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Prove that:

$$\int_0^\infty \left(\frac{\sin(\pi x^2) - x^3\cos(\pi x^2)}{x^6+1} - \frac{\cos(\pi x^2)}{x^3+1}\right)dx = \frac{\pi e^{-\frac{\pi\sqrt{3}}{2}}}{3}$$

Proposed by Angad Singh-India

Solution by proposer

Consider the complex line integral,

$$J = \oint_C \frac{e^{inz^2}}{z^3+a^3}dz$$

where $a > 0$ and $n \geq 0$ and C is a quarter circle of radius R in the first quadrant centered at the origin traversed anti-clockwise. Hence,

$$J = \int_0^R \frac{e^{inx^2}}{x^3+a^3}dx + \int_0^{\frac{\pi}{2}} \frac{e^{in(Re^{i\theta})^2}}{(Re^{i\theta})^3+a^3}Rie^{i\theta}d\theta + \int_R^0 \frac{e^{in(iy)^2}}{(iy)^2+a^3}idy$$

$$\text{Let } I_1 = \int_0^R \frac{e^{inx^2}}{x^3+a^3}dx$$

$$I_2 = \int_0^{\frac{\pi}{2}} \frac{e^{in(Re^{i\theta})^2}}{(Re^{i\theta})^3+a^3}Rie^{i\theta}d\theta$$

$$I_3 = \int_R^0 \frac{e^{in(iy)^2}}{(iy)^2+a^3}idy$$

Now, using the residue theorem,

$$J=2\pi i\frac{\omega e^{ina^2\omega^2}}{3a^2}, \text{where } \omega=-\frac{1}{2}-\frac{i\sqrt{3}}{3}. \text{Thus,}$$

$$Re(j)=\frac{\pi e^{-\frac{na^2\sqrt{3}}{2}}}{3a^2}\left(\sqrt{3}\cos\left(\frac{na^2}{2}\right)-\sin\left(\frac{na^2}{2}\right)\right)$$

Similarly,

$$Re(I_1)=\int_0^R\frac{\cos(nx^2)}{x^3+a^3}dx$$

Now, observe that

$$|I_2|=\left|\int_0^{\frac{\pi}{2}}\frac{e^{in\left(Re^{i\theta}\right)^2}}{(Re^{i\theta})^3+a^3}Rie^{i\theta}d\theta\right|\le+\int_0^{\frac{\pi}{2}}\left|\frac{e^{in\left(Re^{i\theta}\right)^2}}{(Re^{i\theta})^3+a^3}\right|\left|Rie^{i\theta}\right|d\theta\le$$

$$\le\frac{R}{R^3-a^3}\int_0^{\frac{\pi}{2}}e^{-nR^2\sin 2\theta}d\theta$$

Thus,

$$|I_2|\le\frac{\pi R}{2(R^3-a^3)}, \text{since} \lim_{R\to\infty}\frac{\pi R}{2(R^3-a^3)}=0, \qquad \text{thus } I_2 \text{ vanishes.}$$

Now,

$$I_3=\int_R^0\frac{e^{in(iy)^2}}{(iy)^2+a^3}idy\Rightarrow Re(I_3)=-\int_0^R\frac{a^3\sin(n\pi^2)-x^3\cos(nx^2)}{x^6+a^6}dx$$

Finally, we have,

$$Re(J)=Re(I_1)+Re(I_2)+Re(I_3)$$

Thus,

$$\frac{\pi e^{-\frac{na^2\sqrt{3}}{2}}}{3a^2}\left(\sqrt{3}\cos\left(\frac{na^2}{2}\right)-\sin\left(\frac{na^2}{2}\right)\right)=$$

$$=\int_0^R\frac{\cos(nx^2)}{x^3+a^3}dx-\int_0^R\frac{a^3\sin(n\pi^2)-x^3\cos(nx^2)}{x^6+a^6}dx$$

Substituting $n=\pi, a=1$ and letting $R\to\infty$ we complete the proof.