PP34454

MIHÁLY BENCZE - ROMANIA

Prove that:

$$\left(\frac{n}{3}\right)^{4n} < 3^{n^2}; n \geq 6; n \in \mathbb{N}$$

Solution by Daniel Sitaru.

$$\text{Let be } f : [6, \infty) \to \mathbb{R}; f(x) = (x^2 + 4x) \log 3 - 4x \log x$$

$$f'(x) = (2x + 4) \log 3 - 4 \log x - 4$$

$$f''(x) = 2 \log 3 - \frac{4}{x}$$

$$x \geq 6 > \frac{4}{2 \log 3} \text{ because } 12 \log 3 > 4 \Leftrightarrow \log 3 > \frac{1}{3}$$

$$x > \frac{4}{2 \log 3} \Rightarrow 2 \log 3 > \frac{4}{x}; \min f'(x) = f'(6)$$

$$f'(6) = 16 \log 3 - 4 \log 6 - 4 = 4 \log \frac{81}{6e} > 0$$

$$f'(x) > 0; (\forall) x \geq 6 \Rightarrow \min f(x) = f(6)$$

$$f(6) = 60 \log 3 - 24 \log 6 = 12 \log \frac{243}{16} > 0$$

$$f(x) > 0; (\forall) x \geq 6 \Rightarrow f(n) > 0; (\forall) n \geq 6$$

$$(n^2 + 4n) \log 3 - 4n \log n > 0$$

$$n^2 \log 3 + 4n \log \frac{3}{n} > 0$$

$$4n \log \frac{n}{3} < n^2 \log 3$$

$$\log \left(\frac{n}{3}\right)^{4n} < \log 3^{n^2}$$

$$\left(\frac{n}{3}\right)^{4n} < 3^{n^2}$$

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