x}} + a_2^{\sqrt[n]{x}} + \dots + a_p^{\sqrt[n]{x}}}{a_1^{\sqrt[m]{x}} + a_2^{\sqrt[m]{x}} + \dots + a_p^{\sqrt[m]{x}}}\right)^{\frac{1}{\sqrt[n]{x}+\sqrt[m]{x}}}$$

*Florică Anastase*

***Solution.***

***We have:***

$$\lim_{x\to 0}\left(\frac{a_1^{\sqrt[n]{x}}+a_2^{\sqrt[n]{x}}+\cdots+a_p^{\sqrt[n]{x}}}{a_1^{\sqrt[m]{x}}+a_2^{\sqrt[m]{x}}+\cdots+a_p^{\sqrt[m]{x}}}\right)^{\frac{1}{\sqrt[n]{x}+\sqrt[m]{x}}}=$$

$$=\lim_{x\to 0}\left(1+\frac{a_1^{\sqrt[n]{x}}+a_2^{\sqrt[n]{x}}+\cdots+a_p^{\sqrt[n]{x}}-a_1^{\sqrt[m]{x}}-a_2^{\sqrt[m]{x}}-\cdots-a_p^{\sqrt[m]{x}}}{a_1^{\sqrt[m]{x}}+a_2^{\sqrt[m]{x}}+\cdots+a_p^{\sqrt[m]{x}}}\right)^{\frac{1}{\sqrt[n]{x}+\sqrt[m]{x}}}=$$

$$=e^{\frac{1}{p}\lim\limits_{x\to 0}\left(a_1^{\sqrt[m]{x}}\cdot\frac{a_1^{\sqrt[n]{x}-\sqrt[m]{x}}-1}{\sqrt[n]{x}-\sqrt[m]{x}}+a_2^{\sqrt[m]{x}}\cdot\frac{a_2^{\sqrt[n]{x}-\sqrt[m]{x}}-1}{\sqrt[n]{x}-\sqrt[m]{x}}+\cdots+a_p^{\sqrt[m]{x}}\cdot\frac{a_p^{\sqrt[n]{x}-\sqrt[m]{x}}-1}{\sqrt[n]{x}-\sqrt[m]{x}}\right)\cdot\frac{\sqrt[n]{x}-\sqrt[m]{x}}{\sqrt[n]{x}+\sqrt[m]{x}}}=$$

$$=e^{\frac{1}{p}ln(a_1a_2\cdots a_n)\lim\limits_{x\to 0}\frac{\sqrt[n]{x}-\sqrt[m]{x}}{\sqrt[n]{x}+\sqrt[m]{x}}}=L$$

$i)$ $If\ n<m$,

$$\lim_{x\to 0}\frac{\sqrt[n]{x}-\sqrt[m]{x}}{\sqrt[n]{x}+\sqrt[m]{x}}=\lim_{x\to 0}\frac{\sqrt[mn]{\frac{x^m}{x^n}}-1}{\sqrt[mn]{\frac{x^m}{x^n}}+1}=\lim_{x\to 0}\frac{\sqrt[mn]{x^{m-n}}-1}{\sqrt[mn]{x^{m-n}}+1}=-1$$

$ii)$ $If\ n>m$, $\lim\limits_{x\to 0}\frac{\sqrt[n]{x}-\sqrt[m]{x}}{\sqrt[n]{x}+\sqrt[m]{x}}=\lim\limits_{x\to 0}\frac{1-\sqrt[mn]{x^{n-m}}}{1+\sqrt[mn]{x^{n-m}}}=1$

$$\text{So, } L=\begin{cases}\frac{1}{\sqrt[p]{a_1a_2\cdots a_n}}, & n<m\\ 1, & n=m\\ \sqrt[p]{a_1a_2\cdots a_n}, & n>m\end{cases}$$

**Pbl. 79** *For $p\in\mathbb{N}, p>0$ find:*

$$\lim_{p\to\infty}\left[\lim_{n\to\infty}\left(\prod_{k=1}^{n}\left(1+\frac{(k+1)^p}{n^{p+1}}\right)\right)\right]^p$$

*Florică Anastase*

***Solution.***

*We know:* $\frac{x}{x+1}\le ln(1+x)\le x, \forall x\in(-1,\infty)$

$$\frac{(k+1)^p}{(k+1)^p+n^{p+1}}\le ln\left(1+\frac{(k+1)^p}{n^{p+1}}\right)\le\frac{(k+1)^p}{n^{p+1}}$$

$$\sum_{k=1}^{n}\frac{(k+1)^p}{(n+1)^p+n^{p+1}}\le\sum_{k=1}^{n}\frac{(k+1)^p}{(k+1)^p+n^{p+1}}\le ln\left[\prod_{k=1}^{n}\left(1+\frac{(k+1)^p}{n^{p+1}}\right)\right]\le\sum_{k=1}^{n}\frac{(k+1)^p}{n^{p+1}}$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{(k+1)^p}{(n+1)^p+n^{p+1}}=$$

$$=\lim_{n\to\infty}\frac{(n+2)^p}{(n+2)^p-(n+1)^p+(n+1)^{p+1}-n^{p+1}}$$

$$=\lim_{n\to\infty}\frac{(n+2)^p}{(n+2)^p-(n+1)^p+C_{p+1}^0n^{p+1}+C_{p+1}^1n^p+C_{p+1}^2n^{p-1}+\dots+C_{p+1}^{p+1}-n^{p+1}}$$

$$=\frac{1}{p+1}$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{(k+1)^p}{n^{p+1}}=\lim_{n\to\infty}\frac{(n+2)^p}{(n+1)^{p+1}-n^{p+1}}$$

$$=\lim_{n\to\infty}\frac{(n+2)^p}{C_{p+1}^0n^{p+1}+C_{p+1}^1n^p+C_{p+1}^2n^{p-1}+\dots+C_{p+1}^{p+1}-n^{p+1}}=\frac{1}{p+1}$$

$$Then:\lim_{n\to\infty}ln\left[\prod_{k=1}^{n}\left(1+\frac{(k+1)^p}{n^{p+1}}\right)\right]=\frac{1}{p+1}\Rightarrow\lim_{n\to\infty}\left(\prod_{k=1}^{n}\left(1+\frac{(k+1)^p}{n^{p+1}}\right)\right)=e^{\frac{1}{p+1}}$$

**Therefore,**

$$\lim_{p\to\infty}\left[\lim_{n\to\to\infty}\left(\prod_{k=1}^{n}\left(1+\frac{(k+1)^p}{n^{p+1}}\right)\right)\right]^p=\lim_{p\to\infty}\left(e^{\frac{1}{p+1}}\right)^p=e$$

$\boxed{Pbl.80}$ **Find:**

$$\Omega=\lim_{n\to\infty}\left(\lim_{x\to\frac{\pi}{n}}x\left(\frac{tanx}{tan\frac{\pi}{n}}\right)^{tan\frac{nx}{2}}\right)$$

*Florică Anastase*

***Solution.***

$$\lim_{x\to\frac{\pi}{n}}x\left(\frac{tanx}{tan\frac{\pi}{n}}\right)^{tan\frac{nx}{2}}=\lim_{x\to\frac{\pi}{n}}x\left(1+\frac{tanx-tan\frac{\pi}{n}}{tan\frac{\pi}{n}}\right)^{tan\frac{nx}{2}}=\frac{\pi}{n}e^{\lim_{x\to\frac{\pi}{n}}\left(\frac{tanx-tan\frac{\pi}{n}}{tan\frac{\pi}{n}}\cdot tan\frac{nx}{2}\right)}=(*)$$

$$\lim_{x\to\frac{\pi}{n}}\left(\frac{tanx-tan\frac{\pi}{n}}{tan\frac{\pi}{n}}\cdot tan\frac{nx}{2}\right)=\lim_{x\to\frac{\pi}{n}}\left(\frac{-sin\left(x-\frac{\pi}{n}\right)}{cos\frac{\pi}{n}\cdot cosx\cdot\frac{sin\frac{\pi}{n}}{cos\frac{\pi}{n}}}\cdot cot\left(\frac{nx}{2}-\frac{\pi}{2}\right)\right)=$$

$$=-\lim_{x\to\frac{\pi}{n}}\left(\frac{sin\left(x-\frac{\pi}{n}\right)}{cosx\cdot sin\frac{\pi}{n}}\cdot\frac{cos\left(\frac{x-\frac{\pi}{n}}{\frac{2}{n}}\right)}{sin\left(\frac{x-\frac{\pi}{n}}{\frac{2}{n}}\right)}\right)=-\lim_{x\to\frac{\pi}{n}}\left(\frac{cos\left(\frac{x-\frac{\pi}{n}}{\frac{2}{n}}\right)}{cosx\cdot sin\frac{\pi}{n}}\cdot\frac{sin\left(x-\frac{\pi}{n}\right)}{sin\left(\frac{x-\frac{\pi}{n}}{\frac{2}{n}}\right)}\right)=$$

$$=-\frac{2}{sin\frac{2\pi}{n}}\cdot\lim_{x\to\frac{\pi}{n}}\left(\frac{sin\left(x-\frac{\pi}{n}\right)}{x-\frac{\pi}{n}}\cdot\frac{2}{n}\cdot\frac{\frac{x-\frac{\pi}{n}}{\frac{2}{n}}}{sin\left(\frac{x-\frac{\pi}{n}}{\frac{2}{n}}\right)}\right)=-\frac{4}{nsin\frac{2\pi}{n}}\Rightarrow$$

$$(*)=\frac{\pi}{ne^{\frac{4}{nsin\frac{2\pi}{n}}}}$$

$$\Omega=\lim_{n\to\infty}\left(\lim_{x\to\frac{\pi}{n}}x\left(\frac{tanx}{tan\frac{\pi}{n}}\right)^{tan\frac{nx}{2}}\right)=\lim_{n\to\infty}\left(\frac{\pi}{n}\cdot e^{\frac{4}{nsin\frac{2\pi}{n}}}\right)=\lim_{n\to\infty}\left(\frac{\pi}{n}\cdot e^{-\frac{\frac{2\pi}{n}}{sin\frac{2\pi}{n}}\cdot\frac{2n}{\pi}}\right)=0$$

$\boxed{Pbl.81}$ **For $p>1$, find:**

$$\Omega=\lim_{p\to\infty}\left(\lim_{n\to\infty}\left(-n+\sum_{i=1}^{n}\sqrt[p]{1+sin^{-1}\left(\frac{i^{p-1}}{n^p}\right)}\right)\right)^{p+sinp}$$

*Florică Anastase*

***Solution.***

$$\lim_{x\to 0}\frac{(1+sin^{-1}x)^{\frac{1}{p}}-1}{x}=\frac{1}{p}\Rightarrow\ \forall n\in\mathbb{N},\exists\zeta_n>0\ such\ that:$$

$$\left(\frac{1}{p}-\zeta_n\right)\frac{i^{p-1}}{n^p}\le\sqrt[p]{1+\sin^{-1}\left(\frac{i^{p-1}}{n^p}\right)}-1\le\left(\frac{1}{p}+\zeta_n\right)\frac{i^{p-1}}{n^p}$$

$$\left(\frac{1}{p}-\zeta_n\right)\sum_{i=1}^{n}\frac{i^{p-1}}{n^p}\le -n+\sum_{i=1}^{n}\sqrt[p]{1+\sin^{-1}\left(\frac{i^{p-1}}{n^p}\right)}\le\left(\frac{1}{p}+\zeta_n\right)\sum_{i=1}^{n}\frac{i^{p-1}}{n^p}$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i^{p-1}}{n^p}=\int_0^1 x^{p-1}dx=\frac{1}{p},\Rightarrow$$

$$\lim_{n\to\infty}\left(-n+\sum_{i=1}^{n}\sqrt[p]{1+\sin^{-1}\left(\frac{i^{p-1}}{n^p}\right)}\right)=\frac{1}{p^2},$$

$$\Omega=\left(\frac{1}{p^2}\right)^{\lim\limits_{p\to\infty}p+\sin p}=e^{\lim\limits_{p\to\infty}(p+\sin p)\ln\left(\frac{1}{p^2}\right)}=e^{2\lim\limits_{p\to\infty}\left(1+\frac{\sin p}{p}\right)\ln\left(\frac{1}{p^p}\right)}=0$$

**Pbl.82** ***If*** $f:[0,1]\to[0,\infty)$ ***such that*** $\int_0^1 f(x)dx=1$,

$$I_1=\int_0^1\frac{x^2+x+1}{x^2+1}\cdot e^{\tan^{-1}x}dx,\ I_2=\int_0^1\left(x-\int_0^1 tf(t)dt\right)^2 f(x)dx$$

***Then prove:*** $I_1\ge e^{\pi I_2}$

***Florică Anastase***

***Solution.***

$$I_1=\int_0^1\frac{x^2+x+1}{x^2+1}\cdot e^{\tan^{-1}x}dx=\int_0^1\left(1+\frac{x}{1+x^2}\right)\cdot e^{\tan^{-1}x}dx=$$

$$=\int_0^1 e^{\tan^{-1}x}dx+\int_0^1\frac{x}{1+x^2}\cdot e^{\tan^{-1}x}dx=$$

$$=x\cdot e^{\tan^{-1}x}\Big|_0^1-\int_0^1\frac{x}{1+x^2}\cdot e^{\tan^{-1}x}dx+\int_0^1\frac{x}{1+x^2}\cdot e^{\tan^{-1}x}dx=$$

$$=x\cdot e^{\tan^{-1}x}\Big|_0^1=e^{\frac{\pi}{4}};\ (1)$$

$$\int_0^1\left(x-\frac{1}{2}\right)^2 f(x)dx=\int_0^1\left(x-\int_0^1 tf(t)dt+\int_0^1 tf(t)dt-\frac{1}{2}\right)^2 f(x)dx=$$

$$=\int_0^1\left(x-\int_0^1 tf(t)dt\right)^2 f(x)dx+2\int_0^1\left(x-\int_0^1 tf(t)dt\right)\left(\int_0^1 tf(t)dt-\frac{1}{2}\right)f(x)dx+$$

$$+\int_0^1\left(\int_0^1 tf(t)dt-\frac{1}{2}\right)^2 f(x)dx=$$

$$=\int_0^1\left(x-\int_0^1 tf(t)dt\right)^2 f(x)dx+2\left(\int_0^1 tf(t)dt-\frac{1}{2}\right)\int_0^1\left(x-\int_0^1 tf(t)dt\right)f(x)dx+$$

$$+\int_0^1\left(\int_0^1 tf(t)dt-\frac{1}{2}\right)^2 f(x)dx\ge\int_0^1\left(x-\int_0^1 tf(t)dt\right)^2 f(x)dx$$

*Hence,*

$$I_2=\int_0^1\left(x-\int_0^1 tf(t)dt\right)^2 f(x)dx\le\int_0^1\left(x-\frac{1}{2}\right)^2 f(x)dx\le\frac{1}{4}\int_0^1 f(x)dx=\frac{1}{4};\ \ (2)$$

*From* $(1),(2)$ *we get:* $I_1\ge e^{\pi I_2}$

$\boxed{Pbl.83}$ ***If*** $\lambda>1, f:[0,1]\to[1,\lambda]$ ***continuous and convex function such that*** $f(0)=0$ ***then prove:***

$$\lambda^2\int_0^{\frac{1}{\lambda}} f(x)dx\le\int_0^1 f(x)dx\le\lambda(\lambda+1)\int_0^1\frac{dx}{1+f(x)}$$

***Florică Anastase***

***Solution .***

$f:[0,1]\to[1,\lambda]$ ***continuous and convex function, then*** $\frac{f(x)}{x}$ $-$***increasing function on*** $(0,1]$

*Hence,*

$$\frac{f(x)}{x} \le \frac{f\left(\frac{1}{\lambda}\right)}{\frac{1}{\lambda}}, \forall x \in \left(0, \frac{1}{\lambda}\right] \Rightarrow f(x) \le \lambda x f\left(\frac{1}{\lambda}\right) \Rightarrow$$

$$\int_0^{\frac{1}{\lambda}} f(x)dx \le \lambda f\left(\frac{1}{\lambda}\right) \int_0^{\frac{1}{\lambda}} x\,dx = \frac{1}{2\lambda} f\left(\frac{1}{\lambda}\right); \quad (1)$$

$$\frac{f(x)}{x} \ge \frac{f\left(\frac{1}{\lambda}\right)}{\frac{1}{\lambda}}, \forall x \in \left[\frac{1}{\lambda}, 1\right] \Rightarrow f(x) \ge \lambda x f\left(\frac{1}{\lambda}\right) \Rightarrow$$

$$\int_{\frac{1}{\lambda}}^{1} f(x)dx \ge \lambda f\left(\frac{1}{\lambda}\right) \int_{\frac{1}{\lambda}}^{1} x\,dx = \frac{\lambda}{2} f\left(\frac{1}{\lambda}\right)\left(1 - \frac{1}{\lambda^2}\right) = \frac{1}{2\lambda} f\left(\frac{1}{\lambda}\right)(\lambda^2 - 1) \overset{(1)}{\ge}$$

$$\overset{(1)}{\ge} (\lambda^2 - 1) \int_0^{\frac{1}{\lambda}} f(x)dx\,; (2)$$

***From (1),(2) we have:***

$$\int_0^1 f(x)dx = \int_0^{\frac{1}{\lambda}} f(x)dx + \int_{\frac{1}{\lambda}}^1 f(x)dx \ge \int_0^{\frac{1}{\lambda}} f(x)dx + (\lambda^2 - 1)\int_0^{\frac{1}{\lambda}} f(x)dx = \lambda^2 \int_0^{\frac{1}{\lambda}} f(x)dx$$

***Hence,***

$$\lambda^2 \int_0^{\frac{1}{\lambda}} f(x)dx \le \int_0^1 f(x)dx\,; \quad (3)$$

***On the other hand, we have:***

$$f(x) \in [1, \lambda] \Rightarrow \begin{cases} f(x) \ge 1 \\ f(x) \le \lambda \end{cases} \Rightarrow \begin{cases} 1 + f(x) - 2 \ge 0 \\ f(x) - \lambda \le 0 \end{cases} \Rightarrow$$

$$(1 + f(x) - 2)(f(x) - \lambda) \le 0$$

$$\Leftrightarrow f(x) \le \frac{(\lambda + 2)f(x) - 2\lambda}{1 + f(x)} \le \frac{\lambda(\lambda + 1)}{1 + f(x)}$$

***Hence,***

$$\int_0^1 f(x)dx \le \lambda(\lambda+1)\int_0^1 \frac{dx}{1+f(x)};\ \ (4)$$

*From* $(3),(4)$ *it follows that:*

$$\lambda^2\int_0^{\frac{1}{\lambda}} f(x)dx \le \int_0^1 f(x)dx \le \lambda(\lambda+1)\int_0^1 \frac{dx}{1+f(x)}$$

$\boxed{Pbl.84}$ ***If*** $a,b,c\in\left(0,\frac{\pi}{2}\right), a+b+c=\pi$ ***and***

$$I(n)=\sum_{i=1}^{n}\int_i^{i+1}\frac{dx}{(ae^{tana}x^2+be^{tanb}x+ce^{tanc})(e^c tancx^2+e^b tanbx+e^a tana)}$$

***Then find maximum values of expression:***

$$\Omega=\prod_{k=1}^{2020} I(k)$$

*Florică Anastase*

***Solution.***

$$(ae^{tana}x^2+be^{tanb}x+ce^{tanc})(e^c tancx^2+e^b tanbx+e^a tana)=$$

$$(ae^{tana}x^2+be^{tanb}x+ce^{tanc})(e^a tana+e^b tanbx+e^c tancx^2)\overset{BCS}{\ge}$$

$$\ge\left(\sqrt{ae^{a+tana}tana}+\sqrt{be^{b+tanb}tanb}+\sqrt{ce^{c+tanc}tanc}\right)^2x^2\ge$$

$$\ge\left(\sqrt{atana(1+a)(1+tana)}+\sqrt{btanb(1+b)(1+tanb)}+\sqrt{ctanc(1+c)(1+tanc)}\right)^2x^2$$

$$\ge(atana+btanb+ctanc)^2x^2=\left((a+b+c)\left(\frac{atana+btanb+ctanc}{a+b+c}\right)\right)^2x^2\ge$$

$$\ge\left((a+b+c)tan\left(\frac{a^2+b^2+c^2}{a+b+c}\right)\right)^2x^2\ge$$

$$\ge\left((a+b+c)tan\left(\frac{a+b+c}{3}\right)\right)^2x^2=\frac{\pi^2}{3}x^2$$

***Hence,***

$$\frac{1}{(ae^{tana}x^2+be^{tanb}x+ce^{tanc})(e^c tancx^2+e^b tanbx+e^a tana)}\le\frac{3}{\pi^2}\cdot\frac{1}{x^2}$$

*Therefore,*

$$\sum_{i=1}^{n}\int_i^{i+1}\frac{dx}{(ae^{tana}x^2+be^{tanb}x+ce^{tanc})(e^c tancx^2+e^b tanbx+e^a tana)}\le$$

$$\le\frac{3}{\pi^2}\cdot\sum_{i=1}^{n}\int_i^{i+1}\frac{1}{x^2}dx=\frac{3}{\pi^2}\cdot\sum_{i=1}^{n}\left(-\frac{1}{x}\right)\Bigg|_i^{i+1}=\frac{3}{\pi^2}\cdot\sum_{i=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right)=\frac{3n}{(n+1)\pi^2}$$

*Hence,*

$$\sum_{i=1}^{n}\int_i^{i+1}\frac{dx}{(ae^{tana}x^2+be^{tanb}x+ce^{tanc})(e^c tancx^2+e^b tanbx+e^a tana)}\le\frac{3n}{(n+1)\pi^2}$$

*Therefore,*

$$\Omega=\prod_{k=1}^{2020}I(k)\le\prod_{k=1}^{2020}\frac{3k}{(k+1)\pi^2}=\prod_{k=1}^{2020}\frac{3}{\pi^2}\cdot\frac{k}{(k+1)}=$$

$$=\frac{3^{2020}}{\pi^{4040}}\cdot\frac{1}{2}\cdot\frac{2}{3}\cdot\ldots\cdot\frac{2019}{2020}=\frac{3^{2020}}{\pi^{4040}}\cdot\frac{1}{2020}$$

$$Max\{\Omega\}=\frac{3^{2020}}{\pi^{4040}}\cdot\frac{1}{2020}$$

**Pbl. 85** ***If $0<a<b<\frac{\pi}{2}$ then prove:***

$$\frac{3(b-a)\sqrt[3]{4(a+b)}}{\sqrt[3]{4(a+b)-sin4(a+b)}}<3\int_a^b\frac{dx}{\sqrt[3]{1-cos4x}}<cot2a-cot2b+\frac{\pi}{4}$$

*Florică Anastase*

***Solution.***

$$f:\left(0,\frac{\pi}{2}\right)\to\mathbb{R}, f(x)=1-cos4x\ -\text{continuous}$$

$$g:(0,\infty)\to(0,\infty), g(x)=\frac{1}{\sqrt[3]{x}}=x^{-\frac{1}{3}}, g'(x)=-\frac{1}{3}x^{-\frac{4}{3}}<0, g''(x)=\frac{4}{9}x^{-\frac{7}{3}}>0\Rightarrow$$

$g$ *–convex function.*

*Applying Jensen integral inequality, we get:*

$$g\left(\frac{1}{b-a}\int_a^b f(x)dx\right) \le \frac{1}{b-a}\int_a^b g(f(x))dx \Leftrightarrow$$

$$\frac{1}{\sqrt[3]{\frac{1}{b-a}\int_a^b f(x)dx}} \le \frac{1}{b-a}\int_a^b \frac{dx}{\sqrt[3]{f(x)}} \Leftrightarrow$$

$$\frac{1}{\sqrt[3]{\frac{1}{b-a}\int_a^b (1-\cos 4x)dx}} \le \frac{1}{b-a}\int_a^b \frac{dx}{\sqrt[3]{1-\cos 4x}} \Leftrightarrow$$

$$\frac{b-a}{\sqrt[3]{\frac{1}{b-a}\left(b-a-\frac{\sin 4b-\sin 4a}{4}\right)}} \le \int_a^b \frac{dx}{\sqrt[3]{1-\cos 4x}} \Leftrightarrow$$

$$\frac{b-a}{\sqrt[3]{1-\frac{1}{4}\cdot\frac{\sin 4b-\sin 4a}{b-a}}} \le \int_a^b \frac{dx}{\sqrt[3]{1-\cos 4x}} \Leftrightarrow$$

$$\frac{b-a}{\sqrt[3]{1-\frac{1}{2}\cdot\frac{\sin 2(b-a)\cos 2(a+b)}{b-a}}} \le \int_a^b \frac{dx}{\sqrt[3]{1-\cos 4x}}$$

$$u(t) = \frac{\sin t}{t}, t \in \left(0,\frac{\pi}{2}\right) \Rightarrow u'(t) = \frac{t\cos t-\sin t}{t^2}$$

$$v(t) = t\cos t-\sin t \Rightarrow v'(t) = -t\sin t < 0, \forall t \in \left(0,\frac{\pi}{2}\right) \Rightarrow v(t) < v(0) = 0$$

$$\Rightarrow u'(t) < 0 \Rightarrow u(t) = \frac{\sin t}{t} - \textit{decreasing} \Rightarrow \frac{\sin 2(a+b)}{2(a+b)} < \frac{\sin 2(b-a)}{2(b-a)} \Rightarrow$$

$$1-\frac{1}{2}\cdot\frac{\sin 2(b-a)\cos 2(a+b)}{b-a} < 1-\frac{1}{2}\cdot\frac{\sin 2(b+a)\cos 2(a+b)}{b+a}; (*)$$

$$\frac{b-a}{\sqrt[3]{1-\frac{1}{2}\cdot\frac{\sin 2(b-a)\cos 2(a+b)}{b-a}}} =$$

$$= \frac{b-a}{\sqrt[3]{1-\frac{\sin 2(b-a)}{2(b-a)}\cdot\cos 2(a+b)}} \overset{(*)}{\ge} \frac{b-a}{\sqrt[3]{1-\frac{\sin 2(b+a)}{2(b+a)}\cdot\cos 2(a+b)}} =$$

$$=\frac{b-a}{\sqrt[3]{1-\frac{sin4(a+b)}{4(a+b)}}}=\frac{(b-a)\sqrt[3]{4(a+b)}}{\sqrt[3]{4(a+b)-sin4(a+b)}}$$

***Hence,***

$$\frac{(b-a)\sqrt[3]{4(a+b)}}{\sqrt[3]{4(a+b)-sin4(a+b)}}\le\int_a^b\frac{dx}{\sqrt[3]{1-cos4x}};\ (1)$$

$$1+\frac{1}{cos^2x}+\frac{1}{sin^2x}\overset{AGM}{\ge}3\sqrt[3]{\frac{1}{sin^2xcos^2x}}=\frac{6}{\sqrt[3]{2(4sin^2xcos^2x)}}=$$

$$=\frac{6}{\sqrt[3]{2sin^2 2x}}=\frac{6}{\sqrt[3]{1-cos4x}}$$

***Hence,***

$$6\int_a^b\frac{dx}{\sqrt[3]{1-cos4x}}<\int_a^b dx+\int_a^b\frac{1}{cos^2x}dx+\int_a^b\frac{1}{sin^2x}dx\Leftrightarrow$$

$$\int_a^b\frac{dx}{\sqrt[3]{1-cos4x}}<\frac{1}{6}[(b-a)+(tanb-tana)+(cota-cotb)]<$$

$$<\frac{1}{6}(tanb-tana)\left(1+\frac{1}{tana\,tanb}\right)+\frac{\pi}{12}=$$

$$=\frac{1}{6}\left(\frac{sinb}{cosb}-\frac{sina}{cosa}\right)\cdot\frac{1+tana\,tanb}{tana\,tanb}+\frac{\pi}{12}=$$

$$=\frac{1}{6}\cdot\frac{sin(b-a)}{cosa\,cosb}\cdot\frac{cosa\,cosb+sina\,sinb}{sina\,sinb}+\frac{\pi}{12}=$$

$$=\frac{1}{6}\cdot\frac{4sin(b-a)cos(b-a)}{4sina\,sinb\,cosa\,cosb}+\frac{\pi}{12}=\frac{1}{6}\cdot\frac{2sin(2b-2a)}{sin2a\,sin2b}+\frac{\pi}{12}=$$

$$=\frac{1}{3}\cdot\frac{sin2b\cos 2a-sin2a\,cos2b}{sin2a\,sin2b}+\frac{\pi}{12}=\frac{1}{3}\left(\frac{cos2a}{sin2a}-\frac{cos2b}{sin2b}\right)+\frac{\pi}{12}=$$

$$=\frac{1}{3}(cot2a-cot2b)+\frac{\pi}{12}$$

***Hence,***

$$3\int_a^b\frac{dx}{\sqrt[3]{1-cos4x}}<cot2a-cot2b+\frac{\pi}{4};\ (2)$$

***From (1), (2) it follows that:***

$$\frac{(b-a)3\sqrt[3]{4(a+b)}}{\sqrt[3]{4(a+b)}-\sin 4(a+b)} < \cot 2a - \cot 2b + \frac{\pi}{4}$$

**Pbl.86** *If $(a_n)_{n\geq 1}$ is sequence of real number with $a_1 \in (0,1)$ and $a_{n+1} = \sqrt{1+na_n}$. Find:*

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{n^2+a_n}+\frac{2}{n^2+2a_n}+\dots+\frac{n}{n^2+na_n}\right)^{\sqrt[n]{n}-1}$$

*Florică Anastase*

***Solution.***

$$If\ a_1 \in (0,1), for\ induction\ methods\ we\ obtain: a_n \in (n-1,n), \forall n \in \mathbb{N}$$

$$and\ \lim_{n\to\infty}\frac{a_n}{n} = 1$$

$$Let: f:[0,1] \to R, f(x) = \frac{x}{x+1}, and\ division \Delta_n = \left(0.\frac{a_n}{n^2}, \frac{2a_n}{n^2}, \dots, \frac{na_n}{n^2}, 1\right)$$

$$|\Delta_n| = max\left\{\frac{a_n}{n^2}, 1-\frac{na_n}{n^2}\right\} \Rightarrow |\Delta_n| \to 0$$

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{k}{n^2+ka_n}\right)$$

$$= \lim_{n\to\infty}\frac{a_n}{n^2}\sum_{k=1}^{n}\frac{\frac{ka_n}{n^2}}{1+\frac{ka_n}{n^2}}\cdot\frac{1}{a_n^2}$$

$$= \lim_{n\to\infty}\frac{n^2}{a_n^2}\left(\frac{a_n}{n^2}\sum_{k=1}^{n}\frac{\frac{ka_n}{n^2}}{1+\frac{ka_n}{n^2}}\right) = \int_0^1 f(x)dx = \int_0^1 \frac{x}{x+1}dx = 1-\ln 2$$

$$Then: \Omega = 0$$

**Pbl.87** **Prove:**

$$\int_0^n (n+x)^{\frac{1}{\sqrt{n^2+x^2}}}dx \geq n\sqrt[n]{(1+\sqrt{2})^{\log(n\sqrt{2})}}, n \in \mathbb{N}, n \geq 2$$

Florică Anastase

**Solution.**

***Inequality of means about integral forms:***

$$\therefore \ \frac{b-a}{\int_a^b \frac{dx}{\varphi(x)}} \le e^{\frac{1}{b-a}\int_a^b \log\varphi(x)dx} \le \frac{1}{b-a}\int_a^b \varphi(x)dx$$

$$Let\ \varphi:[0,n]\to(0,\infty),\varphi(x)=(n+x)^{\frac{1}{\sqrt{n^2+x^2}}} \to \frac{1}{n}\int_0^n (n+x)^{\frac{1}{\sqrt{n^2+x^2}}}dx$$

$$\ge e^{\frac{1}{n}\int_0^n \frac{\log(n+x)}{\sqrt{n^2+x^2}}dx} \quad (i)$$

$$\int_0^n \frac{\log(n+x)}{\sqrt{n^2+x^2}}dx = \int_0^n \sqrt{n^2+x^2}\cdot\frac{1}{n^2+x^2}\cdot\log(n+x)dx$$

***Applying Chebyshev inequality for function:*** $p(x)=\frac{1}{n^2+x^2}$ *integrable and*

$$f(x)=\sqrt{n^2+x^2}, h(x)=\log(n+x)\ increasing\ , we\ get:$$

$$\left(\int_0^n \frac{1}{n^2+x^2}\right)\cdot\left(\int_0^n \frac{\log(n+x)}{\sqrt{n^2+x^2}}dx\right) \ge \left(\int_0^n \frac{\sqrt{n^2+x^2}}{n^2+x^2}dx\right)\cdot\left(\int_0^n \frac{\log(n+x)}{n^2+x^2}dx\right) \leftrightarrow$$

$$\left(\int_0^n \frac{1}{n^2+x^2}dx\right)\cdot\left(\int_0^n \frac{\log(n+x)}{\sqrt{n^2+x^2}}dx\right) \ge \left(\int_0^n \frac{1}{\sqrt{n^2+x^2}}dx\right)\cdot\left(\int_0^n \frac{\log(n+x)}{n^2+x^2}dx\right) \quad (ii)$$

$$\int_0^n \frac{1}{n^2+x^2}dx = \frac{\pi}{4n}\ and \int_0^n \frac{1}{\sqrt{n^2+x^2}}dx = \log(1+\sqrt{2})$$

$$\int_0^n \frac{\log(n+x)}{n^2+x^2}dx \overset{x=ny,\ dx=ndy}{=} n\int_0^1 \frac{\log(n(1+y))}{n^2(1+y^2)}dy = \int_0^1 \frac{\log(n(1+y))}{n(1+y^2)}dy =$$

$$= \frac{\log n}{n}\int_0^1 \frac{dy}{1+y^2} + \frac{1}{n}\int_0^1 \frac{\log(1+y)}{1+y^2}dy = \frac{\pi\log n}{4n} + \frac{1}{n}\int_0^1 \frac{\log(1+y)}{1+y^2}dy\ , but:$$

$$\int_0^1 \frac{\log(1+y)}{1+y^2}dy \overset{y=tanz, dy=\frac{dz}{cos^2z}}{\cong} \int_0^{\frac{\pi}{4}} \frac{\log(1+tanz)}{\frac{1}{cos^2z}}\cdot\frac{dz}{cos^2z}$$

$$= \int_0^{\frac{\pi}{4}} \log(1+tanz)\,dz = \frac{\pi}{8}\log 2 \quad \rightarrow$$

$$\int_0^n \frac{\log(n+x)}{n^2+x^2}dx = \frac{\pi \log n}{4n} + \frac{\pi}{8n}\log 2 = \frac{\pi\log(n\sqrt{2})}{4n} \quad (iii)$$

***From (ii)+(iii) we get:***

$$\int_0^n \frac{\log(n+x)}{\sqrt{n^2+x^2}}dx \ge \log(n\sqrt{2})\cdot\log(1+\sqrt{2}) = \log(1+\sqrt{2})^{\log(n\sqrt{2})} \quad (iv)$$

***From (i)+(iv), we have:***

$$\int_0^n (n+x)^{\frac{1}{\sqrt{n^2+x^2}}}dx \ge n\sqrt[n]{(1+\sqrt{2})^{\log(n\sqrt{2})}}, n\in\mathbb{N}, n\ge 2$$

$\boxed{Pbl.88}$ **Find:**

$$\Omega = \lim_{n\to\infty}\frac{\int_1^n\left(\frac{tan^{-1}x}{tan^{-1}x-x}\right)^2dx}{2tan^{-1}\left(\frac{\sqrt{1+n^2}-1}{n}\right)-n}$$

*Florică Anastase*

***Solution.***

$$F(x) = \int\left(\frac{tan^{-1}x}{tan^{-1}x-x}\right)^2dx = \int\left(1+\frac{x}{tan^{-1}x-x}\right)^2dx$$

$$= \int dx + 2\int\frac{x}{tan^{-1}x-x}dx + \int\left(\frac{x}{tan^{-1}x-x}\right)^2dt = (*)$$

$$\left(\frac{x}{tan^{-1}x-x}\right)' = \left(\frac{x}{tan^{-1}x-x}\right)^2\cdot\frac{1}{1+x^2} \rightarrow$$

$$\int\left(\frac{x}{tan^{-1}x-x}\right)^2 dt$$

$$=\int\left(\frac{x}{tan^{-1}x-x}\right)'(1+x^2)dx=\frac{1+x^2}{tan^{-1}x-x}-2\int\frac{x}{tan^{-1}x-x}dx$$

$$(*)=x+\frac{1+x^2}{tan^{-1}x-x}+C$$

$$2tan^{-1}\left(\frac{\sqrt{1+n^2}-1}{n}\right)-n\overset{n=tany}{\triangleq}2tan^{-1}\left(\frac{\sqrt{1+tan^2y}-1}{tany}\right)-tany$$

$$=2tan^{-1}\left(\frac{1-cosy}{siny}\right)-tany=2tan^{-1}\left(\frac{2sin^2\frac{y}{2}}{2sin\frac{y}{2}cos\frac{y}{2}}\right)-tany$$

$$=2tan^{-1}\left(tan\frac{y}{2}\right)-tany=y-tany=tan^{-1}n-n$$

$$\Omega=\lim_{n\to\infty}\frac{\int_1^n\left(\frac{tan^{-1}x}{tan^{-1}x-x}\right)^2dx}{2tan^{-1}\left(\frac{\sqrt{1+n^2}-1}{n}\right)-n}=\lim_{n\to\infty}\frac{\frac{1+ntan^{-1}n}{tan^{-1}n-n}-\frac{\pi+4}{\pi-4}}{tan^{-1}n-n}$$

$$=\lim_{n\to\infty}\frac{(\pi-4)(1+ntan^{-1}n)-(\pi+4)(tan^{-1}n-n)}{(\pi-4)(tan^{-1}n-n)^2}$$

$$=\lim_{n\to\infty}\frac{(\pi-4)\left(tan^{-1}n+\frac{1}{n}\right)-(\pi+4)\left(\frac{tan^{-1}n}{n}-1\right)}{(\pi-4)\left(\frac{(tan^{-1}n)^2}{n}-2tan^{-1}n+n\right)}=0$$

**Pbl. 89**

$$a_n=\prod_{k=1}^{n+1}\sqrt[2^k]{tan(2^{k-1}\cdot x)};\ b_n=\sum_{k=1}^{n+1}tan^{-1}(2k^2)$$

***Find:***

$$\Omega=\lim_{n\to\infty}\left(\lim_{x\to\frac{\pi}{2^{n+2}}}\left(\frac{2b_n}{n\pi}\right)^{na_n}\right)$$

*Florică Anastase*

***Solution.***

$$cot(2^ix)-cot(2^{i+1}x)=\frac{1}{sin(2^{i+1}x)}\overset{i=\overline{0,n-1}}{\Longrightarrow}$$

$$\frac{1}{sin2x}+\frac{1}{sin4x}+\cdots+\frac{1}{sin(2^n x)}=cotx-cot(2^n x)\quad (1)$$

***If $\alpha\in\mathbb{R}$ then:***

$$\int cot(\alpha x)dx=\frac{1}{\alpha}log|sin(\alpha x)|+C$$

$$\int \frac{1}{sin(\alpha x)}dx=log\left|tan\frac{\alpha x}{2}\right|^{\frac{1}{\alpha}}+C$$

***For $a,b\in\mathbb{R}, a<b\overset{(1)}{\Rightarrow}$***

$$\int_a^b \frac{1}{sin2x}dx+\int_a^b \frac{1}{sin4x}dx+\cdots+\int_a^b \frac{1}{sin(2^n x)}dx=\int_a^b cotxdx-\int_a^b cot(2^n x)dx\quad (2)$$

***But:***

$$\int_a^b \frac{1}{sin(2^k x)}dx=log\left|tan(2^{k-1}x)\right|^{\frac{1}{2^k}}\Big|_a^b=log\sqrt[2^k]{\left|\frac{tan(2^{k-1}b)}{tan(2^{k-1}a)}\right|}, k=\overline{1,n-1}$$

$$\int_a^b cotxdx=log\left|\frac{sinb}{sina}\right|$$

$$\int_a^b cot(2^n x)dx=log\sqrt[2^n]{\left|\frac{sin(2^n b)}{sin(2^n a)}\right|}$$

$$(2)\Leftrightarrow\sum_{k=1}^n log\sqrt[2^k]{\left|\frac{tan(2^{k-1}b)}{tan(2^{k-1}a)}\right|}=log\left|\frac{sinb}{sina}\right|-log\sqrt[2^n]{\left|\frac{sin(2^n b)}{sin(2^n a)}\right|}\Leftrightarrow$$

$$log\left(\prod_{k=1}^n\sqrt[2^k]{\left|\frac{tan(2^{k-1}b)}{tan(2^{k-1}a)}\right|}\right)=log\left|\frac{sinb}{sina}\right|-log\sqrt[2^n]{\left|\frac{sin(2^n b)}{sin(2^n a)}\right|}\Leftrightarrow$$

$$\prod_{k=1}^n\sqrt[2^k]{\left|\frac{tan(2^{k-1}b)}{tan(2^{k-1}a)}\right|}=\left|\frac{sinb}{sina}\right|\cdot\sqrt[2^n]{\left|\frac{sin(2^n a)}{sin(2^n b)}\right|}\quad (3)$$

***For $b=x, a=\frac{\pi}{2^{n+1}}\Rightarrow(3)$ becomes:***

$$a_n=\prod_{k=1}^{n+1}\sqrt[2^k]{tan(2^{k-1}\cdot x)}=2^{1-\frac{1}{2^{n+1}}}\cdot\frac{sinx}{\sqrt[2^{n+1}]{sin(2^{n+1}x)}}, x\epsilon\left(0,\frac{\pi}{2^{n+1}}\right)$$

$$tan^{-1}(2k^2) = \frac{\pi}{2} - tan^{-1}\left(\frac{1}{2k^2}\right) \xRightarrow{k=\overline{1,n+1}} \sum_{k=1}^{n+1} tan^{-1}\left(\frac{1}{2k^2}\right) = tan^{-1}\left(\frac{n+1}{n+2}\right) \to \frac{\pi}{4}$$

$$\Omega = \lim_{n\to\infty}\left(\lim_{x\to\frac{\pi}{2^{n+2}}}\left(\frac{2b_n}{n\pi}\right)^{na_n}\right)$$

$$= \lim_{n\to\infty}\left(\lim_{x\to\frac{\pi}{2^{n+2}}}\left(\frac{2}{n\pi}\cdot\left(\frac{(n+1)\pi}{2} - \sum_{k=1}^{n+1} tan^{-1}\left(\frac{1}{2k^2}\right)\right)\right)^{na_n}\right)$$

$$= \lim_{n\to\infty}\left(\lim_{x\to\frac{\pi}{2^{n+2}}}\left(1\left(\frac{1}{n} - \frac{2}{n\pi} tan^{-1}\left(\frac{n+1}{n+2}\right)\right)\right)^{na_n}\right)$$

$$= \lim_{n\to\infty} e^{\frac{1}{2}\cdot\lim_{x\to\frac{\pi}{2^{n+2}}}\left(1+\frac{2}{\pi}a_n\right)}$$

$$= \lim_{n\to\infty} e^{\frac{1}{2}\cdot 2^{1-\frac{1}{2^{n+1}}}\cdot\frac{sin\frac{\pi}{2^{n+2}}}{2^{n+1}\sqrt{sin\frac{\pi}{2}}}} = 1$$

$\boxed{Pbl.90}$

$$\omega_n = (2n+1)\left(\sum_{k=0}^{2n} tan\left(x + \frac{k\pi}{2n+1}\right)\right)^{-1}$$

$$Find: \Omega = \lim_{n\to\infty}\left(\lim_{x\to\frac{\pi}{2n+1}}\left(\frac{cotx}{cot\frac{\pi}{2n+1}}\right)^{\omega_n}\right)$$

*Florică Anastase*

***Solution:***

$$Since: \prod_{k=0}^{n-1} cos\left(x + \frac{k\pi}{n}\right) = \frac{sin\left(n\left(x+\frac{\pi}{2}\right)\right)}{2^{n-1}}$$

$$\to \sum_{k=0}^{n-1} log\left(cos\left(x + \frac{k\pi}{n}\right)\right) = log\left(\frac{sin\left(n\left(x+\frac{\pi}{2}\right)\right)}{2^{n-1}}\right)$$

***Now by differentiable both side with respect to x we get:***

$$\sum_{k=0}^{n-1}\frac{d}{dx}\log\left(\cos\left(x+\frac{k\pi}{n}\right)\right)=\frac{d}{dx}\log\left(\frac{\sin\left(n\left(x+\frac{\pi}{2}\right)\right)}{2^{n-1}}\right)\to$$

$$\sum_{k=0}^{n-1}\tan\left(x+\frac{k\pi}{n}\right)=-n\cot\left(n\left(x+\frac{\pi}{2}\right)\right)\to$$

$$\omega_n=\frac{2n+1}{\sum_{k=0}^{n-1}\tan\left(x+\frac{k\pi}{2n+1}\right)}=\frac{2n+1}{-(2n+1)\cot\left((2n+1)(x+\frac{\pi}{2}\right)}=\cot((2n+1)x)$$

$$Let:\varphi_n=\lim_{x\to\frac{\pi}{2n+1}}\left(\frac{\cot x}{\cot\frac{\pi}{2n+1}}\right)^{\omega_n}$$

$$=\exp\left(\lim_{x\to\frac{\pi}{2n+1}}\cot((2n+1)x)\log\left(\frac{\cot x}{\cot\left(\frac{\pi}{2n+1}\right)}\right)\right)=$$

$$=\exp\left(\lim_{x\to\frac{\pi}{2n+1}}\frac{\log\left(\frac{\cot x}{\cot\left(\frac{\pi}{2n+1}\right)}\right)}{\tan((2n+1)x)}\right)\overset{hopital}{\cong}$$

$$=\exp\left(\lim_{x\to\frac{\pi}{2n+1}}\frac{-\sec x\cdot\csc x}{(2n+1)\sec^2((2n+1)x)}\right)=\exp\left(-\frac{\csc\left(\frac{2\pi}{2n+1}\right)}{2n+1}\right)$$

$$Now:\Omega=\lim_{n\to\infty}\left(\lim_{x\to\frac{\pi}{2n+1}}\left(\frac{\cot x}{\cot\frac{\pi}{2n+1}}\right)^{\omega_n}\right)$$

$$=\lim_{n\to\infty}\exp\left(-\frac{\csc\left(\frac{2\pi}{2n+1}\right)}{2n+1}\right)=\exp\left(-\frac{1}{2\pi}\lim_{n\to\infty}\frac{2\pi}{2n+1}\csc\left(\frac{2\pi}{2n+1}\right)\right)=\frac{1}{\sqrt[2\pi]{e}}$$

**Pbl. 91** **For $n\in\mathbb{N}^*, P_n=\prod_{k=1}^{n-1}\sin\frac{k\pi}{n}$, find:**

$$\Omega=\lim_{n\to\infty}\frac{n}{2}\cdot P_n\cdot\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\frac{\cos 3x}{\sin^n x}dx$$

*Florică Anastase*

***Solution:***

The roots of the binomial equation $x^n - 1 = 0$ are $x_j = cos2k\pi + isin2k\pi$,

$j = 1, 2, \dots, n$, $k = 0, 1, \dots, n-1$. So, we have:

$$x^n - 1 = (x - x_1)(x - x_2) \cdot \dots \cdot (x - x_j) \cdot \dots \cdot (x - x_n)$$

$$= (x-1)\left(x - cos\frac{2\pi}{n} - isin\frac{2\pi}{n}\right)\left(x - cos\frac{4\pi}{n} - isin\frac{4\pi}{n}\right) \cdot \dots$$

$$\cdot\left(x - cos\frac{2(n-1)\pi}{n} - isin\frac{2(n-1)\pi}{n}\right), \quad (1)$$

From $\frac{x^n-1}{x-1} = x^{n-1} + x^{n-2} + \dots + 1$ and $x = 1$, the relation $(1)$ becomes:

$$\left(x - cos\frac{2\pi}{n} - isin\frac{2\pi}{n}\right)\left(x - cos\frac{4\pi}{n} - isin\frac{4\pi}{n}\right) \cdot \dots \cdot \left(x - cos\frac{2(n-1)\pi}{n} - isin\frac{2(n-1)\pi}{n}\right) = n, (2)$$

Using the relations $1 - cosa = 2sin^2\frac{a}{2}, sin2a = 2sinacosa \Rightarrow (2)$ becomes:

$$\left(2sin^2\frac{\pi}{n} - 2isin\frac{\pi}{n}cos\frac{\pi}{n}\right)\left(2sin^2\frac{2\pi}{n} - 2isin\frac{2\pi}{n}cos\frac{2\pi}{n}\right) \cdot \dots$$

$$\cdot\left(2sin^2\frac{(n-1)\pi}{n} - 2isin\frac{(n-1)\pi}{n}cos\frac{(n-1)\pi}{n}\right) = n, \qquad (3)$$

Multiplying each factor with $i$ and substracting factor $2^{n-1}$ we get $P_n = \frac{n}{2^{n-1}}$.

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{cos3x}{sin^n x}dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{4cos^3x - 3cosx}{sin^n x}dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{4cos^3x - 3cosx}{sin^n x}dx$$

$$= 4\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{(cos^2x - 1)cosx}{sin^n x}dx + 4\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{cosx}{sin^n x}dx = 4\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{cosx}{sin^n x}dx - 4\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{cosx}{sin^{n-2} x}dx$$

$$= \frac{4}{n-3} \cdot \frac{1}{sin^{n-3}x}\Bigg|_{\frac{\pi}{6}}^{\frac{\pi}{2}} - \frac{4}{n-1} \cdot \frac{1}{sin^{n-1}x}\Bigg|_{\frac{\pi}{6}}^{\frac{\pi}{2}} = \frac{-2^n + 3n - 1}{(n-3)(n-1)}, n \ge 3$$

$$\Omega = \lim_{n\to\infty} \frac{n}{2} \cdot P_n \cdot \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{cos3x}{sin^n x}dx = \lim_{n\to\infty} \frac{n^2}{2^n} \cdot \frac{-2^n + 3n - 1}{(n-3)(n-1)} = -1.$$

**Pbl. 92** If $S_n = \sum_{k=1}^{n} log\left(cos\frac{\pi}{2^{k+2}}\right)$ then find:

$$\Omega = \lim_{n\to\infty}\left(\sqrt[n]{n\cdot S_n}\right)^{\sum_{k=3}^{n} tan\frac{\pi}{k}}$$

*Florică Anastase*

*Solution*

Let $S_n(x) = \sum_{k=1}^{n} log\left(cos\frac{x}{2^k}\right)$, with $cos\frac{x}{2} > 0$ and from $sin2a = 2sinacosa$ we get:

$$log(cosa) = log(sin2a) - log(sina) - log2$$

For $a = \frac{x}{2}, \frac{x}{2^2}, \dots, \frac{x}{2^n}$ we get: $S_n(x) = log(sinx) - nlog2 - log\left(sin\frac{x}{2^n}\right) = log\left(\frac{sinx}{2^n sin\frac{x}{2^n}}\right)$

$$\text{Then: } S_n = S_n\left(\frac{\pi}{4}\right) = log\left(\frac{\frac{1}{\sqrt{2}}}{2^n sin\frac{\pi}{2^{n+2}}}\right) = log\left(\frac{1}{\sqrt{2}\cdot 2^n sin\frac{\pi}{2^{n+2}}}\right)\xrightarrow[n\to\infty]{} log\left(\frac{2\sqrt{2}}{\pi}\right)$$

$$\Omega = \lim_{n\to\infty}\left(\sqrt[n]{n\cdot S_n}\right)^{\sum_{k=3}^{n} tan\frac{\pi}{k}} = e^{\lim\limits_{n\to\infty}\left(\sum_{k=3}^{n} tan\frac{\pi}{k}\right)log\sqrt[n]{n\cdot S_n}}$$

$$= e^{\lim\limits_{n\to\infty}\left(\sum_{k=3}^{n} tan\frac{\pi}{k}\right)\cdot\frac{log(n\cdot S_n)}{n}} = e^{\lim\limits_{n\to\infty}\frac{\left(\sum_{k=3}^{n} tan\frac{\pi}{k}\right)}{\sqrt{n}}\cdot\frac{log(n\cdot S_n)}{\sqrt{n}}}, \quad (1)$$

$$\lim_{n\to\infty}\frac{\left(\sum_{k=3}^{n} tan\frac{\pi}{k}\right)}{\sqrt{n}} \overset{S-Cesaro}{=} \lim_{n\to\infty}\frac{tan\frac{\pi}{n+1}}{\sqrt{n+1}-\sqrt{n}} = \lim_{n\to\infty}\frac{tan\frac{\pi}{n+1}}{\frac{\pi}{n+1}}\cdot\frac{\pi}{(n+1)(\sqrt{n+1}-\sqrt{n})}$$

$$= 0, \qquad (2)$$

$$\lim_{n\to\infty}\frac{log(n\cdot S_n)}{\sqrt{n}} = \lim_{n\to\infty}\frac{logn + logS_n}{\sqrt{n}} = \lim_{n\to\infty}\frac{logn}{\sqrt{n}} \overset{S-Cesaro}{=} \lim_{n\to\infty}\frac{log(n+1) - logn}{\sqrt{n+1}-\sqrt{n}} =$$

$$= \lim_{n\to\infty}\frac{log\left(1+\frac{1}{n}\right)^n}{n(\sqrt{n+1}-\sqrt{n})} = 0, \qquad (3)$$

From $(1), (2), (3)$ we get: $\Omega = e^0 = 1$.

**Pbl. 93** Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{n}{logn}\right)^e\cdot e^{\int_0^e log\left(\frac{log(x+e)}{x^2+ne}\right)dx}$$

*Florică Anastase*

***Solution.***

$$\frac{\beta-\alpha}{\int_\alpha^\beta \frac{1}{f(x)}dx} \le e^{\frac{1}{\beta-\alpha}\int_\alpha^\beta \log(f(x))dx} \le \frac{1}{\beta-\alpha}\int_\alpha^\beta f(x)dx, \alpha$$

$$< \beta\ (*) \text{ means Integral Inequality}$$

For $x \in [0,e] \Rightarrow ne \le x^2 + ne \le e^2 + ne \Rightarrow \frac{\log n}{e^2+ne} \le \frac{\log(x+e)}{x^2+ne} \le \frac{\log(e+n)}{ne}$

Let $f:[0,e] \to \mathbb{R}, f(x) = \frac{\log(x+e)}{x^2+ne}$ we have:

$$\frac{e}{\int_0^e \frac{1}{f(x)}dx} = \frac{e}{\int_0^e \frac{x^2+ne}{\log(x+e)}dx} \ge \frac{e}{\int_0^e \frac{e^2+ne}{\log n}dx} = \frac{1}{e}\cdot\frac{\log n}{e+n}, \quad (1)$$

$$\frac{1}{e}\int_0^e f(x)dx = \frac{1}{e}\int_0^e \frac{\log(x+e)}{x^2+ne}dx \le \frac{1}{e}\int_0^e \frac{\log(e+n)}{ne}dx = \frac{1}{e}\cdot\frac{\log(e+n)}{n}, \quad (2)$$

From $(1),(2)$ we get:

$$\frac{1}{e}\cdot\frac{\log n}{e+n} \le e^{\frac{1}{e}\int_0^e \log\left(\frac{\log(x+e)}{x^2+ne}\right)dx} \le \frac{1}{e}\cdot\frac{\log(e+n)}{n}$$

$$\frac{1}{e}\cdot\frac{n}{e+n} \le \frac{n}{\log n}\cdot e^{\frac{1}{e}\int_0^e \log\left(\frac{\log(x+e)}{x^2+ne}\right)dx} \le \frac{1}{e}\cdot\frac{\log(e+n)}{\log n}$$

$$\frac{1}{e^e}\cdot\left(\frac{n}{e+n}\right)^e \le \left(\frac{n}{\log n}\right)^e \cdot e^{\int_0^e \log\left(\frac{\log(x+e)}{x^2+ne}\right)dx} \le \frac{1}{e^e}\cdot\left(\frac{\log(e+n)}{\log n}\right)^e$$

$$\lim_{n\to\infty}\frac{n}{e+n} = \lim_{n\to\infty}\frac{\log(e+n)}{\log n} = 1$$

So,

$$\Omega = \lim_{n\to\infty}\left(\frac{n}{\log n}\right)^e \cdot e^{\int_0^e \log\left(\frac{\log(x+e)}{x^2+ne}\right)dx} = \frac{1}{e^e}$$

**Pbl. 94** .Let $(x_n)_{n\ge1}, x_1 = 0, x_n = \frac{(1-n)x_{n-1}+1-2n}{nx_{n-1}+2n}$

*Find:*

$$\Omega = \lim_{n\to\infty}\prod_{k=1}^n (2 + x_k)$$

*Florică Anastase*

**Solution.**

$$x_n = \frac{(1-n)x_{n-1}+1-2n}{nx_{n-1}+2n} \Leftrightarrow nx_n(2+x_{n-1}) = 1-n+(1-n)x_n-n$$

$$nx_n(1+x_{n-1})+(n-1)(1+x_{n-1}) = -n-nx_n$$

$$1+x_{n-1} = n(1+x_n)(2+x_{n-1})$$

$$\frac{1}{1+x_n} = \frac{n(2+x_{n-1})}{1+x_{n-1}} \Leftrightarrow \frac{1}{1+x_n} = n\left(1+\frac{1}{1+x_{n-1}}\right)$$

**Let:** $a_n = \frac{1}{1+x_n}, a_1 = 1 \Rightarrow a_n = n(1+a_{n-1}); (1)$

$$2+x_k = 1+\frac{1}{a_k} = \frac{1+a_k}{a_k} = \frac{1}{k+1}\cdot\frac{a_{k+1}}{a_k}$$

$$\prod_{k=1}^{n}(2+x_k) = \prod_{k=1}^{n}\left(\frac{1}{k+1}\cdot\frac{a_{k+1}}{a_k}\right) = \frac{a_{n+1}}{(n+1)!} = \frac{(n+1)a_n}{(n+1)!} \overset{(1)}{=} \frac{1+a_n}{n!}; \quad (2)$$

$$1+a_k = k(1+a_{k-1})+1 \Leftrightarrow \frac{1+a_k}{k!} - \frac{1+a_{k-1}}{(k-1)!} = \frac{1}{k!} \xrightarrow{k=\overline{1,n}}$$

$$\frac{1+a_n}{n!} = 1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!} = E_n; \quad (3)$$

**From (2)+(3) we have:**

$$\Omega = \lim_{n\to\infty}\prod_{k=1}^{n}(2+x_k) = \lim_{n\to\infty}E_n = e$$

**Pbl. 95** **Let $(a_n)_{n\geq1}, a_1 = e, a_n = e^n a_{n-1}^n$ and $(b_n)_{n\geq1}$ such that:**

$$\left(1+\frac{1}{n}\right)^{n+b_n} = \prod_{k=1}^{n}\left(1+\frac{1}{\log a_k}\right)$$

**Find:**

$$\Omega = \lim_{n\to\infty} b_n$$

*Florică Anastase*

**Solution.**

$$a_n = e^n a_{n-1}^n \Leftrightarrow \log a_n = n + n\log a_{n-1} = n(1+\log a_{n-1})$$

**Let:** $x_n = \log a_n;\ x_1 = 1 \Rightarrow x_n = n(1+x_{n-1}), x_1 = 1$

$$1+x_k=k(1+x_{k-1})+1\Rightarrow\frac{1+x_k}{k!}-\frac{1+x_{k-1}}{(k-1)!}=\frac{1}{k!}\Rightarrow$$

$$\frac{1+x_n}{n!}=1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}=E_n$$

$$\prod_{k=1}^{n}\left(1+\frac{1}{loga_k}\right)=\prod_{k=1}^{n}\left(1+\frac{1}{x_k}\right)=\prod_{k=1}^{n}\left(\frac{1}{k+1}\cdot\frac{x_{k+1}}{x_k}\right)=\frac{x_{n+1}}{(n+1)!}=\frac{1+x_n}{n!}$$

$$\lim_{n\to\infty}\prod_{k=1}^{n}\left(1+\frac{1}{loga_k}\right)=\lim_{n\to\infty}\frac{1+x_n}{n!}=e$$

$$\lim_{n\to\infty}b_n=\lim_{n\to\infty}\left(\frac{log\left(1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}\right)}{log\left(1+\frac{1}{n}\right)}-n\right)$$

$$=\lim_{n\to\infty}\left(\frac{log\left(1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}\right)-1}{log\left(1+\frac{1}{n}\right)}+\frac{1}{log\left(1+\frac{1}{n}\right)}-n\right);\quad(1)$$

$$\lim_{n\to\infty}\left(\frac{log\left(1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}\right)-1}{log\left(1+\frac{1}{n}\right)}\right)\overset{L.C-S}{=}\lim_{n\to\infty}\frac{log\left(1+\frac{\frac{1}{(n+1)!}}{1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}}\right)}{log\left(1-\frac{1}{(n+1)^2}\right)}$$

$$=\lim_{n\to\infty}\frac{log\left(1+\frac{\frac{1}{(n+1)!}}{1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}}\right)^{n^2}}{log\left(1-\frac{1}{(n+1)^2}\right)^{n^2}}=0;\quad(2)$$

$$\lim_{n\to\infty}\left(\frac{1}{log\left(1+\frac{1}{n}\right)}-n\right)=\lim_{x\to 0}\left(\frac{1}{log(1+x)}-\frac{1}{x}\right)=\frac{1}{2};(3)$$

*From (1)+(2)+(3) we have:*

$$\Omega=\lim_{n\to\infty}b_n=\frac{1}{2}$$

$\boxed{Pbl.96}$ **If $n\in\mathbb{N},n\geq 2$ prove that:**

$$\frac{n}{n+2}+\int_0^1\left(\tan^{-1}(x^n)\right)^2dx\geq 2\int_0^1\tan^{-1}(x^n)\sqrt[n]{\tan^{-1}x}dx$$

*Florică Anastase*

**Solution.**

$$\tan x\geq x,\forall x\in[0,1]\Rightarrow\tan^{-1}x\leq x,\forall x\in[0,1]$$

$$\left(\sqrt[n]{x}-\tan^{-1}(x^n)\right)^2\geq 0$$

$$\sqrt[n]{x^2}-2\sqrt[n]{x}\tan^{-1}(x^n)+\left(\tan^{-1}(x^n)\right)^2\geq 0$$

$$\sqrt[n]{x^2}+\left(\tan^{-1}(x^n)\right)^2\geq 2\sqrt[n]{x}\tan^{-1}(x^n)\geq 2\sqrt[n]{\tan^{-1}x}\tan^{-1}(x^n)$$

$$\int_0^1\sqrt[n]{x^2}dx+\int_0^1\left(\tan^{-1}(x^n)\right)^2dx\geq 2\int_0^1\sqrt[n]{\tan^{-1}x}\tan^{-1}(x^n)dx$$

$$\frac{n}{n+2}+\int_0^1\left(\tan^{-1}(x^n)\right)^2dx\geq 2\int_0^1\tan^{-1}(x^n)\sqrt[n]{\tan^{-1}x}dx\,;n\in\mathbb{N},n\geq 2$$

**Pbl. 97** **Find:**

$$\Omega=\lim_{n\to\infty}\sum_{1\leq i\leq i\leq n}\frac{(n^2+ni+i^2)(n^2+nj+j^2)}{(n^2+i^2)(n^2+j^2)\sqrt{n^4+i^2+j^2}}\cdot e^{\tan^{-1}\left(\frac{n(i+j)}{n^2-ij}\right)}$$

*Florică Anastase*

**Solution.**

$$\sum_{1\leq i\leq i\leq n}\frac{(n^2+ni+i^2)(n^2+nj+j^2)}{(n^2+i^2)(n^2+j^2)}\cdot e^{\tan^{-1}\left(\frac{n(i+j)}{n^2-ij}\right)}=$$

$$=\sum_{1\leq i\leq j\leq n}\left(1+\frac{ni}{n^2+i^2}\right)\left(1+\frac{nj}{n^2+j^2}\right)e^{\tan^{-1}\left(\frac{i}{n}\right)+\tan^{-1}\left(\frac{j}{n}\right)}=$$

$$=\sum_{1\leq i\leq j\leq n}\left(1+\frac{\frac{i}{n}}{1+\left(\frac{i}{n}\right)^2}\right)e^{\tan^{-1}\left(\frac{i}{n}\right)}\cdot\left(1+\frac{\frac{j}{n}}{1+\left(\frac{j}{n}\right)^2}\right)e^{\tan^{-1}\left(\frac{j}{n}\right)}=$$

$$=\sum_{1\le i\le j\le n} f\left(\frac{i}{n}\right)f\left(\frac{j}{n}\right), \text{where } f:\mathbb{R}\to\mathbb{R}, f(t)=\left(1+\frac{t}{t^2+1}\right)e^{\tan^{-1}t}$$

$$\sum_{1\le i\le j\le n}\frac{f\left(\frac{i}{n}\right)f\left(\frac{j}{n}\right)}{\sqrt{n^4+2n^2}}\le\sum_{1\le i\le j\le n}\frac{f\left(\frac{i}{n}\right)f\left(\frac{j}{n}\right)}{\sqrt{n^4+i^2+j^2}}\le\sum_{1\le i\le j\le n}\frac{f\left(\frac{i}{n}\right)f\left(\frac{j}{n}\right)}{\sqrt{n^4+n^2}}$$

$$\lim_{n\to\infty}\sum_{1\le i\le j\le n}\frac{f\left(\frac{i}{n}\right)f\left(\frac{j}{n}\right)}{\sqrt{n^4+2n^2}}=\lim_{n\to\infty}\frac{1}{n^2}\sum_{1\le i\le j\le n}\frac{f\left(\frac{i}{n}\right)f\left(\frac{j}{n}\right)}{\sqrt{1+\frac{2}{n^2}}}=\frac{1}{2}\left(\int_0^1 f(x)dx\right)^2$$

$$\lim_{n\to\infty}\sum_{1\le i\le j\le n}\frac{f\left(\frac{i}{n}\right)f\left(\frac{j}{n}\right)}{\sqrt{n^4+n^2}}=\lim_{n\to\infty}\frac{1}{n^2}\sum_{1\le i\le j\le n}\frac{f\left(\frac{i}{n}\right)f\left(\frac{j}{n}\right)}{\sqrt{1+\frac{1}{n^2}}}=\frac{1}{2}\left(\int_0^1 f(x)dx\right)^2$$

**Hence,**

$$\lim_{n\to\infty}\sum_{1\le i\le j\le n}\frac{f\left(\frac{i}{n}\right)f\left(\frac{j}{n}\right)}{\sqrt{n^4+i^2+j^2}}=\frac{1}{2}\left(\int_0^1 f(x)dx\right)^2$$

$$\int_0^1 f(x)dx=\int_0^1\frac{x^2+x+1}{x^2+1}\cdot e^{\tan^{-1}x}dx=\int_0^1\left(1+\frac{x}{1+x^2}\right)\cdot e^{\tan^{-1}x}dx=$$

$$=\int_0^1 e^{\tan^{-1}x}dx+\int_0^1\frac{x}{1+x^2}\cdot e^{\tan^{-1}x}dx=$$

$$=x\cdot e^{\tan^{-1}x}\Big|_0^1-\int_0^1\frac{x}{1+x^2}\cdot e^{\tan^{-1}x}dx+\int_0^1\frac{x}{1+x^2}\cdot e^{\tan^{-1}x}dx=$$

$$=x\cdot e^{\tan^{-1}x}\Big|_0^1=e^{\frac{\pi}{4}}$$

**Therefore,**

$$\lim_{n\to\infty}\sum_{1\le i\le j\le n}\frac{f\left(\frac{i}{n}\right)f\left(\frac{j}{n}\right)}{\sqrt{n^4+i^2+j^2}}=\frac{1}{2}e^{\frac{\pi}{2}}$$

$\boxed{Pbl.98}$ **If $0<a<b\le 2a, f:[a,b]\to[0,c), f$ –continuous, $f(a)=0$, $f'(a)\ge 0, f'$ –increasing, then:**

$$\frac{(b-a)\left(c(b-a)+af(b)\right)}{bf(b)+c(b-a)-2\int_a^b f(x)dx}\geq c\left(c-f(b)\right)\int_a^b \frac{dx}{\left(c-f(x)\right)^2}$$

*Florică Anastase*

*Solution.*

$$f' \text{ –increasing} \Rightarrow \begin{cases} f'(x)\geq f'(a)\geq 0, \forall x\in[a,b]\Rightarrow f\nearrow \\ f-\text{convex}\Rightarrow f''(x)\geq 0, \forall x\in[a,b]\end{cases}$$

$$\int_a^b \frac{dx}{c-f(x)}+\int_a^b \frac{xf'(x)dx}{\left(c-f(x)\right)^2}=\int_a^b \frac{dx}{c-f(x)}+\int_a^b x\left(\frac{1}{c-f(x)}\right)'dx=$$

$$=\int_a^b \frac{dx}{c-f(x)}+\frac{x}{c-f(x)}\Big|_a^b-\int_a^b \frac{dx}{c-f(x)}=\frac{b}{c-f(b)}-\frac{a}{c}=\frac{c(b-a)+af(b)}{c(c-f(b)}$$

Hence,

$$\int_a^b \frac{xf'(x)-f(x)+c}{\left(c-f(x)\right)^2}dx=\frac{c(b-a)+af(b)}{c(c-f(b)}$$

Let $g,h:[a,b]\to\mathbb{R}, g(x)=xf'(x)-f(x)+c$ and $h(x)=\frac{1}{(c-x)^2}$

$$g'(x)=xf''(x)\geq 0, \forall x\in[a,b]\Rightarrow g\nearrow [a,b]$$

$$h'(x)=\frac{2}{\left(c-f(x)\right)^3}>0\Rightarrow h\left(f(x)\right)=\frac{1}{\left(c-f(x)\right)^2}>0\Rightarrow h\left(f(x)\right)\nearrow [a,b]$$

$$\int_a^b \frac{xf'(x)-f(x)+c}{\left(c-f(x)\right)^2}dx\overset{\text{Chebyshev}'}{\geq}\frac{1}{b-a}\left(\int_a^b (xf'(x)-f(x)+c)dx\right)\left(\int_a^b \frac{dx}{\left(c-f(x)\right)^2}\right)$$

$$\frac{c(b-a)+af(b)}{c(c-f(b)}\geq\frac{1}{b-a}\left(bf(b)+c(b-a)-2\int_a^b f(x)dx\right)\left(\int_a^b \frac{dx}{\left(c-f(x)\right)^2}\right)$$

Therefore,

$$\frac{(b-a)\left(c(b-a)+af(b)\right)}{bf(b)+c(b-a)-2\int_a^b f(x)dx}\geq c\left(c-f(b)\right)\int_a^b \frac{dx}{\left(c-f(x)\right)^2}$$

,where $bf(b)+c(b-a)-2\int_a^b f(x)dx\overset{f\nearrow}{\geq} bf(b)-c(b-a)-2(b-a)f(b)=$

$$= c(b-a) + (2a-b)f(b) > 0$$

**Pbl. 99** **Find:**

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{\sqrt{n}}\cdot\sqrt[n^2]{\prod_{k=1}^{n} k^k \cdot e^{\sum_{k=1}^{n} k e^{\frac{k}{n^2}}}}\right)$$

*Florică Anastase*

***Solution.***

$$\lim_{x\to 0}\frac{e^x-1}{x} = 1 \to \forall\varepsilon > 0, \exists\delta_\varepsilon > 0 \text{ such that } \left|\frac{e^x-1}{x} - 1\right| < \varepsilon \to$$

$$(1-\varepsilon)\frac{k}{n^2} + 1 < e^{\frac{k}{n^2}} < (1+\varepsilon)\frac{k}{n^2} + 1 \leftrightarrow$$

$$\frac{1}{n^2}\sum_{k=1}^{n}\left[(1-\varepsilon)\frac{k^2}{n^2} + k\right] \le \frac{1}{n^2}\sum_{k=1}^{n} k e^{\frac{k}{n^2}} \le 1/n^2\sum_{k=1}^{n}\left[(1+\varepsilon)\frac{k^2}{n^2} + k\right]$$

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n}\left[(1-\varepsilon)\frac{k^2}{n^2} + k\right] = \lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n}\left[(1+\varepsilon)\frac{k^2}{n^2} + k\right] = \frac{1}{2} \to$$

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n} k e^{\frac{k}{n^2}} = \frac{1}{2} \to \lim_{n\to\infty}\left(\sqrt[n^2]{e^{\sum_{k=1}^{n} k e^{\frac{k}{n^2}}}}\right) = \sqrt{e}; (1)$$

Now,

$$\frac{1}{\sqrt{n}}\cdot\sqrt[n^2]{\prod_{k=1}^{n} k^k} = \sqrt[n^2]{\prod_{k=1}^{n}\left(\frac{k}{n}\right)^k}\cdot\frac{1}{\sqrt{n}}\cdot\sqrt[n^2]{n^{\frac{n(n+1)}{2}}} = \sqrt[n^2]{\prod_{k=1}^{n}\left(\frac{k}{n}\right)^k}\cdot n^{\frac{1}{2n}} = a_n\cdot n^{\frac{1}{2n}}; n^{\frac{1}{2n}} \to 1$$

$$a_n = \sqrt[n^2]{\prod_{k=1}^{n}\left(\frac{k}{n}\right)^k} \to \log a_n = \frac{1}{n}\sum_{k=1}^{n}\frac{k}{n}\log\left(\frac{k}{n}\right) = \sigma_\Delta(f,\xi);$$

$$\Delta = \left(0,\frac{1}{n},\frac{2}{n},\dots,1\right), \xi \in \left\{\frac{1}{n},\frac{2}{n},\dots,1\right\}, f:[0,1]\to\mathbb{R}, f(x) = \begin{cases} x\log x, x > 0 \\ 0, x = 0\end{cases}$$

$$F(x) = \begin{cases}\frac{x^2}{2}\log x - \frac{x^2}{4}, x > 0 \\ 0, x = 0\end{cases} \to \lim_{n\to\infty}\log a_n = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{k}{n}\log\left(\frac{k}{n}\right) = \int_0^1 f(x)dx = -\frac{1}{4}$$

$$\rightarrow \lim_{n\to\infty} a_n = \frac{1}{\sqrt[4]{e}};(2)$$

From (1),(2) it follows that:

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{\sqrt{n}}\cdot\sqrt[n^2]{\prod_{k=1}^{n} k^k\cdot e^{\sum_{k=1}^{n} ke^{\frac{k}{n^2}}}}\right) = \sqrt[4]{e}$$

$\boxed{Pbl.100}$ **Find:**

$$\Omega = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\int_{k-1}^{k} tan^{-1}\left(\frac{n(x-k)}{kx+n^2}\right)dx\right)$$

*Florică Anastase*

*Solution.*

$$\sum_{k=1}^{n}\int_{k-1}^{k} tan^{-1}\left(\frac{n(x-k)}{kx+n^2}\right)dx = \sum_{k=1}^{n}\int_{k-1}^{k} tan^{-1}\left(\frac{x-k}{\frac{k}{n}x+n}\right)dx \overset{x\to nx}{=}$$

$$= n\sum_{k=1}^{n}\int_{\frac{k-1}{n}}^{\frac{k}{n}} tan^{-1}\left(\frac{x-\frac{k}{n}}{1+\frac{k}{n}x}\right) = n\sum_{k=1}^{n}\int_{\frac{k-1}{n}}^{\frac{k}{n}}\left(tan^{-1}x - tan^{-1}\left(\frac{k}{n}\right)\right)dx =$$

$$= n\left(\int_0^1 tan^{-1}x - \frac{1}{n}\sum_{k=1}^{n} tan^{-1}\left(\frac{k}{n}\right)\right)$$

Let $f:\mathbb{R}\to\mathbb{R}, f(x) = tan^{-1}x, f''(x) < 0, \forall x\in\mathbb{R} \Rightarrow f' \searrow$ and let $x \in \left[\frac{k-1}{n},\frac{k}{n}\right) \overset{MVT}{\longrightarrow}$

$\exists c \in \left(\frac{k-1}{n},\frac{k}{n}\right)$ such that $\frac{f(x)-f\left(\frac{k}{n}\right)}{x-\frac{k}{n}} = f'(c) \Rightarrow$

$$f'\left(\frac{k-1}{n}\right) > \frac{f(x)-f\left(\frac{k}{n}\right)}{x-\frac{k}{n}} > f'\left(\frac{k}{n}\right)|\cdot\left(x-\frac{k}{n}\right) < 0 \rightarrow$$

$$\left(x-\frac{k}{n}\right)f'\left(\frac{k-1}{n}\right) < f(x) - f\left(\frac{k}{n}\right) < \left(x-\frac{k}{n}\right)f'\left(\frac{k}{n}\right)$$

$$-\frac{1}{2n^2}\cdot f'\left(\frac{k-1}{n}\right) \le \int_{\frac{k-1}{n}}^{\frac{k}{n}}\left(tan^{-1}x - tan^{-1}\left(\frac{k}{n}\right)\right) \le -\frac{1}{2n^2}f'\left(\frac{k}{n}\right), n\ge 2, k\in\{1,2,\dots,n\}$$

**Thus,**

$$-\frac{1}{2n}\cdot\sum_{k=1}^{n}f'\left(\frac{k-1}{n}\right)\le n\left(\int_0^1 tan^{-1}x-\frac{1}{n}\sum_{k=1}^{n}tan^{-1}\left(\frac{k}{n}\right)\right)\le-\frac{1}{2n}\cdot\sum_{k=1}^{n}f'\left(\frac{k}{n}\right)$$

$$\lim_{n\to\infty}\frac{1}{n}\cdot\sum_{k=1}^{n}f'\left(\frac{k-1}{n}\right)=\lim_{n\to\infty}\frac{1}{n}\cdot\sum_{k=1}^{n}f'\left(\frac{k}{n}\right)=\int_0^1 f'(x)\,dx=\frac{\pi}{4}$$

**Therefore,**

$$\Omega=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\int_{k-1}^{k}tan^{-1}\left(\frac{n(x-k)}{kx+n^2}\right)dx\right)=\lim_{n\to\infty}n\left(\int_0^1 tan^{-1}x-\frac{1}{n}\sum_{k=1}^{n}tan^{-1}\left(\frac{k}{n}\right)\right)$$

$$=-\frac{\pi}{8}$$

**Reference:**

**ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro**