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SECLĂMAN'S SEQUENCE REVISITED

Let $(s_n)_{n\ge 0}, s_0 > 1, s_{n+1} = \frac{s_n^2}{s_{n-1}}$ be Seclăman's sequence and $f:[0,1]\to\mathbb{R}$ continuous and convex function. Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n} s_k\left(f\left(\frac{k}{n}\right) - n\int_{\frac{k-1}{n}}^{\frac{k}{n}} f(x)dx\right)\right)$$

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Solution by proposer

$$s_{n+1} - 1 = \frac{s_n^2 - s_n + 1}{s_n - 1}, \forall n\in\mathbb{N} \to (s_{n+1}-1)(s_n-1) = s_n^2 - s_n + 1 > 0$$

$s_0 > 1$, *from mathematical induction, we get* $s_n > 1, \forall n\in\mathbb{N}$.

$$s_{n+1} - s_n = \frac{s_n}{s_n - 1} > 0 \to (s_n)_{n\ge 0}\nearrow, then\ \exists s = \lim_{x\to\infty} s_n \in\mathbb{R}\ and\ from\ s_n > 1$$

$$\to s > 1\ and\ s^2 - s = s^2 \to s = 0\ (contradiction!)$$

$$\lim_{n\to\infty}\frac{n}{s_n} \overset{L.C-S}{\hat{=}} \lim_{n\to\infty}\frac{n+1-n}{s_{n+1}-s_n} = \lim_{n\to\infty}\frac{1}{s_{n+1}-s_n} = \lim_{n\to\infty}\frac{s_n - 1}{s_n} = 1 - \lim_{n\to\infty}\frac{1}{s_n} = 1; (1)$$

How f' have Darboux property and f' increase function, we obtain that f' is continuous function. We can write:

$$\sum_{k=1}^{n} s_k\left(f\left(\frac{k}{n}\right) - n\int_{\frac{k-1}{n}}^{\frac{k}{n}} f(x)\,dx\right) = n\left(\frac{1}{n}\sum_{k=1}^{n} s_k f\left(\frac{k}{n}\right) - \sum_{k=1}^{n} s_k\int_{\frac{k-1}{n}}^{\frac{k}{n}} f(x)dx\right) =$$

$$= n\left(\sum_{k=1}^{n}\int_{\frac{k-1}{n}}^{\frac{k}{n}} s_k f\left(\frac{k}{n}\right) - \sum_{k=1}^{n} s_k\int_{\frac{k-1}{n}}^{\frac{k}{n}} f(x)dx\right) = n\left(\sum_{k=1}^{n}\int_{\frac{k-1}{n}}^{\frac{k}{n}} s_k\left[f\left(\frac{k}{n}\right) - f(x)\right]dx\right) \overset{M.V.T}{=}$$

$$\overset{M.V.T}{=} n\left(\sum_{k=1}^{n} s_k\int_{\frac{k-1}{n}}^{\frac{k}{n}}\left(\frac{k}{n} - x\right) f'(c_{k,n,x})dx\right) \overset{f'-increase}{\ge}$$

$$= n\cdot\sum_{k=1}^{n} s_k f'\left(\frac{k-1}{n}\right)\int_{\frac{k-1}{n}}^{\frac{k}{n}}\left(\frac{k}{n} - x\right)dx = n\cdot l\cdot\sum_{k=1}^{n} s_k f'\left(\frac{k-1}{n}\right)\cdot\frac{1}{n^2} =$$

$$=\frac{1}{n}\sum_{k=1}^{n}s_kf'\left(\frac{k-1}{n}\right)$$

On the other hand, we have:

$$n\left(\frac{1}{n}\sum_{k=1}^{n}s_kf\left(\frac{k}{n}\right)-\sum_{k=1}^{n}s_k\int_{\frac{k-1}{n}}^{\frac{k}{n}}f(x)dx\right)=$$

$$=n\left(\sum_{k=1}^{n}\int_{\frac{k-1}{n}}^{\frac{k}{n}}s_kf\left(\frac{k}{n}\right)dx-\sum_{k=1}^{n}s_k\int_{\frac{k-1}{n}}^{\frac{k}{n}}f(x)dx\right)=$$

$$=n\left(\sum_{k=1}^{n}s_k\left(f\left(\frac{k}{n}\right)-f(x)\right)dx\right)\overset{M.V.T}{=}n\left(\sum_{k=1}^{n}\int_{\frac{k-1}{n}}^{\frac{k}{n}}\left(\frac{k}{n}-x\right)f'(c_{k,n,x})dx\right)\overset{f'-increase}{\leq}$$

$$\overset{f'-increase}{\leq}n\sum_{k=1}^{n}s_kf'\left(\frac{k}{n}\right)\int_{\frac{k-1}{n}}^{\frac{k}{n}}\left(\frac{k}{n}-x\right)dx=n\sum_{k=1}^{n}s_kf'\left(\frac{k}{n}\right)\cdot\frac{1}{n^2}=\frac{1}{n}\sum_{k=1}^{n}s_kf'\left(\frac{k}{n}\right)$$

So, it follows that:

$$\frac{1}{n}\sum_{k=1}^{n}s_kf'\left(\frac{k-1}{n}\right)\leq\sum_{k=1}^{n}s_k\left(f\left(\frac{k}{n}\right)-n\int_{\frac{k-1}{n}}^{\frac{k}{n}}f(x)\,dx\right)\leq\frac{1}{n}\sum_{k=1}^{n}s_kf'\left(\frac{k}{n}\right)$$

$$\frac{1}{n}\cdot\frac{1}{n}\sum_{k=1}^{n}s_kf'\left(\frac{k-1}{n}\right)\leq\frac{1}{n}\sum_{k=1}^{n}s_k\left(f\left(\frac{k}{n}\right)-n\int_{\frac{k-1}{n}}^{\frac{k}{n}}f(x)\,dx\right)\leq\frac{1}{n}\cdot\frac{1}{n}\sum_{k=1}^{n}s_kf'\left(\frac{k}{n}\right)$$

Therefore,

$$\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n}s_k\left(f\left(\frac{k}{n}\right)-n\int_{\frac{k-1}{n}}^{\frac{k}{n}}f(x)\,dx\right)\right)=\frac{(f(1)-f(0))}{2}\cdot\lim_{n\to\infty}\frac{s_n}{n}\overset{(1)}{=}\frac{(f(1)-f(0))}{2}$$