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Find a closed form:

$$\Omega=\sum_{n=1}^{\infty}\frac{x^n}{n!}\cdot\left[\frac{n!}{e}\right], x\in[-1,1), [*]-GIF$$

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$$\Omega=\sum_{n=1}^{\infty}\frac{x^n}{n!}\cdot\left[\frac{n!}{e}\right]=\frac{1}{e}\cdot\frac{x}{1-x}+\sum_{n=1}^{\infty}\frac{x^n}{n!}\cdot\left\{\frac{n!}{e}\right\}; x\in[-1,1]$$

We have:

$$\frac{1}{e}=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}=\sum_{k=0}^{n}\frac{(-1)^k}{k!}+\sum_{k=n+1}^{\infty}\frac{(-1)^k}{k!}$$

For $k\le n$: $k!\mid n!$, **then** $n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}\in\mathbb{Z}$

$$\left\{\frac{n!}{e}\right\}=\left\{n!\sum_{k=n+1}^{\infty}\frac{(-1)^k}{k!}\right\}=\left\{n!\sum_{p=0}^{\infty}\frac{(-1)^{n+1+p}}{(n+1+p)!}\right\}=\left\{(-1)^{n+1}\sum_{p=1}^{\infty}\frac{(-1)^p}{p!}\cdot\frac{n!\,p!}{(n+1+p)!}\right\}$$

We know that:

$$\frac{n!\,p!}{(n+1+p)!}=\frac{\Gamma(n+1)\Gamma(p+1)}{\Gamma(n+1+p+1)}=\int_0^1 t^n(1-t)^p dt$$

$$\text{So: }\left\{\frac{n!}{e}\right\}=\left\{(-1)^{n+1}\sum_{p=1}^{\infty}\frac{(-1)^p}{p!}\cdot\int_0^1 t^n(1-t)^p dt\right\}=$$

$$=\left\{(-1)^{n+1}\int_0^1 t^n\sum_{p=0}^{\infty}\frac{(t-1)^p}{p!}dt\right\}=\left\{\frac{(-1)^{n+1}}{e}\int_0^1 t^n e^t\,dt\right\}$$

$$(\therefore 0\le t\le 1\Rightarrow 0\le t^n\le 1; 1\le e^t\le e\Rightarrow t^n\le t^n e^t\le e t^n)$$

$$\left(\therefore 0<\frac{t^n}{e}\le\frac{1}{e}t^n e^t\le t^n<1\right)$$

$$\therefore\left\{\frac{n!}{e}\right\}=\begin{cases}\frac{1}{e}\int_0^1 t^{2p+1}e^t dt, \text{ if } n\in 2\mathbb{N}+1\\ 1-\frac{1}{e}\int_0^1 t^{2p}e^t dt, \text{ if } n\in 2\mathbb{N}\end{cases}$$

For $x\in[-1,1]$, we have:

$$\Omega_1=\sum_{n=1}^{\infty}\frac{x^{2n}}{(2n)!}\left(1-\frac{1}{e}\int_0^1 t^{2n}e^t dt\right)+\frac{1}{e}\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}\left(\int_0^1 t^{2n+1}e^t dt\right)=$$

$$=\cosh(x)-1-\frac{1}{e}\int_0^1(\cosh(xt)-1)e^t dt+\frac{1}{e}\int_0^1 \sinh(xt)e^t dt=$$

$$=\cosh(x)-1-\frac{1}{e}\int_0^1(\cosh(xt)-\sinh(xt))e^t dt+\frac{1}{e}\int_0^1 e^t dt=$$

$$=\cosh(x)-1-\frac{e^{1-x}-1}{e(1-x)}+\frac{e-1}{e}$$

$$\Omega = \sum_{n=1}^{\infty} \frac{x^n}{n!} \cdot \left[\frac{n!}{e}\right] = \frac{e^{1-x} - 1}{e(1-x)} - cosh(x)$$

Note. For $x = -1$ the series is not convergent.

Note by editor:
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