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PROBLEMS FOR JUNIORS

JP.376. Let ΔABC be an acute triangle. Prove that:

$$\sqrt{\frac{\sin A}{\sin B\cdot\sin C}}+\sqrt{\frac{\sin B}{\sin C\cdot\sin A}}+\sqrt{\frac{\sin C}{\sin A\cdot\sin B}}\geq\sqrt[4]{108}$$

Proposed by George Apostolopoulos - Greece

JP.377. Let a,b,c be positive real numbers such that $ab+bc+ca\leq a+b+c$. Prove that:

$$\frac{a}{b^2c^2}+\frac{b}{c^2a^2}+\frac{c}{a^2b^2}\geq 3$$

Proposed by George Apostolopoulos - Greece

JP.378. Determine all triplets (x,y,z) of positive integers which satisfy the following two equations:

$$xy+z^2=31, x+yz^2=53$$

Proposed by George Apostolopoulos - Greece

JP.379. If $ABCD$ tetrahedron $AB=a, AD=b, AC=c$, $BD=d, DC=e, CB=f; F$ - total area, then:

$$a^4+b^4+c^4+d^4+e^4+f^4\geq 2F^2$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru - Romania

JP.380. If $a,b,c,d>0, \sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d}=4$ then:

$$\sum_{cyc}\frac{1}{a^4+b^4+c^4+5}\leq\frac{1}{2\sqrt{abcd}}$$

Proposed by Daniel Sitaru - Romania

JP.381. If ABC and UVW are two triangles then:

$$\sum_{cyc}\frac{\cos\frac{A}{2}}{1+\sin\frac{A}{2}}\left(1+\sin\frac{U}{2}\right)\geq\sum_{cyc}\cos\frac{U}{2}$$

Proposed by Cristian Miu - Romania

JP.382. In acute $\Delta ABC, D \in (BC), E \in (AC), F \in (AB)$. Prove that:

$$\sqrt{\frac{AD^3+BE^3}{AD^5+BE^5}}+\sqrt{\frac{BE^3+CF^3}{BE^5+CF^5}}+\sqrt{\frac{CF^3+AD^3}{CF^5+AD^5}}\le\frac{1}{r}$$

Proposed by Marian Ursărescu - Romania

JP.383. In ΔABC the following relationship holds:

$$\sum_{cyc}\sqrt{\frac{1}{\sin^2 A}+\frac{1}{\sin^2 B}+\frac{1}{\sin A\sin B}}\ge 6$$

Proposed by Marian Ursărescu - Romania

JP.384. Solve for real numbers:

$$2^x+9^{\frac{1}{x}}+2^x\cdot 9^{\frac{1}{x}}=19$$

Proposed by Marian Ursărescu - Romania

JP.385 If $a,b,c>0$ are such that $\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}=\frac{3}{4}$ then:

$$\sum_{cyc}\frac{a+2b}{a^2+2b^2}+\sum_{cyc}\frac{b+2a}{b^2+2a^2}\le 3$$

Proposed by Daniel Sitaru - Romania

JP.386. Let ΔABC be any triangle. Prove that for any $n\in\mathbb{N}\setminus\{0,1\}$ the following inequality holds:

$$\frac{|a-b||a-c|^n a^{3n}}{b^{3n-3}|b-c|^{n-1}}+\frac{|b-c||b-a|^n b^{3n}}{c^{3n-3}|c-a|^{n-1}}+$$

$$+\frac{|a-c||c-b|^n c^{3n}}{a^{3n-3}|a-b|^{n-1}}\ge 16sr^2(4R^2+5Rr+r^2-s^2)$$

Proposed by George Florin Şerban- Romania

JP.387. Let ΔABC be any triangle. Prove that for any $n\in\mathbb{N}\setminus\{0,1\}$ the following inequality holds:

$$\frac{|a-b||a-c|^n a^{2n}}{b^{2n-2}|b-c|^{n-1}}+\frac{|b-c||b-a|^n b^{2n}}{c^{2n-2}|c-a|^{n-1}}+$$

$$+\frac{|a-c||c-b|^n c^{2n}}{a^{2n-2}|a-b|^{n-1}}\ge 4r^2[(4R+r)^2-3s^2]$$

Proposed by George Florin Şerban- Romania

JP.388. Solve for complex numbers:

$$\begin{cases} |x-y| \geq \sqrt{3}|z| \\ |y-z| \geq \sqrt{3}|x| \\ |z-x| \geq \sqrt{3}|y| \end{cases}$$

Proposed by Ionuţ Florin Voinea - Romania

JP.389. A right parallelipiped $ABCDA'B'C'D'$ has the basis $ABCD$ rhombus, and areas of the two diagonals sections of the parallelipiped are F_1 and F_2 respectively. Let R be the circumradius of $\Delta ABC, R_2$ circumradius of ΔABD and V volume of the right parallelipiped. Prove that: $R_1R_2F_1F_2 \geq V^2$.

Proposed by Radu Diaconu - Romania

JP.390. Let $x \in \mathbb{R}$ and ABC a triangle with F area. Prove that:

$$\frac{a^3}{\sqrt{b^2\sin^2 x + c^2\cos^2 x}} + \frac{b^3}{\sqrt{c^2\sin^2 x + a^2\cos^2 x}} + \\ + \frac{c^3}{\sqrt{a^2\sin^2 x + b^2\cos^2 x}} \geq 4\sqrt{3}F$$

Proposed by D.M. Bătineţu - Giurgiu - Romania

PROBLEMS FOR SENIORS

SP.376. Let r, r_a, r_b, r_c and R be, respectively, the inradius, the exradii, and the circumradius of triangle ABC with side lengths a, b, c. Prove that:

$$36\sqrt{3}\frac{r^3}{R^2} \leq \frac{a^2+b^2}{a+b} + \frac{b^2+c^2}{b+c} + \frac{c^2+a^2}{c+a} + 4r\left(\frac{r_a}{b+c} + \frac{r_b}{c+a} + \frac{r_c}{a+b}\right) \leq \frac{9\sqrt{3}R^2}{4r}$$

Proposed by George Apostolopoulos - Greece

SP.377. Let w_a, w_b, w_c be the internal bisectors, r_a, r_b, r_c the exradii, r the inradius and R the circumradius of a triangle ABC. Prove that:

$$\left(\frac{w_a}{r_a}\right)^2 + \left(\frac{w_b}{r_b}\right)^2 + \left(\frac{w_c}{r_c}\right)^2 \leq 27\left(\frac{R}{2r}\right)^4 - 24$$

Proposed by George Apostolopoulos - Greece

SP.378. Let m_a, m_b, m_c be the lengths of the medians of a triangle ABC with area F. Prove that:

$$m_a^n + m_b^n + m_c^n \geq 3^{\frac{n}{4}+1} \cdot F^{\frac{n}{2}} \text{ for each integer } n \geq 1$$

Proposed by George Apostolopoulos - Greece

SP.379. If $x, y, z \in (0,1)$ then:

$$\frac{x}{(y+z)^2(1-x^2)}+\frac{y}{(z+x)^2(1-y^2)}+\frac{z}{(x+y)^2(1-z^2)}\geq\frac{9\sqrt{3}}{8}$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru - Romania

SP.380. Let a, b, c be the sides of an arbitrary triangle. Denote by m_a, w_a, h_a the lengths of the median, the internal bisector and the altitude corresponding to the side a and ω its Brocard angle. Prove that:

$$\frac{1}{\sin\omega}\geq 2\sqrt[4]{\frac{m_a^2m_b^2m_c^2}{w_aw_bw_ch_ah_bh_c}}$$

Proposed by Vasile Jiglău - Romania

SP.381. Find all continuous functions $f:(0,\infty)\to\mathbb{R}$ such that:

$$f(a^x)+f(a^{2x})+f(a^{4x})=x, \forall x\in\mathbb{R}, a>0, a\neq 1 \text{ - fixed}$$

Proposed by Marian Ursărescu - Romania

SP.382. $z_1, z_2, z_3 \in \mathbb{C}^*$ - different in pairs such that:

$$|z_1|=|z_2|=|z_3|=1, A(z_1), B(z_2), C(z_3)$$

Prove that:

$$\sum_{cyc}\frac{z_2z_3}{(z_2-z_3)^2[z_2(z_1-z_3)^2-z_3(z_1+z_2)^2]}=\frac{1}{4z_1z_2z_3}\Rightarrow AB=BC=CA$$

Proposed by Marian Ursărescu - Romania

SP.383. If $a, b, c > 0$ then:

$$\frac{a^{10}c^5+b^{10}a^5+c^{10}b^5}{a^2b+b^2c+c^2a}\geq a^4b^4c^4$$

Proposed by Daniel Sitaru - Romania

SP.384. Let $(a_n)_{n\geq 1}$ be sequence of real numbers with $a_n > 0, \forall n \in \mathbb{N}$ and

$$a_0=1, a_n^2+a_ne^{a_n}=(n+1)(n+1+e^{a_n}).$$

Find:

$$\Omega=\lim_{n\to\infty}a_n\cdot\sqrt[n]{a_n\cdot\sin 1\cdot\sin\frac{1}{2}\cdot\ldots\cdot\sin\frac{1}{n}}$$

Proposed by Florică Anastase - Romania

SP.385. Let $(a_n)_{n\geq 1}$ be sequence of positive real numbers such that:

$$a_{n+1}^3 - (a_n + a_1)a_{n+1}^2 + (a_{n+1} - a_1)a_n^2 + a_1 a_n a_{n+1} = 0$$

$\forall n \in \mathbb{N}^*, n > 1$. Prove that:

$$\sum_{k=1}^{n} \log_3\left(\left(\frac{a_k}{a_{k+1}}\right)^2 + \left(\frac{a_k}{a_{k+1}}\right) + 1\right) \geq n$$

Proposed by Florică Anastase - Romania

SP.386. Solve for real numbers:

$$\log_2(5^x - 3) = \log_7(3^x + 4)$$

Proposed by Ionuţ Florin Voinea - Romania

SP.387. Given $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = -2x^2 + 6x - 3$, then find:

$$\Omega = \lim_{n\to\infty} \int_1^2 f^n(x)dx$$

where

$$f^n(x) = \underbrace{f(f(f(\dots f(x))))}_{n \text{ - times}}$$

Proposed by Rajeev Rastogi-India

SP.388. Given $\{a_n\}$ is a sequence of real numbers satisfying $a_1 = 0$ and

$$\frac{4}{4 - a_{n+1}} - \frac{4}{4 - a_n} = 2n + 1, \forall n \geq 1$$

Define $b_n = \frac{4-a_n}{4}$ for $n \geq 1$, then find:

$$\Omega = \lim_{n\to\infty}\left[4 \cdot \sin^{-1}\frac{b_{n+1}}{b_n} - \left(\pi - \frac{1}{2} + \tanh^{-1}\sqrt{b_n}\right)\left(\prod_{k=2}^{n} a_k\right)\right]$$

Proposed by Rajeev Rastogi-India

SP.389. Given $f : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying the functional equation:

$$f(x + y) - 3^y f(x) = 3^x f(y); \forall x, y \in \mathbb{R}$$

then find:

$$\Omega = \lim_{n\to\infty}\left[\frac{f(x)}{f'(x)} + \frac{f'(x)}{f''(x)} + \dots + \frac{f^{n-1}(x)}{f^n(x)}\right]$$

(where $f^n(x)$ denotes the n^{th} derivative of $f(x)$ with respect to x)

Proposed by Rajeev Rastogi-India

SP.390. Given $f(x)$ be a non-constant function satisfying the integral equation

$$f(x)=2x^2-\int_0^2(f(t)-x)^2dt$$

then find:

$$\Omega=\lim_{n\to\infty}\left[\lim_{n\to\infty}\left(\frac{\sum_{r=0}^{n}f(\frac{x}{2^r})}{x}\right)\right]$$

Proposed by Rajeev Rastogi-India

UNDERGRADUATE PROBLEMS

UP.376. If $a,b>0$ then find:

$$\Omega=\lim_{n\to\infty}\left(\frac{1}{\sqrt[n]{(2n-1)!!}}\cdot\sum_{k=1}^{n}\sqrt{\frac{1}{b^2}+\frac{1}{(a+bn)^2}+\frac{1}{(a+b(n+1))^2}}\right)$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru - Romania

UP.377. Let $(x_n)_{n\geq 0}$ sequence of positive real numbers such that:

$$nx_n^2=ax_{n+1}^2+(an-1)x_{n+1}x_n;a>0,x_0>0\text{ - fixed. Find:}$$

$$\Omega=\lim_{n\to\infty}\left(\lim_{m\to\infty}\left(\frac{\sum_{k=1}^{n}x_k^{\frac{1}{\sqrt{m}}}}{n}\right)^{\frac{1}{\tan(\frac{1}{\sqrt{m}})}}\right)$$

Proposed by Florică Anastase - Romania

UP.378. If $(a_n)_{n\geq 1},(b_n)_{n\geq 1}$ are positive real sequences such that:

$$b_n=a_1\cdot\sqrt{a_2!}\cdot\sqrt[3]{a_3!}\cdot\ldots\cdot\sqrt[n]{a_n!}\text{ and }\lim_{n\to\infty}\frac{a_{n+1}}{n\cdot a_n}=\pi.\text{ Find:}$$

$$\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{b_{n+1}}}-\frac{n^2}{\sqrt[n]{b_n}}\right)$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu - Romania

UP.379. If $S_n=-2\sqrt{n}+\sum_{k=1}^{n}\frac{1}{\sqrt{k}}$ is the Ioachimescu's sequence with $\lim_{n\to\infty}s_n=s$, then compute:

$$\lim_{n\to\infty}(s_n-s)\sqrt[2n]{(2n-1)!!}$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu - Romania

UP.380. Let be $E(n)=\Gamma(\frac{1}{n})\cdot\Gamma(\frac{2}{n})\cdot\ldots\cdot\Gamma(\frac{n-1}{n}), n\geq 2, n\in\mathbb{N}^*$. Find:

$$\Omega=\lim_{n\to\infty}\left(\frac{E(n)}{\sin\frac{1}{\sqrt{n}}}\cdot\sin\frac{1}{\sqrt{(2\pi)^{n-1}}}\right)$$

Proposed by Florică Anastase - Romania

UP.381. If $a,b\in\mathbb{R}_+, \gamma_n(a,b)=-\log(n+a)+\sum_{k=1}^{n}\frac{1}{k+b}$, $\lim_{n\to\infty}\gamma_n(a,b)=\gamma(a)\in\mathbb{R}$, then find:

$$\Omega=\lim_{n\to\infty}\left(\log\Big(\frac{e}{n+a}\Big)+\sum_{k=1}^{n}\frac{1}{k+b}-\gamma(a,b)\right)^n$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu - Romania

UP.382. For $t>0$ find:

$$\Omega=\lim_{n\to\infty}n^{1-t}\left(\frac{(\sqrt[n+1]{(n+1)!})^{2t}}{(n+1)^t}-\frac{(\sqrt[n]{n!})^{2t}}{n^t}\right)$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu - Romania

UP.383. Let R be the circumradius of ΔABC having the length of the sides a,b,c. Prove that:

$$\Delta=\begin{vmatrix}3\sqrt{3}R & a & b & c\\ a & 3\sqrt{3}R & c & b\\ b & c & 3\sqrt{3}R & a\\ c & b & a & 3\sqrt{3}R\end{vmatrix}>0$$

Proposed by Daniel Sitaru - Romania

UP.384. Find:

$$\Omega=\lim_{n\to\infty}\left(1+\Big(1+\frac{1}{n}\Big)^{n+1}-e\right)^n$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru - Romania

UP.385. In any ΔABC let x,y,z be the distances from the incentre to the sides of triangle $u\geq 1$ - fixed. Prove that:

$$u^x+u^y+u^z\leq u^{\sqrt{\frac{bc(s-a)}{s}}}+u^{\sqrt{\frac{ca(s-b)}{s}}}u^{\sqrt{\frac{ab(s-c)}{s}}}$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru - Romania

UP.386. If $0<a\leq b$ then:

$$\int_a^b\int_a^b\frac{dxdy}{xy+1}\leq\frac{2(b-a)^2}{(a+1)(b+1)}$$

Proposed by Daniel Sitaru - Romania

UP.387. If in $\Delta ABC, 2s = 3$ then:

$$\frac{m_a + m_b}{m_c} + \frac{a^2b(m_b + m_c)}{m_a} + \frac{bc^2(m_c + m_a)}{m_b} \geq 8\sqrt{3}F$$

Proposed by Daniel Sitaru - Romania

UP.388. If $x, y, z, p, q, r > 0; x + y + z = p + q + r = 3$ then:

$$x^p + x^q + x^r + y^p + y^q + y^r + z^p + z^q + z^r \geq 9$$

Proposed by D.M. Bătineţu - Giurgiu, Daniel Sitaru - Romania

UP.389. If $0 < a \leq b$ then:

$$\int_a^b \int_a^b \frac{dxdy}{(xy + 1)^2} \leq \frac{4(b - a)^2(a^2 + b^2 + ab + 3a + 3b + 3)}{3(a + 1)^3(b + 1)^3}$$

Proposed by Daniel Sitaru - Romania

UP.390. If $x, y, z \geq 1; x + y + z = 6$ then in ΔABC the following relationship holds:

$$(x^x + y^x + z^x)a^4 + (x^y + y^y + z^y)b^4 + (x^z + y^z + z^z)c^4 \geq 5184r^4$$

Proposed by D.M. Bătineţu - Giurgiu, Daniel Sitaru - Romania

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