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In acute ΔABC the following relationship holds:

$$\cos(A-B)\cos(B-C)\cos(C-A) \le \sqrt[3]{\frac{8m_a^2m_b^2m_c^2}{(m_a^2+m_b^2)(m_b^2+m_c^2)(m_c^2+m_a^2)}}$$

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Solution 1 by Soumava Chakraborty-Kolkata-India, Solution 2 by Tran Hong-Dong Thap-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\cos(A-B)\cos(B-C)\cos(C-A)$$

$$= \left(2\cos^2\frac{A-B}{2}-1\right)\left(2\cos^2\frac{B-C}{2}-1\right)\left(2\cos^2\frac{C-A}{2}-1\right)$$

$$\overset{(a)}{=} 8\prod\cos^2\frac{B-C}{2} - 4\left(\prod\cos^2\frac{B-C}{2}\right)\sum\sec^2\frac{B-C}{2} + 2\sum\cos^2\frac{B-C}{2} - 1$$

$$\text{Now, } \sum\cos^2\frac{B-C}{2} = \sum\frac{(b+c)^2\sin^2\frac{A}{2}}{16R^2\sin^2\frac{A}{2}\cos^2\frac{A}{2}} = \frac{1}{16R^2s}\sum\frac{bc(b+c)^2}{s-a}$$

$$= \frac{1}{16R^2s}\sum\frac{bc(s+s-a)^2}{s-a}$$

$$= \frac{1}{16R^2s}\sum\left\{\frac{bcs^2}{s-a} + 2sbc + bc(s-a)\right\} = \frac{1}{16R^2s}\left\{s^3\sum\sec^2\frac{A}{2} + 3s\sum ab - 3abc\right\}$$

$$= \frac{1}{16R^2 s}\left[s^3\left\{\frac{s^2+(4R+r)^2}{s^2}\right\} + 3s(s^2+4Rr+r^2) - 12Rrs\right] = \frac{4s^2+(4R+r)^2+3r^2}{16R^2}$$

$$\Rightarrow \sum \cos^2\frac{B-C}{2} \overset{(1)}{=} \frac{4s^2+(4R+r)^2+3r^2}{16R^2}$$

$$\text{Again, } \sum \sec^2\frac{B-C}{2} = \sum \frac{16R^2\sin^2\frac{A}{2}\cos^2\frac{A}{2}}{(b+c)^2\sin^2\frac{A}{2}} = \sum \frac{16R^2 s(s-a)a}{4Rrs(b+c)^2}$$

$$= \frac{2R}{r}\sum \frac{a(b+c-a)}{(b+c)^2} \overset{(2)}{=} \frac{2R}{r}\left\{\sum \frac{a}{b+c} - \sum \frac{a^2}{(b+c)^2}\right\}$$

$$\text{Now, } \sum \frac{a}{b+c} = \frac{\sum a(c+a)(a+b)}{\prod(b+c)} = \frac{\sum a(\sum ab + a^2)}{2s(s^2+2Rr+r^2)}$$

$$= \frac{2s(s^2+4Rr+r^2)+2s(s^2-6Rr-3r^2)}{2s(s^2+2Rr+r^2)} \overset{(3)}{=} \frac{2s^2-2Rr-2r^2}{s^2+2Rr+r^2}$$

$$\text{and, } \sum \frac{a^2}{(b+c)^2} = \sum \frac{\left(2s-(b+c)\right)^2}{(b+c)^2}$$

$$= \sum \frac{4s^2-4s(b+c)+(b+c)^2}{(b+c)^2} \overset{(i)}{=} 4s^2\left[\frac{\sum\{(c+a)^2(a+b)^2\}}{\{\prod(b+c)\}^2}\right]$$

$$-4s\left[\frac{\sum(c+a)(a+b)}{\prod(b+c)}\right] + 3$$

$$\sum\{(c+a)^2(a+b)^2\} = \sum\left(\sum ab + a^2\right)^2 = \sum\left\{\left(\sum ab\right)^2 + 2a^2\sum ab + a^4\right\}$$

$$= 3\left(\sum ab\right)^2 + 2\left(\sum ab\right)(\sum a^2) + (\sum a^2)^2 - 2\sum a^2b^2$$

$$= \left(\sum ab\right)^2 + 2\left(\sum ab\right)(\sum a^2) + (\sum a^2)^2 + 2\sum a^2b^2 + 4abc(2s) - 2\sum a^2b^2$$

$$= \left(\sum ab + \sum a^2\right)^2 + 32Rrs^2$$

$$= (3s^2-4Rr-r^2)^2 + 32Rrs^2$$

$$\therefore \sum\{(c+a)^2(a+b)^2\} \overset{(ii)}{=} (3s^2-4Rr-r^2)^2 + 32Rrs^2$$

$$\text{Again, } \sum(c+a)(a+b)=\sum\left(\sum ab+a^2\right)=3\sum ab+\sum a^2$$

$$=\sum a^2+2\sum ab+\sum ab=4s^2+s^2+4Rr+r^2$$

$$\therefore \sum(c+a)(a+b)\overset{(iii)}{=}5s^2+4Rr+r^2$$

$$\because \prod(b+c)=s^2+2Rr+r^2 \therefore (i),(ii),(iii)\Rightarrow \sum\frac{a^2}{(b+c)^2}$$

$$=\frac{4s^2\{(3s^2-4Rr-r^2)^2+32Rrs^2\}}{4s^2(s^2+2Rr+r^2)^2}-\frac{4s(5s^2+4Rr+r^2)}{2s(s^2+2Rr+r^2)}+3$$

$$=\frac{(3s^2-4Rr-r^2)^2+32Rrs^2-2(5s^2+4Rr+r^2)(s^2+2Rr+r^2)+3(s^2+2Rr+r^2)^2}{(s^2+2Rr+r^2)^2}$$

$$=\frac{2s^4-s^2(8Rr+12r^2)+12R^2r^2+8Rr^3+2r^4}{(s^2+2Rr+r^2)^2}$$

$$\Rightarrow \sum\frac{a^2}{(b+c)^2}\overset{(4)}{=}\frac{2s^4-s^2(8Rr+12r^2)+12R^2r^2+8Rr^3+2r^4}{(s^2+2Rr+r^2)^2}$$

$$(2),(3),(4)\Rightarrow \sum\sec^2\frac{B-C}{2}$$

$$=\frac{2R}{r}\left\{\frac{2s^2-2Rr-2r^2}{s^2+2Rr+r^2}\right.$$

$$\left.-\frac{2s^4-s^2(8Rr+12r^2)+12R^2r^2+8Rr^3+2r^4}{(s^2+2Rr+r^2)^2}\right\}$$

$$\overset{(5)}{=}\frac{2R}{r}\left[\frac{(2s^2-2Rr-2r^2)(s^2+2Rr+r^2)-\{2s^4-s^2(8Rr+12r^2)+12R^2r^2+8Rr^3+2r^4\}}{(s^2+2Rr+r^2)^2}\right]$$

$$\text{Also, } 8\prod\cos^2\frac{B-C}{2}=8\prod\frac{(b+c)^2\sin^2\frac{A}{2}}{a^2}$$

$$=8\left\{\frac{4s^2(s^2+2Rr+r^2)^2}{16R^2r^2s^2}\right\}\left(\frac{r^2}{16R^2}\right)\overset{(6)}{=}\frac{(s^2+2Rr+r^2)^2}{8R^4}$$

$$(a),(1),(5),(6)\Rightarrow \cos(A-B)\cos(B-C)\cos(C-A)=\frac{(s^2+2Rr+r^2)^2}{8R^4}$$

$$-\left\{\frac{(s^2+2Rr+r^2)^2}{16R^4}\right\}\frac{2R}{r}\left[\frac{(2s^2-2Rr-2r^2)(s^2+2Rr+r^2)-\{2s^4-s^2(8Rr+12r^2)+12R^2r^2+8Rr^3+2r^4\}}{(s^2+2Rr+r^2)^2}\right]$$

$$+\frac{4s^2+(4R+r)^2+3r^2}{8R^2}-1$$

$$\Rightarrow \cos(A-B)\cos(B-C)\cos(C-A) \overset{(m)}{\cong} \frac{r(s^2+2Rr+r^2)^2-R\sigma+R^2r\{4s^2+(4R+r)^2+3r^2\}-8R^4r}{8R^4r}$$

$$(\text{where } \sigma = (2s^2-2Rr-2r^2)(s^2+2Rr+r^2) - \{2s^4-s^2(8Rr+12r^2)+12R^2r^2+8Rr^3+2r^4\})$$

Moreover, Tereshin $\Rightarrow m_a \geq \frac{b^2+c^2}{4R} \Rightarrow \frac{4RSm_a}{S} \geq b^2+c^2 \Rightarrow \frac{abcm_a}{S} \geq b^2+c^2$

$$\Rightarrow \frac{am_a}{S} \geq \frac{b}{c}+\frac{c}{b}$$

applying which on a triangle with sides $\frac{2m_a}{3}, \frac{2m_b}{3}, \frac{2m_c}{3}$ whose area of course $= \frac{S}{3}$ and medians of course $= \frac{a}{2}, \frac{b}{2}, \frac{c}{2}$ we get :

$$\frac{\left(\frac{2m_a}{3}\right)\left(\frac{a}{2}\right)}{\frac{S}{3}} \geq \frac{\frac{2m_b}{3}}{\frac{2m_c}{3}}+\frac{\frac{2m_c}{3}}{\frac{2m_b}{3}} \Rightarrow \frac{2m_a}{\left(\frac{2S}{a}\right)} \geq \frac{m_b}{m_c}+\frac{m_c}{m_b} \leq \frac{2m_a}{h_a} \overset{\text{Panaitopol}}{\leq} \frac{R}{r} \Rightarrow \frac{m_b^2+m_c^2}{m_bm_c} \leq \frac{R}{r}$$

$$\Rightarrow \frac{m_bm_c}{m_b^2+m_c^2} \geq \frac{r}{R} \text{ and analogs}$$

$$\overset{\text{multiplying together}}{\Rightarrow} \sqrt[3]{\frac{8m_a^2m_b^2m_c^2}{(m_a^2+m_b^2)(m_b^2+m_c^2)(m_c^2+m_a^2)}} \overset{(n)}{\geq} \frac{2r}{R}$$

$$\therefore (m), (n) \Rightarrow \text{it suffices to prove}$$

$$: \frac{r(s^2+2Rr+r^2)^2-R\sigma+R^2r\{4s^2+(4R+r)^2+3r^2\}-8R^4r}{8R^4r}-\frac{2r}{R} \leq 0$$

$$\Leftrightarrow \frac{r(s^2+2Rr+r^2)^2-R\sigma+R^2r\{4s^2+(4R+r)^2+3r^2\}-8R^4r-16R^3r^2}{8R^4r} \leq 0$$

$$\Leftrightarrow s^4+8R^4-s^2(6R^2+8Rr-2r^2)+8R^3r+22R^2r^2+8Rr^3+r^4 \overset{(u)}{\leq} 0$$

$\because \Delta$ ABC is acute − angled, Walker and Gerretsen

$$\Rightarrow (s^2-2R^2-8Rr-3r^2)(s^2-4R^2-4Rr-3r^2)\leq 0$$

$\Rightarrow$ in order to prove (u),

it suffices to prove : $s^4+8R^4-s^2(6R^2+8Rr-2r^2)+8R^3r+22R^2r^2+8Rr^3+r^4$

$$\leq (s^2-2R^2-8Rr-3r^2)(s^2-4R^2-4Rr-3r^2)$$

$$\Leftrightarrow (R+2r)s^2 \overset{(v)}{\lessgtr} 8R^3+7R^2r+7Rr^2+2r^3$$

Now, $(R+2r)s^2 \overset{\text{Gerretsen}}{\lessgtr} (R+2r)(4R^2+4Rr+3r^2) \overset{?}{\lessgtr} 8R^3+7R^2r+7Rr^2+2r^3$

$$\Leftrightarrow 4t^3-5t^2-4t-4 \overset{?}{\gtrless} 0 \left(\text{where } t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(4t^2+3t+2) \overset{?}{\gtrless} 0 \to \text{true} \because t \overset{\text{Euler}}{\gtrless} 2 \Rightarrow (v) \Rightarrow (u) \text{ is true}$$

$$\therefore \cos(A-B)\cos(B-C)\cos(C-A) \leq \sqrt[3]{\frac{8m_a^2m_b^2m_c^2}{(m_a^2+m_b^2)(m_b^2+m_c^2)(m_c^2+m_a^2)}} \text{ (Proved)}$$

Solution 2 by Tran Hong-Dong Thap-Vietnam

In any ΔABC (acute): $cos(b-c)\leq \frac{h_a}{m_a}$ $(and\ analogs)$

$$\prod_{cyc} cos(B-C) \leq \frac{h_ah_bh_c}{m_am_bm_c} \overset{m_am_bm_c\geq s^2r}{\leq} \frac{\frac{2s^2r^2}{R}}{s^2r}=\frac{2r}{R}$$

We need to prove:

$$\frac{2r}{R}\leq \sqrt[3]{\frac{8m_a^2m_b^2m_c^2}{(m_a^2+m_b^2)(m_b^2+m_c^2)(m_c^2+m_a^2)}} \Leftrightarrow \frac{8m_a^2m_b^2m_c^2}{(m_a^2+m_b^2)(m_b^2+m_c^2)(m_c^2+m_a^2)} \leq \left(\frac{R}{r}\right)^3$$

Let $x=m_a^2, y=m_b^2, z=m_c^2$

$$x+y+z=\frac{3}{4}(a^2+b^2+c^2)=\frac{3}{4}\cdot 2(s^2-4Rr-r^2)$$

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\leq \frac{1}{h_a^2}+\frac{1}{h_b^2}+\frac{1}{h_c^2}=\frac{s^2-4Rr-r^2}{2s^2r^2}$$

$$LHS=\frac{(x+y)(y+z)(z+x)}{xyz}=\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}+\frac{y}{x}+\frac{z}{y}+\frac{x}{z}+2\right)=$$

$$=(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)-1\le\frac{3(s^2-4Rr-r^2)^2}{4s^2r^2}-1\overset{(1)}{\le}\left(\frac{R}{r}\right)^3$$

$$(1)\Leftrightarrow r[3(s^2-4Rr-r^2)^2-4s^2r^2]\le 4s^2R^3\Leftrightarrow$$

$$[(4R^3+24Rr^2+10r^3)-3rs^2]s^3\ge r(48R^2r^2+3r^4+24Rr^3);(2)$$

But: $16Rr-5r^2\le s^2\le 4R^2+4Rr+3r^2(Gerretsen)$

$$(4R^3+24Rr^2+10r^3)-3rs^2\ge(4R^3+24Rr^2+10r^3)-3r(4R^2+4Rr+3r^2)=$$

$$=4R^2+12Rr^2-12R^2r+r^3\overset{(3)}{\ge}0\left(\therefore t=\frac{R}{r}\ge 2\ (Euler)\right)\Leftrightarrow$$

$$(3)\Leftrightarrow 4t^3-12t^2+12t+1\ge 0\Leftrightarrow 4t\left[\left(t-\frac{3}{2}\right)^2+\frac{3}{4}\right]+1>0\ (true\ for\ t\ge 2)$$

So, we need to prove:

$$(4R^3+12Rr^2-12rR^2+r^3)(16Rr-5r^2)\ge 3r^3(16R^2+r^2+8Rr);\left(t=\frac{R}{r}\ge 2\right)$$

$4(t-2)(16t^3-21t^2+9t+1)\ge 0$ **which is true from** $t\ge 2$ **and**

$16t^3-21t^2+9t+1>0\Rightarrow(2)$ **proved⇒(1) proved.**

Note by editor:
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