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PROBLEMS FOR JUNIORS

JP.361. Find $x, y, z > 0$ such that:

$$\begin{cases} x^3y + y^3z + z^3x = xyz(x+y+z) \\ 2x + 3y + 5z = 10 \end{cases}$$

Proposed by Daniel Sitaru - Romania

JP.362. Let ABC be a triangle with inradius r, and circumradius R. Equilateral triangles with AB, BC and CA, are drawn externally to triangle ABC. Let K, L and M be the centroids of the equilateral triangles, respectively. Prove that:

$$\frac{2r}{R} \le \frac{[KLM]}{[ABC]} \le \left(\frac{R}{2r}\right)^2$$

Proposed by George Apostolopoulos - Greece

JP.363. Let a, b, c be positive real numbers with $a^2 + b^2 + c^2 = 12$. Prove that:

$$\frac{a^4}{\sqrt{a^3+1}} + \frac{b^4}{\sqrt{b^3+1}} + \frac{c^4}{\sqrt{c^3+1}} \ge 16$$

Proposed by George Apostolopoulos - Greece

JP.364. If $x, y, z > 0$ then:

$$\sqrt{x^2 - xy\sqrt{3} + y^2} + \sqrt{y^2 - yz\sqrt{2} + z^2} = \sqrt{x^2 + z^2 - \frac{xz(\sqrt{6}-\sqrt{2})}{2}} \Leftrightarrow$$

$$\Leftrightarrow \frac{2\sqrt{2}}{x} + \frac{2}{z} = \frac{\sqrt{2}+\sqrt{6}}{y}$$

Proposed by Daniel Sitaru - Romania

JP.365. If in ΔABC exists the relationship $\frac{4}{w_a} = \frac{1}{r} + \frac{1}{r_a}$ then prove that $AH \ge 2r$.

Proposed by Marian Ursărescu - Romania

JP.366. If acute ΔABC the following relationship holds:

$$\cos A + \sqrt{\cos A \cos B} + \sqrt[3]{\cos A \cos B \cos C} < 2$$

Proposed by Marian Ursărescu - Romania

JP.367. $a,b,c\in\mathbb{C}^*$ **- different in pairs,** $A(a),B(b),C(c)$;
$|a|=|b|=|c|=1$. **If**

$$(ab)^3+(bc)^3+(ca)^3=3(abc)^2$$

then ΔABC **is equilateral.**

Proposed by Marian Ursărescu - Romania

JP.368. $a,b,c\in\mathbb{C}^*$ **- different in pairs,** $A(a),B(b),C(c)$;
$|a|=|b|=|c|=1$. **If**

$$|a-b|\left(\frac{1}{a}+\frac{1}{b}\right)+|b-c|\left(\frac{1}{b}+\frac{1}{c}\right)+|c-a|\left(\frac{1}{c}+\frac{1}{a}\right)=0$$

then ΔABC **is equilateral.**

Proposed by Marian Ursărescu - Romania

JP.369. Find $x,y,z\in(0,\frac{\pi}{2})$ **such that:**

$$\frac{\cos(5x)}{\cos x}+\frac{\cos(5y)}{\cos y}+\frac{\cos(5z)}{\cos z}+\frac{15}{4}=0$$

Proposed by Daniel Sitaru - Romania

JP.370. In ΔABC **the following relationship holds:**

$$\sqrt{ab}+\sqrt{bc}+\sqrt{ca}+\sqrt{\frac{a^2+b^2}{2}}+\sqrt{\frac{b^2+c^2}{2}}+\sqrt{\frac{c^2+a^2}{2}}\le 6\sqrt{3}R$$

Proposed by Daniel Sitaru - Romania

JP.371. Solve for real positive numbers the equation:

$$x^{\log 3}+x^{\log 4}+x^{\log 5}=x^{\log 6}$$

Proposed by D.M. Bătineţu - Giurgiu, Neculai Stanciu - Romania

JP.372. If in $\Delta ABC: a^2+b^2=2c^2$ **then:**

$$2am_a+m_c^2\cdot\sqrt{\frac{ab}{m_am_b}}\le\frac{\sqrt{3}}{2}(a^2+b^2+c^2)$$

Proposed by Daniel Sitaru - Romania

JP.373. If $a,b,c<0; a+b+c=3; F_n$ **- Fibonacci numbers;** L_n **- Lucas numbers;** P_n **- Pell numbers;** $n\in\mathbb{N}; n\ge 2$ **then:**

$$\frac{a^2(P_n-F_n)(P_n-L_n)}{F_nL_n}+\frac{b^2(F_n-L_n)(F_n-P_n)}{L_nP_n}+$$
$$+\frac{c^2(L_n-P_n)(L_n-F_n)}{P_nF_n}\ge 9$$

Proposed by Daniel Sitaru - Romania

JP.374. Solve for complex numbers:

$$57x^6-180x^5+234x^4-159x^3+60x^2-12x+1=0$$

Proposed by Daniel Sitaru - Romania

JP.375. Let a,b,c be positive real numbers such that $a+b+c=3$. Prove that:

$$9(a^2+b^2+c^2)-2(a^3+b^3+c^3)\geq 21$$

Proposed by George Apostolopoulos - Greece

PROBLEMS FOR SENIORS

SP.361. Let $(x_n)_{n\geq 1}, x_1=1$ such that:

$$n^2[2(x_{n+1}-x_n-1)-n^2]+2x_n=n[3(n^2+x_n)-x_{n+1}]$$

Find:

$$\lim_{n\to\infty}\left(1+\frac{1}{2^n}\sum_{k=0}^{n}\frac{\binom{n}{k}}{2k+1}\right)^{\frac{x_n}{n^2}}$$

Proposed by Florică Anastase - Romania

SP.362. Let m_a,m_b,m_c be the lengths of the medians of a triangle ΔABC. Prove:

$$\frac{4\sqrt{3}}{3R}\leq\frac{\csc A}{m_a}+\frac{\csc B}{m_b}+\frac{\csc C}{m_c}\leq\frac{\sqrt{3}R}{3r^2}$$

Proposed by George Apostolopoulos - Greece

SP.363. Triangle ABC has $|BC|=a,|CA|=b,|AB|=c$, inradius r and circumradius R. Equilateral triangles A_1BC,B_1CA and C_1AB with centroids K,L and M respectively, are drawn externally to triangle ABC. Prove that:

$$3\sqrt{3}\leq[ALM]+[BMK]+[CKL]\leq\frac{3\sqrt{3}}{4}R^2$$

where $[XYZ]$ represents the area of triangle XYZ.

Proposed by George Apostolopoulos - Greece

SP.364. Let ABC be non-right triangle with circumradius R. Squares with sides AB, BC, CA and centroids K, L, M respectively, are drawn externally to triangle ABC. Let α, β, γ be the distance from the vertices A, B, C to the segments $\overline{KM}, \overline{KL}, \overline{LM}$, respectively. Prove that:

$$\left(\frac{\cot A}{\alpha}\right)^2+\left(\frac{\cot B}{\beta}\right)^2+\left(\frac{\cot C}{\gamma}\right)^2 \geq \frac{8\cdot\tan 75^\circ}{3R^2}$$

Proposed by George Apostolopoulos - Greece

SP.365. Let $f, F : [a, b] \to \mathbb{R}$, such that $F(x) = -f(x) + \cos f(x)$. If F - is Riemann integrable, prove that f - is Riemann integrable.

Proposed by Cristian Miu - Romania

SP.366. Let m_a, m_b, m_c be the medians, r_a, r_b, r_c the exradii, r inradius and R circumradius of a triangle ABC. Prove that:

$$\frac{3}{2}\left(\frac{R}{2r}\right)^{-2} \leq \frac{r_a^2}{m_b^2+m_c^2}+\frac{r_b^2}{m_c^2+m_a^2}+\frac{r_c^2}{m_a^2+m_b^2} \leq 2\left(\frac{R}{2r}\right)^2-\frac{1}{2}\left(\frac{R}{2r}\right)$$

Proposed by George Apostolopoulos - Greece

SP.367. Let m_a, m_b, m_c be the medians, r_a, r_b, r_c the exradii, r the inradius and R the circumradius of a triangle ABC. Prove that:

$$\frac{8r}{R^2} < \frac{r_a+r_b}{m_am_b}+\frac{r_b+r_c}{m_bm_c}+\frac{r_c+r_a}{m_cm_a} \leq \frac{1}{r}\left(3\left(\frac{R}{2r}\right)^4-1\right)$$

Proposed by George Apostolopoulos - Greece

SP.368. If $0 < a < b < 1$, then prove:

$$\frac{a(2b-a)}{b\sqrt{a^2+b^2}} < \int_a^b \frac{dx}{x\sqrt{(x^2+a^2)^3}}+\frac{\sqrt{2}}{2} < \frac{a}{\sqrt{a^2+b^2}}+\frac{b-a}{a\sqrt{2}}$$

Proposed by Florică Anastase - Romania

SP.369. Let $(L_n)_{n\geq 0}, L_0 = 2, L_1 = 1, L_{n+2} = L_{n+1} + L_n, \forall n \in \mathbb{R}$, be the Lucas' sequences, and $a, b, c \in \mathbb{R}_+^*$ such that $abc = 1$. Prove that:

$$\frac{1}{a^6(bL_n+cL_{n+1})^2}+\frac{1}{b^6(cL_n+aL_{n+1})^2}+\frac{1}{c^6(aL_n+bL_{n+1})^2} \geq \frac{3}{L_{n+2}^2}, \forall n \in \mathbb{N}$$

Proposed by D.M. Bătineţu - Giurgiu, Neculai Stanciu - Romania

SP.370. If ABC is a triangle with inradius r and circumradius R, then for any point M in the plane of triangle, $M \notin \{A, B, C\}$, holds the inequality:

$$\frac{MA}{MB+MC}+\frac{MB}{MC+MA}+\frac{MC}{MA+MB}\geq\frac{R+r}{R}\geq\frac{3r}{R}$$

Proposed by D.M. Bătineţu - Giurgiu, Neculai Stanciu - Romania

SP.371. Let $ABCD$ be a tetrahedron, and let M be a point in space, $M \notin \{A, B, C\}$. Prove:

$$\frac{MA}{MB+MC+MD}+\frac{MB}{MC+MD+MA}+\frac{MC}{MD+MA+MB}+$$
$$+\frac{MD}{MA+MB+MC}\geq\frac{R+r}{R}\geq\frac{4r}{R}$$

Proposed by D.M. Bătineţu - Giurgiu, Neculai Stanciu - Romania

SP.372. If $f:\mathbb{R}_+^*\to\mathbb{R}_+^*$ with $\lim_{n\to\infty}\frac{f(x)}{x}=a\in\mathbb{R}_+^*$, $(b_n)_{n\geq1}$ is an arithmetic progression with $b_1, r\in\mathbb{R}_+^*$ and $u, v\in\mathbb{R}$ satisfy $u+v=1$, then compute:

$$\lim_{n\to\infty}\Big((n+1)^u\sqrt[n+1]{(f(b_1)f(b_2)\dots f(b_n)f(b_{n+1}))^v}-$$
$$-n^u\sqrt[n]{(f(b_1)f(b_2)\dots f(b_n))^v}\Big)$$

Proposed by D.M. Bătineţu - Giurgiu, Neculai Stanciu - Romania

SP.373. If $f:\mathbb{R}_+^*\to\mathbb{R}_+^*$ is a function such that $\lim_{x\to\infty}\frac{f(x)}{x}=c\in\mathbb{R}_+^*$ and $(a_n)_{n\geq1}$ is a positive sequence such that $\lim_{n\to\infty}(a_{n+1}-a_n)=a\in\mathbb{R}_+^*$, then compute:

$$\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{f(a_1)f(a_2)\dots f(a_n)f(a_{n+1})}}-\frac{n^2}{\sqrt[n]{f(a_1)f(a_2)\dots f(a_n)}}\right)$$

Proposed by D.M. Bătineţu - Giurgiu, Neculai Stanciu - Romania

SP.374. Let m_a, m_b, m_c be the lengths of the medians of a triangle with circumradius R and area F. Prove that:

$$\frac{4}{9R^2}\leq\frac{1}{m_a(m_b+2m_c)}+\frac{1}{m_b(m_c+2m_a)}+\frac{1}{m_c(m_a+2m_b)}\leq\frac{\sqrt{3}}{3F}$$

Proposed by D.M. Bătineţu - Giurgiu, Neculai Stanciu - Romania

SP.375. If $x, y, z \in (0,1)$ then in any ABC triangle with the area F the following inequality holds:

$$\frac{xa^4}{(y+z)^2(1-x^2)} + \frac{yb^4}{(z+x)^2(1-y^2)} + \frac{zc^4}{(x+y)^2(1-z)} \geq 6\sqrt{3}F^2$$

Proposed by D.M. Bătineţu - Giurgiu, Neculai Stanciu - Romania

UNDERGRADUATE PROBLEMS

UP.361. Prove that:

$$\int_0^1 \frac{\tan^{-1} x}{x\sqrt{1-x^2}} dx = \log_2(\sqrt{2}-1) \int_0^{\frac{\pi}{2}} \log(\sin x) dx$$

Proposed by Florică Anastase - Romania

UP.362.

$$\omega_n = \sum_{k=1}^{2n} \cot \frac{k\pi}{2n+1} \cdot \left(\sin \frac{2k\pi}{2n+1} + i \cos \frac{2k\pi}{2n+1} \right)$$

Find:

$$\Omega = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k^{\omega_n + 2 - (k+n)}}$$

Proposed by Florică Anastase - Romania

UP.363. Let be $(a_n)_{n\geq 1}; (b_n)_{n\geq 1} \subset (0,\infty)$ such that:

$$\lim_{n\to\infty} (a_{n+1} - a_n) = a \in (0,\infty); b_n = \left(\prod_{k=1}^{n} a_{2k-1} \right)^{\frac{1}{n}}$$

Find:

$$\Omega = \lim_{n\to\infty} \left(\frac{a_{n+1} \cdot b_{n+1}}{n+1} - \frac{a_n \cdot b_n}{n} \right)$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru - Romania

UP.364. In ΔABC the following relationship holds:

$$\left(\sum_{cyc} \frac{1}{a} \right) \left(\sum_{cyc} \frac{1}{a^2} \right) \cdot \ldots \cdot \left(\sum_{cyc} \frac{1}{a^n} \right) \geq \frac{3^n}{\sqrt{(R\sqrt{3})^{n^2+n}}}; n \in \mathbb{N}^*$$

Proposed by D.M. Bătineţu - Giurgiu, Daniel Sitaru - Romania

UP.365. Let be

$$a_n = \sum_{k=1}^{n} \tan^{-1}\left(\frac{1}{k^2+k+1}\right); n \geq 1$$

Find:

$$\Omega = \lim_{n\to\infty} n^2(e^{a_{n+1}} - e^{a_n})$$

Proposed by D.M. Bătineţu - Giurgiu, Daniel Sitaru - Romania

UP.366. If $s_n = 1 + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}} - 2\sqrt{n}; n \geq 1$ **find:**

$$\Omega = \lim_{n\to\infty} (1 + e^{s_{n+1}} - s^{s_n})^{n\sqrt{n}}$$

Proposed by D.M. Bătineţu - Giurgiu, Daniel Sitaru - Romania

UP.367. Find:

$$\Omega = \lim_{n\to\infty} \sqrt[n]{\frac{((2n)!!)^n}{(2an)!!}}; a \in \mathbb{N}$$

$$0!! = 1, (2k)!! = 2 \cdot 4 \cdot \ldots \cdot (2k), k \in \mathbb{N}^*$$

Proposed by D.M. Bătineţu - Giurgiu, Daniel Sitaru - Romania

UP.368. Let be

$$\Omega_n = \int_n^{n+1} \frac{x^n}{e^x + 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!}} dx, n \in \mathbb{N}^*$$

Prove that: $\Omega_n < n!$

Proposed by D.M. Bătineţu - Giurgiu, Daniel Sitaru - Romania

UP.369. Let $f : \mathbb{R} \to \mathbb{R}$ **be a continuous function;** $a, b > 0$; $a < b; a + b = s$;

$$f(s - x) + f(x) = c; \forall x \in \mathbb{R}; c > 0. \text{ Find:}$$

$$\Omega = \int_a^b (x^2 - sx + s^2) f(x) dx$$

Proposed by D.M. Bătineţu - Giurgiu, Daniel Sitaru - Romania

UP.370. If $a, b > 0$**, then:**

$$\int_0^{\frac{\pi}{4}} \frac{dx}{(x+1)(a^2\cos^2 x + b^2 \sin^2 x)} < \frac{1}{ab(\pi+4)}\left(\pi\frac{b}{a} + 4\tan^{-1}\left(\frac{b}{a}\right)\right)$$

Proposed by Florică Anastase - Romania

UP.371. **Let be**

$$(a_n)_{n\geq 1}; a_n = \prod_{k=1}^{n}((2k-1)!!)^{\frac{1}{k}}. \text{ Find:}$$

$$\Omega = \lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{a_{n+1}}} - \frac{n^2}{\sqrt[n]{a_n}}\right)$$

Proposed by D.M. Bătineţu - Giurgiu, Daniel Sitaru - Romania

UP.372. $(x)_{n\geq 1}$ **- be a positive sequence of real numbers such that** $x_1 > 0$ **and** $x_{n+1} = \frac{x_n^2}{2x_n - \log(1+x_n)}$. **Find:**

$$\Omega = \lim_{n\to\infty}\frac{(2n)!!}{(2n-1)!!}\cdot\frac{nx_n}{\sqrt{2n+1}}$$

Proposed by Florică Anastase - Romania

UP.373. $k \in \mathbb{N}, k > 0$ **and** $x_1 > k, x_{n+1} = \frac{x_n^2}{x_n - k}; n \in \mathbb{N}^*$. **Find:**

$$\Omega = \lim_{n\to\infty}\frac{n^2}{\log(\log n)}\cdot\frac{\sqrt[n]{H_n} - 1}{x_n}$$

Proposed by Florică Anastase - Romania

UP.374. **Calculate the integral:**

$$\int_0^1 \frac{x\ln x}{x^3 + x\sqrt{x} + 1}dx$$

Proposed by Vasile Mircea Popa - Romania

UP.375. **In any convex polygon** $A_1A_2...A_n, n \geq 3$ **with the area** F **and the sides lengths** $A_kA_{k+1} = a_k, k = \overline{1,n}, A_{n+1} = A_1$ **the following inequality holds:**

$$\sum_{k=1}^{n}(a_k - \sqrt{a_ka_{k+1}} + a_{k+1})^2 \geq 4F\cdot\tan\frac{\pi}{n}$$

Proposed by D.M. Bătineţu - Giurgiu - Romania

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