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UP.358 If $0 < a \leq b < \frac{\pi}{2}$ then:

$$\left(\int_a^b \frac{\sin x}{x}dx\right)^2 + \left(\int_a^b \frac{\cos x}{x}dx\right)^2 \leq \log^2\left(\frac{b}{a}\right)$$

Proposed by DanielSitaru-Romania

Solution 1 by proposer, Solution 2 by Gabriel Brehuescu-Romania, Solution 3 by Mohammad Rostami-Kabul-Afghanistan, Solution 4 by Ravi Prakash-New Delhi-India

Solution 1 by proposer

$$A = \int_a^b \frac{\sin x}{x}dx, B = \int_a^b \frac{\cos x}{x}dx,\ u = |u|(\cos t + i\sin t), t \in [0, 2\pi),$$

$$A = |u|\cos t, B = |u|\sin t, |u|^2 = A^2 + B^2 = A \cdot A + B \cdot B =$$

$$= |u|\cos t \cdot A + |u|\sin t \cdot B =$$

$$= |u|\left(\cos t\int_a^b \frac{\sin x}{x}dx + \sin t\int_a^b \frac{\cos x}{x}dx\right) \leq$$

$$\le |u|\int_a^b \sqrt{\left(\frac{sinx}{x}\right)^2+\left(\frac{cosx}{x}\right)^2}dx = |u|\int_a^b \frac{1}{x}dx = |u|log\left(\frac{b}{a}\right)$$

$$|u|^2 \le |u|log\left(\frac{b}{a}\right) \to |u| \le log\left(\frac{b}{a}\right) \to |u|^2 \le log^2\left(\frac{b}{a}\right) \to A^2+B^2 \le log^2\left(\frac{b}{a}\right)$$

$$\left(\int_a^b \frac{sinx}{x}dx\right)^2+\left(\int_a^b \frac{cosx}{x}dx\right)^2 \le log^2\left(\frac{b}{a}\right)$$

Solution 2 by Gabriel Brehuescu-Romania

Applying BCS inequality, we get:

$$\left(\int_a^b \frac{sinx}{x}dx\right)^2 = \left(\int_a^b \left(\frac{sinx}{\sqrt{x}}\right)\cdot\frac{1}{\sqrt{x}}dx\right)^2 \le \int_a^b \left(\frac{sinx}{\sqrt{x}}\right)^2 dx \cdot \int_a^b \left(\frac{1}{\sqrt{x}}\right)^2 dx$$
$$= \int_a^b \frac{sin^2x}{x^2}dx \cdot \int_a^b \frac{dx}{x};\ (1)$$

$$\left(\int_a^b \frac{cosx}{x}dx\right)^2 = \left(\int_a^b \left(\frac{cosx}{\sqrt{x}}\right)\cdot\frac{1}{\sqrt{x}}dx\right)^2 \le \int_a^b \left(\frac{cosx}{\sqrt{x}}\right)^2 dx \cdot \int_a^b \left(\frac{1}{\sqrt{x}}\right)^2 dx$$
$$= \int_a^b \frac{cos^2x}{x^2}dx \cdot \int_a^b \frac{dx}{x};\ (2)$$

From $(1), (2)$ we get:

$$\left(\int_a^b \frac{sinx}{x}dx\right)^2+\left(\int_a^b \frac{cosx}{x}dx\right)^2 \le \int_a^b \frac{dx}{x}\cdot\int_a^b \left(\frac{sin^2x+cos^2x}{x}\right)dx = \left(\int_a^b \frac{dx}{x}\right)^2$$
$$= log^2\left(\frac{b}{a}\right)$$

Solution 3 by Mohammad Rostami-Kabul-Afghanistan

$$\begin{cases} U = \int_a^b \frac{sinx}{x} dx \\ V = \int_a^b \frac{cosx}{x} dx \end{cases} \rightarrow \begin{cases} W = |W|(cos\theta + isin\theta) \\ |w|^2 = U^2 + V^2 \\ U = |W|cos\theta \\ V = |W|sin\theta \end{cases}$$

$$|W|^2 = U^2 + V^2 = |W| \cdot Ucos\theta + |W| \cdot Vsin\theta =$$

$$= |W|\left(cos\theta \int_a^b \frac{sinx}{x} dx + sin\theta \int_a^b \frac{cosx}{x} dx\right) \le$$

$$\le |W| \int_a^b \sqrt{\left(\frac{sinx}{x}\right)^2 + \left(\frac{cosx}{x}\right)^2} dx = |W| \int_a^b \frac{1}{x} dx = |W| log\left(\frac{b}{a}\right)$$

$$\Rightarrow |W|^2 \le |W| log^2\left(\frac{b}{a}\right) \Rightarrow |W| \le log\left(\frac{b}{a}\right) \Rightarrow |W|^2 \le log^2\left(\frac{b}{a}\right)$$

$$U^2 + V^2 \le log^2\left(\frac{b}{a}\right)$$

Therefore,

$$\left(\int_a^b \frac{sinx}{x} dx\right)^2 + \left(\int_a^b \frac{cosx}{x} dx\right)^2 \le log^2\left(\frac{b}{a}\right)$$

Solution 4 by Ravi Prakash-New Delhi-India

Let $A = \int_a^b \frac{sinx}{x} dx, B = \int_a^b \frac{cosx}{x} dx$

Now, $A^2 + B^2 = |B + iA| = \left|\int_a^b \frac{1}{x}(cosx + isinx) dx\right|^2 = \left|\int_a^b \frac{e^{ix}}{x} dx\right|^2 \le$

$$\le \left(\int_a^b \frac{1}{x} |e^{ix}| dx\right)^2 = \left(\int_a^b \frac{1}{x} dx\right)^2 = log^2\left(\frac{b}{a}\right)$$

Therefore,

$$\left(\int_a^b \frac{sinx}{x} dx\right)^2 + \left(\int_a^b \frac{cosx}{x} dx\right)^2 \le log^2\left(\frac{b}{a}\right)$$

Note by editor:
Many thanks to Florică Anastase-Romania for typed solutions.