The vertex removal trick

Aditya Guha Roy

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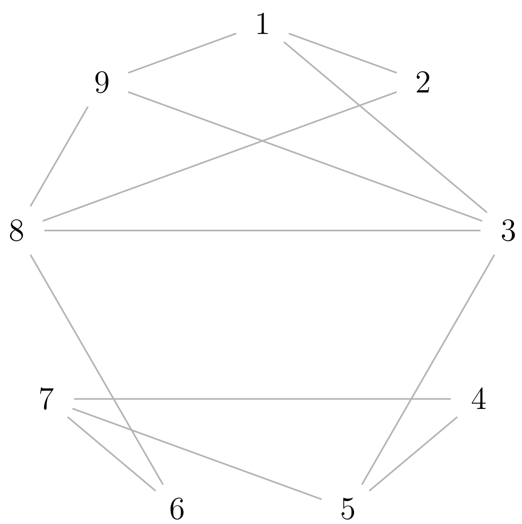
Introduction

A ***graph*** refers to a collection of vertices, some of which may be joined by edges. Formally a graph can be represented by a pair (V, E) where V corresponds to a set of ***vertices*** (also called ***nodes***) and E is a multi-set whose elements are of the form $\{a, b\}$ for $a, b \in V$. When, in addition we want to give directions to the edges, we get ***directed graphs*** and for directed graphs $E \subseteq V \times V$. We call the elements of E as the ***edges*** of the graph G. Unless mentioned otherwise, by "graphs" we shall mean undirected graphs. For a graph G, vertices a, b of G, an edge $e = \{a, b\}$ in G is said to have ***end-points*** a, b. Furthermore, two or more edges having the same endpoints are called ***multi-edges.*** A graph having no multi-edges is called a ***simple graph***. For a graph G, and a vertex a of G, the number of edges of G, having an endpoint a, is called the ***degree*** of a. A graph G' is said to be a ***subgraph*** of G if the vertex set of G' is a subset of the vertex set of G, and the edge set of G' is a subset of the edge set of G. If G' is a subgraph of G, then we say that G is a ***supergraph*** of G'.
Below is an example of a graph, where the numbers denote the vertices and the straight lines denote the edges. It is a simple graph because it contains no multi-edges.



We call a class $\mathcal{C}$ of graphs a ***subgraph invariant*** class of graphs if for any $G \in \mathcal{C}$, and any subgraph H of G, we have $H \in \mathcal{C}$. Many important classes of graphs are subgraph invariants. Examples include planar graphs, graphs not containing C_t for a fixed t, or graphs not containing K_t for a fixed t, etc.. In fact, such classes frequently arise in topics such as extremal graph theory.

1 The vertex removal trick

A common breed of problems in extremal graph theory asks to prove some properties about graphs when some relation between the order and the size of the graph is provided. Often, one is able to translate the to-be-proved criteria into a form which concerns some subgraph invariant class of graphs. In this setting, a possible way to approach such problems is to look for construction of a subgraph which obeys the given relation between its size and order, and then induce on the order (or sometimes size) of the subgraph.

The big picture of the procedure can be summarised as :

* formulate an induction hypothesis carefully, and prove the base case ;
* obtain a subgraph of the given graph which obeys the given type of relation between its size and order ;
* use the induction hypothesis and you are through.

2 Examples

Let us now see an example of how this method can be used.

Example 2.1. Let G be a simple and connected graph having n vertices and $n-1$ edges. Prove that G has no cycles.

Solution. We shall use the method discussed in the previous section. Let $P(n)$ be the assertion that if connected simple graph has n vertices and $n-1$ edges, then it is acyclic[1].
Clearly $P(2)$ is true.
Now, let ℓ be a positive integer such that $k \leq \ell \implies$ the assertion $P(k)$ is true. Let $n = \ell + 1$, and let G be a simple connected graph on n vertices and $n-1$ edges. Assume to the contrary that G has a cycle. Then notice that, since $2 \cdot (n-1) \leq 2n$, so there must exist at least one vertex in G which has degree $= 1$. Now, let v_0 be such a vertex in G. We remove v_0 from G. Then, we will have a graph G' on $n-1$ vertices having $n-2$ edges, which is connected.
Furthermore, we notice that G has no cycles if and only if G' has no cycles (since, the node v_0 having degree 1 does not participate in any cycle).
Now, by the induction hypothesis, we have G' is acyclic. Hence G must also be acyclic.

This completes the solution.

□

[1] a graph having no cycles is called an acyclic graph

Example 2.2 (Erdős)**.** Let G be a simple graph having $n \geq 4$ vertices, such that between any 4 vertices of G, there are at most 4 edges. Prove that the size of this graph is $\leq n^2/4$.

Solution. First of all notice that, by a straightforward verification, we have the to-be-proved assertion true for the case $n = 4$. Now, assume that there is a positive integer $m \geq 5$ such that one can find a simple graph G_m of order m such that between any four vertices of G_m, there are at most 4 edges, and such that the size of G_m is $> m^2/4$. Take m to be the smallest such positive integer. Then, there exists a real number $r > 1$, such that the size of G_m is $= r \cdot \frac{m^2}{4}$.
Now, we take a minimum degree vertex v in G_m and we remove it. Notice that, this gives us a graph G which is a subgraph of G_m, and has $m - 1$ vertices, and $r \cdot \frac{m^2}{4} -$ degree of v.
Now, notice that the degree of v is $\leq r \cdot \frac{2m^2}{4m} = r \cdot \frac{m}{2}$. This would mean that G has $\geq r \cdot \lfloor (m-1)^2/4 \rfloor$ many edges. But, notice that $r \cdot \lfloor (m-1)^2/4 \rfloor > \lfloor (m-1)^2/4 \rfloor$. And by minimality of m, G must have $\leq \frac{(m-1)^2}{4}$ many edges, and so in effect, G must have $\leq \lfloor (m-1)^2/4 \rfloor$ many edges.
This leads us to a contradiction, thereby showing that our assumption is false.
This, completes the solution.

□

3 Problems

Below are some problems left for the reader to get a hands-on experience of dealing with the result and its variants (along the lines commented earlier) related to removal of vertices from a graph.

1. (Cambridge Mathematics Tripos) Let G be a simple graph with $n \geq 6$ vertices and having $\geq \lfloor n^2/4 \rfloor + 1$ edges. Prove that G must contain a C_5.
2. a. Let G be a simple graph with $n \geq 4$ vertices and $> \frac{n}{4} \cdot (1 + \sqrt{4n-3})$ many edges. Prove that G must contain a C_4.

   b. More generally, show that if G is a simple graph having n vertices and with number of edges $> \frac{n}{4} \cdot (1 + \sqrt{1 + (4t-1)(n-1)})$, then G must contain the complete bipartite graph $K_{2,t}$.
3. Prove that for $n \geq 5$, every graph with n vertices and $\lfloor n^2/4 \rfloor + 2$ edges contains two triangles with precisely one vertex in common.

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