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If $a, b, c > 1$ then prove that:

$$\log_{ab^2c^2} a + \log_{a^2bc^2} b + \log_{a^2b^2c} c \ge \frac{3}{5}$$

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$$\log_{ab^2c^2} a + \log_{a^2bc^2} b + \log_{a^2b^2c} c \ge \frac{3}{5}$$

$$\Leftrightarrow \sum_{cyc} \frac{\log a}{\log ab^2c^2} \ge \frac{3}{5} \Leftrightarrow \sum_{cyc} \frac{\log a}{\log a + 2\log b + 2\log c} \ge \frac{3}{5}; (1)$$

Let us denote: $\log a = x, \log b = y, \log c = z; x, y, z > 0; (2)$

From $(1), (2)$ we must show that:

$$\sum_{cyc} \frac{x}{x + 2y + 2z} \ge \frac{3}{5}; (3)$$

$$\sum_{cyc} \frac{x}{x + 2y + 2z} = \sum_{cyc} \frac{x^2}{x^2 + 2xy + 2xz} \overset{Bergstrom}{\ge} \frac{(x + y + z)^2}{x^2 + y^2 + z^2 + 4xy + 4yz + 4zx}; (4)$$

From $(3), (4)$ we must show that:

$$\frac{(x + y + z)^2}{x^2 + y^2 + z^2 + 4xy + 4yz + 4zx} \ge \frac{3}{5} \Leftrightarrow$$

$$5(x + y + z)^2 \ge 3(x^2 + y^2 + z^2) + 12(xy + yz + zx) \Leftrightarrow$$

$$5(x^2+y^2+z^2)+10(xy+yz+zx)\geq 3(x^2+y^2+z^2)+12(xy+yz+zx)\Leftrightarrow$$

$$x^2+y^2+z^2\geq xy+yz+zx \text{ true}.$$