

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro



UP.339 Prove that for any positive real numbers a, b, c

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} + \frac{1}{2}(a + b + c) \geq \frac{9(a^2 + b^2 + c^2)}{2(a + b + c)}$$

Proposed by Nguyen Viet Hung-Hanoi-Vietnam

Solution 1 by proposer, Solution 2 by Tran Hong-Dong Thap-Vietnam, Solution 3 by SanongHuayrerai-Nakon Pathom-Thailand, Solution 4 by Abdul Hannan-Tezpur-India, Solution 5 by Marin Chirciu-Romania

Solution 1 by proposer

Lemma 1. For any positive real numbers a, b, c then

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq \frac{(a^2 + b^2 + c^2)(a + b + c)}{ab + bc + ca}$$

Proof. The inequality is equivalent to

$$(ab + bc + ca) \left(\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \right) \geq (a^2 + b^2 + c^2)(a + b + c)$$

Or

$$\frac{a^3c}{b} + \frac{b^3a}{c} + \frac{c^3b}{a} \geq a^2b + b^2c + c^2a$$

This follows from the AM-GM inequality as follows

$$\frac{a^3c}{b} + \frac{b^3a}{c} \geq 2a^2b,$$

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\frac{b^3a}{c} + \frac{c^3b}{a} \geq 2b^2c,$$

$$\frac{c^3b}{a} + \frac{a^3c}{b} \geq 2c^2a.$$

Using the above lemma, it's enough to show that

$$\frac{(a^2 + b^2 + c^2)(a + b + c)}{ab + bc + ca} + \frac{1}{2}(a + b + c) \geq \frac{9(a^2 + b^2 + c^2)}{2(a + b + c)}$$

This is equivalent to

$$\begin{aligned} \frac{(a^2 + b^2 + c^2)(a + b + c)}{ab + bc + ca} - \frac{3(a^2 + b^2 + c^2)}{a + b + c} &\geq \frac{3(a^2 + b^2 + c^2)}{2(a + b + c)} - \frac{1}{2}(a + b + c) \\ \frac{(a^2 + b^2 + c^2)[(a + b + c)^2 - 3(ab + bc + ca)]}{(ab + bc + ca)(a + b + c)} &\geq \frac{3(a^2 + b^2 + c^2) - (a + b + c)^2}{2(a + b + c)} \\ \frac{(a^2 + b^2 + c^2)(a^2 + b^2 + c^2 - ab - bc - ca)}{ab + bc + ca} &\geq a^2 + b^2 + c^2 - ab - bc - ca \\ (a^2 + b^2 + c^2 - ab - bc - ca) \left(\frac{a^2 + b^2 + c^2}{ab + bc + ca} - 1 \right) &\geq 0 \end{aligned}$$

The last inequality is clearly true and we are done.

Solution 2 by Tran Hong-Dong Thap-Vietnam

$$\begin{aligned} \frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} - (a + b + c) &= \left(\frac{a^2}{b} - 2a + b \right) + \left(\frac{b^2}{c} - 2b + c \right) + \left(\frac{c^2}{a} - 2c + a \right) = \\ &= \frac{(a - b)^2}{b} + \frac{(b - c)^2}{c} + \frac{(c - a)^2}{a} \\ \frac{9(a^2 + b^2 + c^2)}{2(a + b + c)} - \frac{3(a + b + c)}{2} &= \frac{9(a^2 + b^2 + c^2) - 3(a + b + c)^2}{2(a + b + c)} = \\ &= \frac{6(a^2 + b^2 + c^2 - ab - bc - ca)}{2(a + b + c)} = \frac{3[(a - b)^2 + (b - c)^2 + (c - a)^2]}{2(a + b + c)} \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} + \frac{1}{2}(a + b + c) &\geq \frac{9(a^2 + b^2 + c^2)}{2(a + b + c)} \Leftrightarrow \\ \frac{(a - b)^2}{b} + \frac{(b - c)^2}{c} + \frac{(c - a)^2}{a} &\geq \frac{3[(a - b)^2 + (b - c)^2 + (c - a)^2]}{2(a + b + c)} \Leftrightarrow \end{aligned}$$

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned} & \left(\frac{1}{b} - \frac{3}{2(a+b+c)}\right)(a-b)^2 + \left(\frac{1}{c} - \frac{3}{2(a+b+c)}\right)(b-c)^2 \\ & \quad + \left(\frac{1}{a} - \frac{3}{2(a+b+c)}\right)(c-a)^2 \geq 0 \Leftrightarrow \\ & \frac{2(a+c)-b}{2b(a+b+c)}(a-b)^2 + \frac{2(a+b)-c}{2c(a+b+c)}(b-c)^2 + \frac{2(b+c)-a}{2a(a+b+c)}(c-a)^2 \geq 0 \Leftrightarrow \\ & S = \sum \frac{2(a+c)-b}{2b(a+b+c)}(a-b)^2 \geq 0; (1) \end{aligned}$$

Let us denote: $S_a = \frac{2(a+c)-b}{2b(a+b+c)}$, $S_b = \frac{2(a+b)-c}{2c(a+b+c)}$, $S_c = \frac{2(b+c)-a}{2a(a+b+c)}$

If $a \geq b \geq c$ then $S_a > 0, S_c > 0$

$$S_a + 2S_b = \frac{2(a+c)-b}{2b(a+b+c)} + 2 \cdot \frac{2(b+c)-a}{2a(a+b+c)} > 0 \Leftrightarrow$$

$$a(2a+2b-c) + 2c(2b+2c-a) \geq 0 \Leftrightarrow$$

$$2a^2 + 2ab + 2bc + 4c^2 - 3ac \geq 0 \text{ true by } 2a^2 + 2c^2 = 2(a^2 + c^2) \geq 4ac > 3ac \Rightarrow$$

$$2a^2 + 2ab + 2bc + 4c^2 > 2a^2 + 2c^2 > 3ac$$

Similarly: $S_c + 2S_b > 0 \xrightarrow{S.O.S.} S \geq 0; (2)$

If $a \leq b \leq c$ then $S_b > 0$

$$S_a + S_b = \frac{2(a+c)-b}{2b(a+b+c)} + \frac{2(a+b)-c}{2c(a+b+c)} > 0 \Leftrightarrow$$

$$a(2a+2b-c) + c(2b+2c-a) > 0 \Leftrightarrow$$

$$2a^2 + 2c^2 - 2ac + 2ab + 2bc \geq 0 \text{ true by } 2a^2 + 2b^2 \geq 4ac > 2ac \Rightarrow$$

$$2a^2 + 2c^2 - 2ac + 2ab + 2bc > 2a^2 + 2c^2 - 2ac > 0$$

Similarly: $S_b + S_c > 0 \xrightarrow{S.O.S.} S \geq 0; (3) \Rightarrow S \geq 0, \forall a, b, c > 0 \Rightarrow (1) \text{ is true.}$

Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand

For $a, b, c > 0$, we have $\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} + \frac{1}{2}(a+b+c) \geq \frac{9}{2} \cdot \frac{a^2+b^2+c^2}{a+b+c}$

$$(a+b+c) \left(\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} + \frac{1}{2}(a+b+c) \right) \geq \frac{9}{2}(a^2+b^2+c^2)$$

$$2(a+b+c) \left(\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \right) + (a+b+c)^2 \geq 9(a^2+b^2+c^2)$$

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\frac{a^3}{b} + \frac{b^3}{c} + \frac{c^3}{a} + \frac{a^2c}{b} + \frac{c^2b}{a} + \frac{b^2a}{c} + ab + bc + ca \geq 3(a^2 + b^2 + c^2)$$

and since

$$\left(\frac{a^3}{b} + ab\right) + \left(\frac{b^3}{c} + bc\right) + \left(\frac{c^3}{a} + ca\right) \geq 2(a^2 + b^2 + c^2)$$

We will show

$$\frac{a^2c}{b} + \frac{c^2b}{a} + \frac{b^2a}{c} \geq a^2 + b^2 + c^2$$

$$\text{Iffa}^3c^2 + c^3b^2 + b^3a^2 \geq a^3bc + b^3ca + c^3ab$$

Select $a \leq b \leq c$; $a, ax, axy, x, y \geq 1$

$$\begin{aligned} a^3(axy)^2 + (axy)^2 + (axy)^2(ax)^2 + (ax)^3a^2 \\ \geq a^3(axaxy) + (ax)^3(aaxy) + (axy)^3(aax) \end{aligned}$$

$$\text{Iff}x^2y^2 + x^5y^3 + x^3 \geq x^2y + x^4y + x^4y^3$$

$$y^2 + x^3y^3 + x \geq y + x^2y + x^2y^3$$

$$y(y-1) + x^2y^3(x-1) \geq x(xy-1) \geq xy(x-1) \text{ true.}$$

$$y(y-1) + (x-1)(x^2y^3 - xy) = (x-1)xy(xy^2 - 1) \geq 0 \text{ true.}$$

Solution 4 by Abdul Hannan-Tezpur-India

$$\text{Claim: } \frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq \frac{(a+b+c)(a^2+b^2+c^2)}{ab+bc+ca}; \quad (1)$$

$$\text{Proof: } (1) \Leftrightarrow \left(\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a}\right)(ab + bc + ca) \geq (a^2 + b^2 + c^2)(a + b + c)$$

$$\Leftrightarrow \sum a^3 + \sum a^2c + \sum \frac{a^3c}{b} \geq \sum a^3 \sum a^2b + \sum a^2c$$

$$\Leftrightarrow \sum \frac{a^3c}{b} \geq \sum a^2b \Leftrightarrow \sum a^4c^2 \geq a^3b^2c$$

Which is true by $\sum x^2 \geq \sum xy$, $x = a^2c, y = b^2a, z = c^2b$

So, it is enough to prove that

$$\frac{(a+b+c)(a^2+b^2+c^2)}{ab+bc+ca} + \frac{a+b+c}{2} \geq \frac{9(a^2+b^2+c^2)}{2(a+b+c)}$$

$$\Leftrightarrow (a+b+c)^2 \left(\frac{a^2+b^2+c^2}{ab+bc+ca} + \frac{1}{2}\right) \geq \frac{9(a^2+b^2+c^2)}{2}$$

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

Let $k = \frac{a^2+b^2+c^2}{ab+bc+ca} \geq 1$. To prove:

$$(k+2)\left(k+\frac{1}{2}\right) \geq \frac{9k}{2} \Leftrightarrow (k-1)^2 \geq 0 \text{ which is true.}$$

Solution 5 by Marin Chirciu-Romania

Lemma. If $a, b, c > 0$ then $\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq \frac{(a+b+c)(a^2+b^2+c^2)}{ab+bc+ca}$

Proof. We have: $\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq \frac{(a+b+c)(a^2+b^2+c^2)}{ab+bc+ca} \Leftrightarrow$

$$\left(\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a}\right)(ab+bc+ca) \geq (a+b+c)(a^2+b^2+c^2) \Leftrightarrow$$

$$\sum a^3 + \sum ab^2 + \sum \frac{ab^3}{c} \geq \sum a^3 + \sum ab^2 + \sum a^2b \Leftrightarrow$$

$$\sum \frac{ab^3}{c} \geq \sum a^2b \Leftrightarrow \sum a^2b^4 \geq abc \sum a^2b \Leftrightarrow$$

$$a^2(b^2-ca)^2 + b^2(c^2-ab)^2 + c^2(a^2-bc)^2 \geq 0$$

Equality holds if and only if $a = b = c$.

Let's get back to the main problem.

$$LHS = \frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} + n(a+b+c) \geq \frac{(a+b+c)(a^2+b^2+c^2)}{ab+bc+ca} + n(a+b+c) \geq$$

$$\stackrel{(1)}{\geq} 3(n+1) \frac{a^2+b^2+c^2}{a+b+c} = RHS,$$

$$(1) \Leftrightarrow \frac{(a+b+c)(a^2+b^2+c^2)}{ab+bc+ca} + n(a+b+c) \geq 3(n+1) \frac{a^2+b^2+c^2}{a+b+c}$$

$$\Leftrightarrow \frac{\sum a \sum a^2}{\sum bc} + n \sum a \geq 3(n+1) \frac{\sum a^2}{\sum a} \Leftrightarrow \sum a \left(\frac{\sum a^2}{\sum bc} + n \right) \geq 3(n+1) \frac{\sum a^2}{\sum a} \Leftrightarrow$$

$$\left(\sum a\right)^2 \left(\frac{\sum a^2}{\sum bc} + n\right) \geq 3(n+1) \sum a^2 \Leftrightarrow$$

$$\frac{(\sum a)^2}{\sum a^2} \left(\frac{\sum a^2}{\sum bc} + n\right) \geq 3(n+1) \Leftrightarrow \frac{\sum a^2 + 2\sum bc}{\sum a^2} \left(\frac{\sum a^2}{\sum bc} + n\right) \geq 3(n+1)$$

$$\Leftrightarrow \left(1 + \frac{2}{t}\right)(t+n) \geq 3(n+1) \Leftrightarrow (t+2)(t+n) \geq 3(n+1)t$$

$$\Leftrightarrow t^2 - (2n+1)t + 2n \geq 0, \text{ which follows from } t \geq 1 \geq 2n, \text{ where}$$

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$t = \frac{\sum a^2}{\sum bc}, \text{ equality for } t = 1 \Leftrightarrow \sum a^2 = \sum bc \Leftrightarrow$$

$$(a - b)^2 + (b - c)^2 + (c - a)^2 = 0 \Leftrightarrow a = b = c.$$

Note.

For $n = \frac{1}{2}$, we obtain Problem UP.339,RMM, proposed by Nguyen Viet Hung-Hanoi-Vietnam.

Note by editor:

Many thanks to Florică Anastase-Romania for typed solutions.