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If $a,b,c>0$ such that $abc=1$ and $n\in\mathbb{N},\lambda\geq 1$ then:

$$\frac{1}{a^n+\lambda}+\frac{1}{b^n+\lambda}+\frac{1}{c^n+\lambda}\leq\frac{3}{\lambda+1}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Marian Ursărescu-Romania, Solution 2 by Fayssal Abdelli-Bejaia-Algerie

Solution 1 by Marian Ursărescu-Romania

We must show: $\frac{1}{a^n+\lambda}+\frac{1}{b^n+\lambda}+\frac{1}{c^n+\lambda}\leq\frac{3}{\lambda+1}\Leftrightarrow\frac{\lambda}{a^n+\lambda}+\frac{\lambda}{b^n+\lambda}+\frac{\lambda}{c^n+\lambda}\leq\frac{3\lambda}{\lambda+1}$

$$\Leftrightarrow\sum_{cyc}\frac{a^n+\lambda-a^n}{a^n+\lambda}\leq\frac{3\lambda}{\lambda+1}\Leftrightarrow 3-\sum_{cyc}\frac{a^n}{a^n+\lambda}\leq\frac{3\lambda}{\lambda+1}\Leftrightarrow\sum_{cyc}\frac{a^n}{a^n+\lambda}\geq\frac{3}{\lambda+1};(1)$$

Because $abc=1\Rightarrow\exists x,y,z\in\mathbb{R}_+^*$ such that $a=\frac{x}{y},b=\frac{y}{z},c=\frac{z}{x};(2)$

From (1),(2) we must show that:

$$\sum_{cyc}\frac{\frac{x^n}{y^n}}{\frac{x^n}{y^n}+\lambda}\geq\frac{3}{\lambda+1}\Leftrightarrow\sum_{cyc}\frac{x^n}{x^n+\lambda y^n}\geq\frac{3}{\lambda+1};(3)$$

Let $x^n=u,y^n=v,z^n=w;u,v,w>0;(4)$

From (3),(4) we must show:

$$\sum_{cyc} \frac{u}{u+\lambda v} \ge \frac{3}{\lambda+1}; (5)$$

$$\sum_{cyc} \frac{u}{u+\lambda v} = \sum_{cyc} \frac{u^2}{u^2+\lambda uv} \overset{Bergstrom}{\ge} \frac{(u+v+w)^2}{\sum u^2 + \lambda \sum uv}; (6)$$

From (5),(6) we must show:

$$\frac{(u+v+w)^2}{\sum u^2 + \lambda \sum uv} \ge \frac{3}{\lambda+1} \Leftrightarrow (\lambda+1)(u+v+w)^2$$

$$\ge 3(u^2+v^2+w^2) + 3\lambda(uv+vw+wu)$$

$$\Leftrightarrow (\lambda+1)\sum u^2 + 2(\lambda+1)\sum uv > 3\sum u^2 + 3\lambda \sum uv \Leftrightarrow$$

$(\lambda-2)\sum u^2 \ge (\lambda-2)\sum uv \Leftrightarrow \sum u^2 \ge \sum uv$ **which is clearly true.**

Solution 2 by Fayssal Abdelli-Bejaia-Algerie

$$\frac{1}{a^n+\lambda} + \frac{1}{b^n+\lambda} + \frac{1}{c^n+\lambda} \le \frac{3}{\lambda+1}; (A) \Leftrightarrow \frac{\lambda}{a^n+\lambda} + \frac{\lambda}{b^n+\lambda} + \frac{\lambda}{c^n+\lambda} \le \frac{3\lambda}{\lambda+1}$$

$$\Leftrightarrow \sum_{cyc} \frac{a^n+\lambda-a^n}{a^n+\lambda} \le \frac{3\lambda}{\lambda+1} \Leftrightarrow 3 - \sum_{cyc} \frac{a^n}{a^n+\lambda} \le \frac{3\lambda}{\lambda+1} \Leftrightarrow \sum_{cyc} \frac{a^n}{a^n+\lambda} \overset{(?)}{\ge} \frac{3}{\lambda+1}$$

$$\sum_{cyc} \frac{a^n}{a^n+\lambda} \ge \frac{\left(\sum \sqrt{a^n}\right)^2}{3\lambda + \sum a^n}$$

But: $\frac{1}{3}\sum \sqrt{a^n} \ge \sqrt[3]{\sqrt{a^n b^n c^n}} = 1 \Rightarrow \sum \sqrt{a^n} \ge 3 \Rightarrow \left(\sum \sqrt{a^n}\right)^2 \ge 9$**, hence**

$$\sum_{cyc} \frac{a^n}{a^n+\lambda} \ge \frac{9}{3\lambda + \sum a^n} \overset{(?)}{\ge} \frac{3}{\lambda+1}$$

$$\frac{3}{3\lambda + \sum a^n} \ge \frac{1}{\lambda+1} \Leftrightarrow 3\lambda + 3 \ge 3\lambda + a^n + b^n + c^n \Leftrightarrow a^n + b^n + c^n \le 3; (1)$$

But: $\frac{a^n+b^n+c^n}{3} \ge \sqrt[3]{a^n b^n c^n} \Rightarrow a^n + b^n + c^n \ge 3; (2)$

From $(1), (2)$ **it follows that** $\frac{9}{3\lambda+3} \ge \frac{3}{\lambda+1}$ **true.**

Therefore,

$$\frac{1}{a^n+\lambda} + \frac{1}{b^n+\lambda} + \frac{1}{c^n+\lambda} \le \frac{3}{\lambda+1}$$