PP35317

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In all triangles ABC holds:
1. $\prod_{cyc}(r_a - r) \geq \frac{8sr^2}{3\sqrt{3}}$
2. $\prod_{cyc}(h_a - r) \geq \frac{16r^4}{R}$

Solution by Daniel Sitaru.

$$1.\ \prod_{cyc}(r_a - r) = \prod_{cyc}\left(\frac{F}{s-a} - \frac{F}{s}\right) =$$
$$= F^3 \prod_{cyc}\left(\frac{s-s+a}{s(s-a)}\right) =$$
$$= \frac{F^3}{s^3} \cdot \frac{abc}{(s-a)(s-b)(s-c)} = \frac{Fabc}{s^2F^2} =$$
$$= \frac{F \cdot 4RF}{s^2} = \frac{4Rr^2s^2}{s^2} \geq \frac{8sr^2}{3\sqrt{3}}$$
$$\Leftrightarrow 3\sqrt{3}R \geq 2s \Leftrightarrow s \leq \frac{3\sqrt{3}}{2}R \text{ (MITRINOVIC)}$$
$$2.\ \prod_{cyc}(h_a - r) = \prod_{cyc}\left(\frac{2F}{a} - \frac{F}{s}\right) =$$
$$= F^3 \prod_{cyc}\left(\frac{2}{a} - \frac{1}{s}\right) = F^3 \prod_{cyc} \frac{(2s-a)}{sa} =$$
$$= \frac{F^3}{s^3} \cdot \frac{(b+c)(c+a)(a+b)}{abc} =$$
$$= \frac{r^3}{abc} \cdot (b+c)(c+a)(a+b) =$$
$$= \frac{r^3}{4Rrs} \cdot 2s(s^2 + r^2 + 2Rr) =$$
$$= \frac{r^2}{2R} \cdot (s^2 + r^2 + 2Rr) \geq$$
$$\overset{\text{MITRINOVIC}}{\geq} \frac{r^2(27r^2 + r^2 + 2Rr)}{2R} \overset{\text{EULER}}{\geq}$$
$$\geq \frac{r^2(27r^2 + r^2 + 4r^2)}{2R} = \frac{32r^4}{2R} = \frac{16r^4}{R}$$

Equality holds for $a = b = c$.

□

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