Number 24

Spring 2022

R M M

ROMANIAN MATHEMATICAL MAGAZINE

Founding Editor
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Available online
www.ssmrmh.ro

ISSN-L 2501-0099

PROBLEMS FOR JUNIORS

JP.346. Find all values of k such that the following inequality:

$$\frac{a^2}{b}+\frac{b^2}{a}+\frac{kab}{a+b}\geq\left(1+\frac{k}{4}\right)(a+b)$$

holds for all positive real numbers a,b.

Proposed by Nguyen Viet Hung - Vietnam

JP.347. Let a,b,c be non-negative real numbers, not two of which are zero. Prove that:

$$\frac{2(a+b)(a+c)}{b+c}+\frac{2(b+c)(b+a)}{c+a}+\frac{2(c+a)(c+b)}{a+b}>$$
$$>\frac{(a+b+c)^3}{ab+bc+ca}+\frac{ab+bc+ca}{a+b+c}$$

Proposed by Nguyen Viet Hung - Vietnam

JP.348. If $a,b,c>0$ then:

$$\left(\frac{a^4}{b^4}+\frac{b^4}{c^4}+\frac{c^4}{a^4}\right)\left(\frac{a^3}{b^3}+\frac{b^3}{c^3}+\frac{c^3}{a^3}\right)\geq\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right)^2$$

Proposed by Daniel Sitaru - Romania

JP.349. Let a,b,c be positive real numbers such that $a+b+c=3$. Prove that:

$$\frac{a^6}{a^2+b}+\frac{b^6}{b^2+c}+\frac{c^6}{c^2+a}\geq\frac{3}{2}$$

Proposed by George Apostolopoulos - Greece

JP.350. For $1\leq a,b,c\leq\frac{2\sqrt{3}}{3}$, prove that:

$$\sqrt{4-3a^2}+\sqrt{4-3b^2}+\sqrt{4-3c^2}+(a+b+c)^2-3(a+b+c)\leq 3$$

Proposed by George Apostolopoulos - Greece

JP.351. In ΔABC the following relationship holds:

$$\prod_{cyc}\sin^2 A\geq 4\prod_{cyc}\cos A-5\prod_{cyc}\cos^2 A$$

Proposed by Cristian Miu - Romania

JP.352. If $a, b, c \in \mathbb{C}; |a| = |b| = |c| = 1$ then:

$$3|a+b+c| + 2(|a-b| + |b-c| + |c-a|) \geq 9$$

Proposed by Daniel Sitaru - Romania

JP.353. In $\Delta ABC, P \in Int(\Delta ABC), \mu(\widehat{ABP}) = 20^\circ$, $\mu(\widehat{PBC}) = \mu(\widehat{PCB}) = 10^\circ, \mu(\widehat{PCA}) = 40^\circ$. Prove that:

$$|AP| + |BC| = \sqrt{3}|AB|$$

Proposed by Mehmet Şahin - Turkey

JP.354. In acute $\Delta ABC, O$ - circumcenter, $F, K \in (AB), M, L(BC)$, $E, N \in (CA), \overline{FOE}, \overline{MON}, \overline{LOK}$ - are the antiparallels. Let ρ_a, ρ_b, ρ_c - inradii of $\Delta AFE, \Delta BLK, \Delta CMN$. Prove that: $\rho_a + \rho_b + \rho_c = R$

Proposed by Mehmet Şahin - Turkey

JP.355. In $\Delta ABC, A_1, B_1, C_1$ are contact points by inscribed circle. Prove that:

$$\left(\frac{AB}{A_1B_1}\right)^2 + \left(\frac{BC}{B_1C_1}\right)^2 + \left(\frac{CA}{C_1A_1}\right)^2 \geq \frac{6R}{r}$$

Proposed by Marian Ursărescu - Romania

JP.356. In $\Delta ABC, I$ - incenter and R_a, R_b, R_c - circumradius in $\Delta IBC, \Delta IAB, \Delta IAC$. Prove that:

$$\left(\frac{R_a}{a}\right)^2 + \left(\frac{R_b}{b}\right)^2 + \left(\frac{R_c}{c}\right)^2 \geq 1$$

Proposed by Marian Ursărescu - Romania

JP.357. In $\Delta ABC, N_a$ - Nagel's point. Prove that incribed circle of ΔABC passes through point N_a if and only if $s^2 + 4r^2 = 16Rr$.

Proposed by Marian Ursărescu - Romania

JP.358. If $x, y, z > 0, xyz = 1$ then in ΔABC the following relationship holds:

$$a^4\left(y + 1 + \frac{1}{x}\right) + b^4\left(z + 1 + \frac{1}{y}\right) + c^4\left(x + 1 + \frac{1}{z}\right) \geq 1296r^4$$

Proposed by Daniel Sitaru - Romania

JP.359. Find all n natural numbers such that:

$$\sqrt[3]{\frac{n+27}{(n+8)(n+1)}} \in \mathbb{Q}$$

Proposed by George Florin Şerban - Romania

JP.360. If $x, y, z \in (0, \frac{\pi}{2})$ then:

$$\sum_{cyc} \frac{\tan^2 x}{\tan^3 x + \cot x} + \sum_{cyc} \frac{\cot^2 x}{\cot^3 x + \tan x} \geq 2\sum_{cyc} \frac{1}{\tan^2 x + \cot^2 x}$$

Proposed by Daniel Sitaru - Romania

PROBLEMS FOR SENIORS

SP.346. Determine all functions $f : (0, \infty) \to \mathbb{R}$ such that:

$$f(xy) \leq xf(x) + yf(y) \leq \log(xy), \forall x, y > 0$$

Proposed by Marian Ursărescu - Romania

SP.347. If $A, B \in M_3(\mathbb{R})$ such that $Tr((AB - BA)^2) = 0$. Prove that:

$$\det((AB - BA)^2 + AB - BA + I_3) = (1 - \det(AB - BA))^2$$

Proposed by Marian Ursărescu - Romania

SP.348. In ΔABC prove that incribed circle of ΔABC passes through to G if and only if $s^2 = 16Rr + 4r^2$.

Proposed by Marian Ursărescu - Romania

SP.349. If $a \in (0, \frac{\pi}{2})$ then prove:

$$(\sin a)^{\sqrt{\log_{\sin a} \cos a}} + (\cos a)^{\sqrt{\log_{\cos a} \sin a}} \leq \sqrt{2}$$

Proposed by Ionuţ Florin Voinea - Romania

SP.350. If $x, y, z > 0, xy + yz + zx = 1$ and $\lambda \geq \frac{2}{3}$ then:

$$\frac{1}{x^2(x^2 + \lambda)} + \frac{1}{y^2(y^2 + \lambda)} + \frac{1}{z^2(z^2 + \lambda)} \geq \frac{27}{3\lambda + 1}$$

Proposed by Marin Chirciu - Romania

SP.351. If $x, y \in (0, \frac{\pi}{2})$; $\sqrt[3]{1 + \tan x} + \sqrt[3]{1 + \tan y} = 2\sqrt[3]{2}$ then:

$$\sqrt[3]{1 - \tan x} + \sqrt[3]{1 - \tan y} \leq 4 - 2\sqrt[3]{2}$$

Proposed by Daniel Sitaru - Romania

SP.352. Let $(x_n)_{n\geq 1}, (y_n)_{n\geq 1}$ be sequences of real numbers with $x_1 = 0, y_1 = 1, x_{n+1} = \frac{ax_n+by_n}{a+b}, y_{n+1} = \frac{cx_n+dy_n}{c+d}, \forall n \geq 1$, $a,b,c,d > 0, ad \neq bc$. Prove that if $(z_n)_{n\geq 1}, z_n = y_n - x_n$, then $(z_n)_{n\geq 1}$ - geometric progression, and if $q < 1, q$ - ratio of progression, then:

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n.$$

Proposed by Marin Chirciu - Romania

SP.353. Let $\lambda > 0$ fixed. Solve for real numbers:

$$\begin{cases} \lambda x = \sqrt{\lambda^2 y^2 - 1} + \sqrt{\lambda^2 z^2 - 1} \\ \lambda y = \sqrt{\lambda^2 z^2 - 1} + \sqrt{\lambda^2 x^2 - 1} \\ \lambda z = \sqrt{\lambda^2 x^2 - 1} + \sqrt{\lambda^2 y^2 - 1} \end{cases}$$

Proposed by Marin Chirciu - Romania

SP.354. If $x,y,z \geq 1$ then:

$$y \cdot x^x + z \cdot y^y + x \cdot z^z \geq x + y + z + \ln(x^{xy} \cdot y^{yz} \cdot z^{zx})$$

Proposed by Daniel Sitaru - Romania

SP.355. Let I_a, I_b, I_c and r_a, r_b, r_c denote the excenters and exradii of the triangle ABC, respectively. Let ρ_a - be the radius of the circle that lies inside and touches internally the excircle opposite A and touches the sides I_aI_b, I_aI_c of triangle $I_aI_bI_c$ externally. Let ρ_b, ρ_c be defined similarly. Prove that:

$$\frac{\rho_a}{r_a} + \frac{\rho_b}{r_b} + \frac{\rho_c}{r_c} \leq 1$$

Proposed by Mehmet Şahin - Turkey

SP.356. If $a,b,c > 0$ such that $abc = 1$ then:

$$\frac{(1+ab)^3}{(c+a)(a+b)} + \frac{(1+bc)^3}{(a+b)(b+c)} + \frac{(1+ca)^3}{(b+c)(c+a)} \geq \frac{54}{(a^2+b^2+c^2)^2}$$

Proposed by Pedro Pantoja - Brazil

SP.357. Let $S(n)$ be the sum of the digits of the positive integer n. Determine all pairs of positive integers $(a,b), a \geq b$ such that the equation $S(a^2) = (b-2018)^4$ has only finite solutions in positive integers.

Proposed by Pedro Pantoja - Brazil

SP.358. If $x, y, z > 0$ then:

$$4\sum_{cyc}\frac{x^3}{(y+1)(z+1)}+3\ge 6\sqrt[3]{xyz}$$

Proposed by Daniel Sitaru - Romania

SP.359. If $A \in M_3(\mathbb{R}), p \in \mathbb{R}^*$ such that $\det(A^2 - pA + p^2 I_3) = 0$. Prove that: $2\det(A^2 + p^2 I_3) \ge (\det A + p^3)^2$.

Proposed by Marian Ursărescu - Romania

SP.360. $z_1, z_2, z_3 \in \mathbb{C}^*$ different in pairs such that $|z_1| = |z_2| = |z_3|$. If

$$\sum_{cyc}\left|\frac{2z_1 - z_2 - z_3}{(z_1 - z_2)|z_1 - z_3| + (z_1 - z_3)|z_1 - z_2|}\right| = \frac{1}{|z_1 - z_2|}+\frac{1}{|z_2 - z_3|}+\frac{1}{|z_3 - z_1|},$$

then z_1, z_2, z_3 are affixes on equilateral triangle.

Proposed by Marian Ursărescu - Romania

UNDERGRADUATE PROBLEMS

UP.346. Solve for real numbers:

$$2\int_0^x \frac{x^2 \cdot e^{\arctan x}}{\sqrt{1+x^2}}dx = 1$$

Proposed by Daniel Sitaru - Romania

UP.347. Solve for complex numbers:

$$\begin{cases} \frac{|x|^2}{3}+\frac{|y|^2}{5}=\frac{|x+y|^2}{8} \\ 10x + y = 7 + 14i \end{cases}$$

Proposed by Daniel Sitaru - Romania

UP.348. If $x, y, z \in \mathbb{R}_+^* = (0, \infty)$ and ABC is a triangle with the area F, then:

$$\frac{(x+3y)(y+4z+3x)}{(y+3z)(z+3x)}\cdot a^4+\frac{(y+3z)(z+4x+3y)}{(z+3x)(x+3y)}\cdot b^4+$$

$$+\frac{(z+3x)(x+4y+3z)}{(x+3y)(y+3z)}\cdot c^4 \ge 32F^2$$

Proposed by D.M Bătineţu - Giurgiu - Romania

UP.349. If $m \geq 0; x, y, z > 0$, then in any ΔABC with the area F the following inequailty holds:

$$\frac{y+z}{x \cdot h_a^{m+1}} + \frac{z+x}{y \cdot h_b^{m+1}} + \frac{x+y}{z \cdot h_c^{m+1}} \geq \frac{2}{(\sqrt{F})^{m+1}} (\sqrt[4]{3})^{3-m}$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru- Romania

UP.350. If $x, y, z \in \mathbb{R}_+^* = (0, \infty)$, then in any ΔABC with the area F the following inequality holds:

$$\sum_{cyc} \frac{y+z}{x + \sqrt{(x+2y)(x+2z)}} a^2 \geq 2\sqrt{3}F$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru- Romania

UP.351. If $x, y, z \in \mathbb{R}_+^* = (0, \infty)$ and ABC is a triangle, then:

$$\frac{(x+2y)(2x+y+3z) \cdot a^2}{(y+z)(z+2x) \cdot h_a^2} + \frac{(y+2z)(2y+z+3x) \cdot b^2}{(z+2x)(x+2y) \cdot h_b^2} +$$

$$+ \frac{(z+2x)(2z+x+3y) \cdot c^2}{(x+2y)(y+2z) \cdot h_c^2} \geq 8$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru- Romania

UP.352. If $x, y, z > 0$ and ΔABC have the semiperimeter s the following inequality holds:

$$\frac{(y+z)a}{x \cdot h_a(s-a)} + \frac{(z+x)b}{y \cdot h_b(s-b)} + \frac{(x+y)c}{z \cdot h_c(s-c)} \geq \frac{12\sqrt{3}}{s}$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru- Romania

UP.353. Let $(x_n)_{n \geq 1}$ be a sequence of real numbers with $x_1 = \sqrt[3]{a^3 - a}, a \geq 2$ and $x_{n+1} = \sqrt[3]{a^3 - a + x_n}, n \geq 1$. Prove that $(x_n)_{n \geq 1}$ - convergent and find

$$\Omega_1 = \lim_{n \to \infty} x_n, \Omega_2 = \lim_{n \to \infty} \{x_n\}, \text{ where } \{*\} \text{ - fractional part.}$$

Proposed by Marin Chirciu - Romania

UP.354. Find:

$$\Omega = \lim_{n \to \infty} \left(1 + \left(\sum_{k=1}^{n-1} \frac{k}{n} \sin \frac{k\pi}{n} \right)^{-1} \right)^n$$

Proposed by Florică Anastase - Romania

UP.355. Find:

$$\Omega = \lim_{n\to\infty}\left(\lim_{x\to\frac{\pi}{2n}}\left(\frac{\cot x}{2}\cdot\left(\sum_{k=1}^{n-1}\frac{k}{n}\sin\frac{k\pi}{n}\right)^{-1}\right)^{\frac{1}{\tan(2nx)}}\right)$$

Proposed by Florică Anastase - Romania

UP.356. Find:

$$\Omega = \lim_{n\to\infty}\sqrt[n]{\sum_{k=1}^{n}\cos\frac{(n-1)k\pi}{n}\cdot\cos^{n-1}\left(\frac{k\pi}{n}\right)}$$

Proposed by Florică Anastase - Romania

UP.357. If $f:[0,\frac{\pi}{2})\to\mathbb{R}$,

$$f(x) = -\int_0^x \log(\cos y)dy$$

Prove that:

$$\lim_{\substack{x\to\frac{\pi}{2}\\ x<\frac{\pi}{2}}} f(x) = -\int_0^{\frac{\pi}{2}}\log(\sin y)dy$$

Proposed by Florică Anastase - Romania

UP.358. If $0 < a \le b < \frac{\pi}{2}$ then:

$$\left(\int_a^b \frac{\sin x}{x}dx\right)^2 + \left(\int_a^b \frac{\cos x}{x}dx\right)^2 \le \log^2\left(\frac{b}{a}\right)$$

Proposed by Daniel Sitaru - Romania

UP.359. Find:

$$\Omega = \int_0^\infty \frac{\tan^{-1}x}{x^3+1}dx$$

Proposed by Vasile Mircea Popa - Romania

UP.360. Find $x, y > 0$ such that:

$$\begin{cases} x+y=\frac{2}{3} \\ \frac{(x+1)^2}{3x^2-2x+1}+\frac{(y+1)^2}{3y^2-2y+1}=\frac{16}{3}\end{cases}$$

Proposed by Daniel Sitaru - Romania

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