R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

ABOUT AN INEQUALITY BY D.M.BĂTINEȚU-GIURGIU-II

By Marin Chirciu-Romania

1) In ΔABC the following relationship holds:

$$\frac{1}{h_a h_b}+\frac{1}{h_b h_c}+\frac{1}{h_c h_a}\ge\frac{\sqrt{3}}{S}$$

Proposed by D.M. Bătinețu – Giurgiu – Romania

Solution. *We prove the following lemma:*

Lemma.

2) In ΔABC the following relationship holds:

$$\frac{1}{h_a h_b}+\frac{1}{h_b h_c}+\frac{1}{h_c h_a}=\frac{s^2+r^2+4Rr}{4s^2r^2}$$

Proof. *Using* $h_a=\frac{2S}{a}$ *we obtain* $\sum\frac{1}{h_b h_c}=\sum\frac{1}{\frac{2S}{b}\cdot\frac{2S}{c}}=\sum\frac{bc}{4S^2}=\frac{s^2+r^2+4Rr}{4s^2r^2}$

Let's get back to the main problem. Using the Lemma the inequality from enunciation can be written:

$\frac{s^2+r^2+4Rr}{4s^2r^2}\ge\frac{\sqrt{3}}{sr}\Leftrightarrow s^2+r^2+4Rr\ge 4sr\sqrt{3}$ *which follows from Mitrinovic's inequality:*

$s\le\frac{3R\sqrt{3}}{2}$*. It remains to prove that:* $s^2+r^2+4Rr\ge 4\cdot\frac{3R\sqrt{3}}{2}\cdot r\sqrt{3}\Leftrightarrow$

$\Leftrightarrow s^2+r^2+4Rr\ge 18Rr\Leftrightarrow s^2\ge 14Rr-r^2$*, which follows from Gerretsen's inequality* $s^2\ge 16Rr-5r^2$*. It remains to prove that:*

$16Rr-5r^2\ge 14Rr-r^2\Leftrightarrow R\ge 2r$ *(Euler's inequality)*

Equality holds if and only if the triangle is equilateral.

Remark. *The inequality can be strengthened:*

3) In ΔABC the following relationship holds:

$$\frac{1}{h_a h_b}+\frac{1}{h_b h_c}+\frac{1}{h_c h_a}\ge\frac{5R-r}{Ss}$$

Solution. *Using Lemma the inequality can be written:*

$\frac{s^2+r^2+4Rr}{4s^2r^2}\ge\frac{5R-r}{rs^2}\Leftrightarrow s^2\ge 16Rr-5r^2$ *(Gerretsen's inequality)*

Equality holds if and only if the triangle is equilateral.

Remark. *Inequality 3) is stronger than inequality 1).*

4) In ΔABC the following relationship holds:

$$\frac{1}{h_ah_b}+\frac{1}{h_bh_c}+\frac{1}{h_ch_a}\ge\frac{5R-r}{Ss}\ge\frac{\sqrt{3}}{S}$$

Solution. *See inequality 3) and* $\frac{5R-r}{Ss}\ge\frac{\sqrt{3}}{S}\Leftrightarrow 5R-r\ge s\sqrt{3}$, *which follows from Mitrinovic's inequality* $s\le\frac{3R\sqrt{3}}{2}$. *It remains to prove that*

$$5R-r\ge\frac{3R\sqrt{3}}{2}\cdot\sqrt{3}\Leftrightarrow R\ge 2r\ (Euler)$$

Remark. *Inequality 3) can be developed.*

5) In ΔABC the following relationship holds:

$$\frac{1}{h_ah_b}+\frac{1}{h_bh_c}+\frac{1}{h_ch_a}\ge\frac{nR+(9-2n)r}{Ss},\ where\ n\le 5.$$

Solution. *Using Lemma the inequality can be written:*

$\frac{s^2+r^2+4Rr}{4s^2r^2}\ge\frac{nR+(9-2n)r}{rs^2}\Leftrightarrow s^2\ge Rr(4n-4)+r^2(35-8n)$, *which follows from Gerretsen's inequality* $s^2\ge 16Rr-5r^2$. *It remains to prove that:*

$16Rr-5r^2\ge Rr(4n-4)+r^2(35-8n)\Leftrightarrow R(5-n)\ge 2r(5-n)$, *obviously from Euler's inequality* $R\ge 2r$ *and the condition from hypothesis* $n\le 5$.

Equality holds if and only if the triangle is equilateral.

Remark. *Let's find an inequality having an opposite sense:*

6) In ΔABC the following inequality holds:

$$\frac{1}{h_ah_b}+\frac{1}{h_bh_c}+\frac{1}{h_ch_a}\le\frac{1}{3r^2}$$

Solution. *Using Lemma the inequality from enunciation can be written:*

$\frac{s^2+r^2+4Rr}{4s^2r^2}\le\frac{1}{3r^2}\Leftrightarrow s^2\ge 12Rr+3r^2$, *which follows from Gerretsen's inequality.*

$s^2\ge 16Rr-5r^2$. *It remains to prove that:*

$$16Rr-5r^2\ge 12Rr+3r^2\Leftrightarrow R\ge 2r\ (Euler's\ inequality)$$

Equality holds if and only if the triangle is equilateral.

Remark. *Inequality 6) can be strengthened.*

7) InΔABC the following inequality holds:

$$\frac{1}{h_ah_b}+\frac{1}{h_bh_c}+\frac{1}{h_ch_a}\le\frac{1}{4r^2}\left(1+\frac{9Rr}{2s^2}\right)$$

Solution. *Using Lemma the inequality can be written:*

$$\frac{s^2+r^2+4Rr}{4s^2r^2} \le \frac{1}{4r^2}\left(1+\frac{9Rr}{2s^2}\right) \Leftrightarrow R \ge 2r \text{ (Euler's inequality)}$$

Equality holds if and only if the triangle is equilateral.

Remark. *Inequality 7) is stronger than inequality 6)*

8) In ΔABC the following inequality holds:

$$\frac{1}{h_a h_b}+\frac{1}{h_b h_c}+\frac{1}{h_c h_a} \le \frac{1}{4r^2}\left(1+\frac{9Rr}{2s^2}\right) \le \frac{1}{3r^2}$$

Solution. *See inequality 7) and* $\frac{1}{4r^2}\left(1+\frac{9Rr}{2s^2}\right) \le \frac{1}{3r^2} \Leftrightarrow 2s^2 \ge 27Rr$*, which follows from Gerretsen's inequality* $s^2 \ge 16Rr-5r^2$*. It remains to prove that:*

$$2(16Rr-5r^2) \ge 27Rr \Leftrightarrow R \ge 2r \text{ (Euler's inequality)}$$

Equality holds if and only if the triangle is equilateral.

Remark. *We can write the double inequality:*

9) In ΔABC the following inequality holds:

$$\frac{5R-r}{Ss} \le \frac{1}{h_a h_b}+\frac{1}{h_b h_c}+\frac{1}{h_c h_a} \le \frac{1}{4r^2}\left(1+\frac{9Rr}{2s^2}\right)$$

Proposed by Marin Chirciu – Romania

Solution. *See inequalities 3) and 7). Equality holds if and only if the triangle is equilateral.*

Remark. *We can write the sequence of inequalities.*

10) In ΔABC the following inequality holds:

$$\frac{\sqrt{3}}{S} \le \frac{5R-r}{Ss} \le \frac{1}{h_a h_b}+\frac{1}{h_b h_c}+\frac{1}{h_c h_a} \le \frac{1}{4r^2}\left(1+\frac{9Rr}{2s^2}\right) \le \frac{1}{3r^2}$$

Solution. *See inequalities 4) and 8). Equality holds if and only if the triangle is equilateral.*

Reference:

Romanian Mathematical Magazine-www.ssmrmh.ro