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Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{\log(n+1)}\sqrt[n]{\prod_{k=1}^{n}\frac{1}{2k+1}}\right)$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Sergio Esteban-Argentina, Solution 2 by Asmat Quatea-Kabul-Afganistan, Solution 3 by Florentin Vişescu-Romania, Solution 4 by Ravi Prakah-New Delhi-India, Solution 5 by Abdallah El Farissi-Bechar-Algerie, Solution 6 by Remus Florin Stanca-Romania

Solution 1 by Sergio Esteban-Argentina

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{\log(n+1)}\sqrt[n]{\prod_{k=1}^{n}\frac{1}{2k+1}}\right) = \lim_{n\to\infty}\left(\frac{1}{\log(n+1)}\sqrt[n]{\frac{1}{(2n+1)!!}}\right) =$$

$$= \lim_{n\to\infty}\left(\frac{1}{\log(n+1)}\sqrt[n]{\frac{2^n n!}{(2n+1)!}}\right) = \lim_{n\to\infty}\left(\frac{2}{\log(n+1)}\sqrt[n]{\frac{n!}{(2n+1)!}}\right) \overset{by\ Stirling's}{=}$$

$$= \lim_{n\to\infty}\left(\frac{2}{\log(n+1)}\cdot\frac{\frac{n}{e}}{\left(\frac{2n+1}{e}\right)^{\frac{2n+1}{n}}}\right) = \frac{2}{e}\cdot\lim_{n\to\infty}\frac{n}{\log(n+1)}\cdot\frac{e^2}{(2n+1)^2} = 0$$

Solution 2 by Asmat Quatea-Kabul-Afganistan

$$\frac{1\cdot 3\cdot 5\cdot \ldots\cdot (2n-1)\cdot 2n}{2\cdot 4\cdot 6\cdot \ldots\cdot (2n)}=\frac{(2n)!}{2^n\cdot n!}$$

$$\frac{1}{1\cdot 3\cdot 5\cdot \ldots\cdot (2n-1)}=\frac{2^n\cdot n!}{(2n)!}$$

$$\frac{1}{1\cdot 3\cdot 5\cdot \ldots\cdot (2n-1)\cdot (2n+1)}=\frac{2^n\cdot n!}{(2n)!\,(2n+1)}$$

$$n!\approx\sqrt{2n\pi}\left(\frac{n}{e}\right)^n;\ (2n)!=\sqrt{4n\pi}\left(\frac{2n}{e}\right)^{2n}$$

$$\frac{1}{1\cdot 3\cdot 5\cdot \ldots\cdot (2n-1)\cdot (2n+1)}\approx\frac{2^n\cdot\sqrt{2n\pi}\left(\frac{n}{e}\right)^n}{(2n+1)\sqrt{4n\pi}\left(\frac{2n}{e}\right)^{2n}}$$

$$\frac{1}{1\cdot 3\cdot 5\cdot \ldots\cdot (2n-1)\cdot (2n+1)}\approx\frac{2^n\cdot\sqrt{2}\left(\frac{n}{e}\right)^n}{(4n+2)\left(\frac{2n}{e}\right)^{2n}}$$

$$\left(\frac{1}{1\cdot 3\cdot 5\cdot \ldots\cdot (2n-1)\cdot (2n+1)}\right)^n\approx\frac{2\cdot\left(\sqrt{2}\right)^{\frac{1}{n}}\cdot\frac{n}{e}}{(4n+2)^{\frac{1}{n}}\left(\frac{4n^2}{e^2}\right)}$$

$$\left(\frac{1}{1\cdot 3\cdot 5\cdot \ldots\cdot (2n-1)\cdot (2n+1)}\right)^n\approx\frac{2\cdot\frac{n}{e}}{\frac{4n^2}{e^2}}=\frac{e}{2n}$$

$$\lim_{n\to\infty}\left(\frac{1}{log(n+1)}\cdot\frac{e}{2n}\right)=0$$

$$\Omega=\lim_{n\to\infty}\left(\frac{1}{log(n+1)}\sqrt[n]{\prod_{k=1}^{n}\frac{1}{2k+1}}\right)=0$$

Solution 3 by Florentin Vişescu-Romania

$$\Omega=\lim_{n\to\infty}\left(\frac{1}{log(n+1)}\sqrt[n]{\prod_{k=1}^{n}\frac{1}{2k+1}}\right)=\lim_{n\to\infty}\left(\sqrt[n]{\frac{1}{log^n(n+1)}\cdot\prod_{k=1}^{n}\frac{1}{2k+1}}\right)$$

$$a_n=\frac{1}{log^n(n+1)}\cdot\prod_{k=1}^{n}\frac{1}{2k+1}$$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{1}{log^{n+1}(n+2)}\cdot\prod_{k=1}^{n+1}\frac{1}{2k+1}\cdot\frac{log^n(n+1)}{1}\cdot\frac{1}{\prod_{k=1}^{n}\frac{1}{2k+1}}=$$

$$=\lim_{n\to\infty}\frac{1}{(2n+3)log(n+2)}\cdot\left(\frac{log(n+1)}{log(n+2)}\right)^n=0$$

$$\lim_{n\to\infty}\left(\frac{log(n+1)}{log(n+2)}\right)^n=\lim_{n\to\infty}\left(1+\frac{log\left(\frac{n+1}{n+2}\right)}{log(n+2)}\right)^n=e^{\lim_{n\to\infty}\left(\frac{log\left(\frac{n+1}{n+2}\right)^n}{log(n+2)}\right)}=$$

$$=e^{\lim_{n\to\infty}\left(\frac{log\left[\left(1+\frac{-1}{n+2}\right)^{n+2}\right]^{\frac{n}{n+2}}}{log(n+2)}\right)}=e^0=1$$

Solution 4 by Ravi Prakah-New Delhi-India

$$a_n=\prod_{k=1}^{n}\frac{1}{2k+1}$$

For $1\le k\le n\Rightarrow\frac{1}{2n+1}\le\frac{1}{2k+1}\le\frac{1}{3}$

$$\frac{1}{(2n+1)^n}\le\frac{1}{(2k+1)^n}\le\frac{1}{3^n}$$

$$\frac{1}{2n+1}\le\sqrt[n]{\prod_{k=1}^{n}\frac{1}{2k+1}}\le\frac{1}{3}$$

$$\frac{1}{(2n+1)log(n+1)}\le\frac{1}{log(2n+1)}\sqrt[n]{\prod_{k=1}^{n}\frac{1}{2k+1}}\le\frac{1}{(2n+1)log(n+1)}$$

$$\lim_{n\to\infty}\frac{1}{(2n+1)log(n+1)}=\lim_{n\to\infty}\frac{1}{(2n+1)log(n+1)}=0$$

Therefore,

$$\Omega=\lim_{n\to\infty}\left(\frac{1}{log(n+1)}\sqrt[n]{\prod_{k=1}^{n}\frac{1}{2k+1}}\right)=0$$

Solution 5 by Abdallah El Farissi-Bechar-Algerie

$$\frac{1}{n+2}=\frac{n}{\sum_{k=1}^{n}(2k+1)}\le\sqrt[n]{\prod_{k=1}^{n}\frac{1}{2k+1}}\le\frac{\sum_{k=1}^{n}\frac{1}{2k+1}}{n}\le\frac{1}{3}$$

$$\frac{1}{(n+2)log(n+1)}\le\frac{1}{log(n+1)}\sqrt[n]{\prod_{k=1}^{n}\frac{1}{2k+1}}\le\frac{1}{3log(n+1)}$$

Therefore,

$$\Omega=\lim_{n\to\infty}\left(\frac{1}{log(n+1)}\sqrt[n]{\prod_{k=1}^{n}\frac{1}{2k+1}}\right)=0$$

Solution 6 by Remus Florin Stanca-Romania

$$\Omega=\lim_{n\to\infty}\left(\frac{1}{log(n+1)}\sqrt[n]{\prod_{k=1}^{n}\frac{1}{2k+1}}\right)=\lim_{n\to\infty}\left(\sqrt[n]{\frac{1}{log^{n}(n+1)}\cdot\prod_{k=1}^{n}\frac{1}{2k+1}}\right)\overset{C-D}{=}$$

$$=\lim_{n\to\infty}\frac{\prod_{k=1}^{n+1}\frac{1}{2k+1}}{log^{n+1}(n+2)}\cdot\frac{log^{n}(n+1)}{\prod_{k=1}^{n}\frac{1}{2k+1}}=\lim_{n\to\infty}\frac{1}{(2n+3)log(n+2)}\cdot\left(\frac{log(n+1)}{log(n+2)}\right)^{n}=$$

$$=\lim_{n\to\infty}\frac{1}{(2n+3)log(n+2)}\cdot\left(1+\frac{log(n+1)-log(n+2)}{log(n+2)}\right)^{n}=$$

$$=\lim_{n\to\infty}\frac{1}{(2n+3)log(n+2)}\cdot\lim_{n\to\infty}\left(1+\frac{log(n+1)-log(n+2)}{log(n+2)}\right)^{n}=$$

$$=\lim_{n\to\infty}\frac{1}{(2n+3)log(n+2)}\cdot\lim_{n\to\infty}e^{\frac{n}{log(n+2)}log\left(\frac{n+1}{n+2}\right)}=$$

$$=\lim_{n\to\infty}\frac{1}{(2n+3)log(n+2)}\cdot\lim_{n\to\infty}e^{\frac{-1}{log(n+2)}}=0$$

Therefore,

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{log(n+1)}\sqrt[n]{\prod_{k=1}^{n}\frac{1}{2k+1}}\right) = 0$$

Note by editor:
Many thanks to Florică Anastase-Romania for typed solutions.