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If $x, y, z \in (0,1)$, then prove:

$$\sum \frac{z^2}{(1-x^2)(y^2+yz+z^2)} \geq \frac{3\sqrt{3}}{2} \cdot \frac{xy+yz+zx}{x+y+z}$$

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Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$From\ AM-GM, we\ have:\ x^2 + \frac{\sqrt{3}}{9x} + \frac{\sqrt{3}}{9x} \geq 3\sqrt[3]{x^2 . \frac{\sqrt{3}}{9x} . \frac{\sqrt{3}}{9x}} = 1$$

$$\rightarrow 1 - x^2 \leq \frac{2\sqrt{3}}{9x}\ \ (And\ analogs) \rightarrow$$

$$\sum \frac{z^2}{(1-x^2)(y^2+yz+z^2)} \geq \frac{9}{2\sqrt{3}} \sum \frac{xz^2}{y^2+yz+z^2} = \frac{3\sqrt{3}}{2} \sum \frac{(zx)^2}{xy^2+xyz+xz^2} \geq$$

$$\overset{CBS}{\geq} \frac{3\sqrt{3}}{2} . \frac{(\sum zx)^2}{\sum(xy^2+xyz+xz^2)} = \frac{3\sqrt{3}}{2} . \frac{(xy+yz+zx)^2}{(xy+yz+zx)(x+y+z)}$$

$$Therefore,\ \ \sum \frac{z^2}{(1-x^2)(y^2+yz+z^2)} \geq \frac{3\sqrt{3}}{2} . \frac{xy+yz+zx}{x+y+z}$$