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In ΔABC the following relationship holds:

$$\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}+\frac{27r^2}{s^2}\ge 4$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } s-a=x,\ s-b=y \text{ and } s-c=z \therefore s=x+y+z$$

$$\text{Now, }\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}+\frac{27r^2}{s^2}=\frac{s-b}{s-a}+\frac{s-c}{s-b}+\frac{s-a}{s-c}+\frac{27s(s-a)(s-b)(s-c)}{s^4}$$

$$=\frac{y}{x}+\frac{z}{y}+\frac{x}{z}+\frac{27xyz}{(x+y+z)^3}\ge 4$$

$$\Leftrightarrow (x^2y+y^2z+z^2x)(x+y+z)^3+27x^2y^2z^2\ge 4xyz(x+y+z)^3$$

$$\Leftrightarrow \sum x^5y+3\sum x^4y^2+\sum x^2y^4+3\sum x^3y^3+12x^2y^2z^2\overset{(1)}{\ge} xyz\sum x^3$$
$$+5xyz\sum x^2y+6xyz\sum xy^2$$

$$\text{Now, }3\sum x^3y^3+9x^2y^2z^2\underbrace{\overset{\text{Schur}}{\ge}}_{(i)}3xyz\sum x^2y+3xyz\sum xy^2$$

$$\text{Again, }\sum x^4y^2+\sum x^2y^4+6x^2y^2z^2\overset{A-G}{\ge}2\sum x^3y^3+6x^2y^2z^2\underbrace{\overset{\text{Schur}}{\ge}}_{(ii)}2xyz\sum x^2y$$
$$+2xyz\sum xy^2$$

$$\text{Also, } \sum x^5y = xyz\left(\frac{x^4}{z}+\frac{y^4}{x}+\frac{z^4}{y}\right)$$

$$= xyz\left(\frac{x^6}{x^2z}+\frac{y^6}{xy^2}+\frac{z^6}{yz^2}\right) \overset{\text{Bergstrom}}{\geq} xyz\left\{\frac{(\sum x^3)^2}{\sum xy^2}\right\} \overset{(iii)}{\geq} xyz\left\{\frac{(\sum x^3)^2}{\sum x^3}\right\}$$

$$= xyz\sum x^3$$

$$\left(\because x^3+y^3+y^3 \overset{A-G}{\geq} 3xy^2, y^3+z^3+z^3 \overset{A-G}{\geq} 3yz^2 \text{ and } z^3+x^3\right.$$

$$\left.+x^3 \overset{A-G}{\geq} 3zx^2 \text{ and summing up, we get } \sum xy^2 \leq \sum x^3\right)$$

$$\text{Moreover, } \sum x^4y^2$$

$$= x^4y^2+y^4z^2+z^4x^2 \underbrace{\overset{\because \sum m^2 \geq mn+np+pm}{\geq}}_{(iv)} x^2y.y^2z+y^2z.z^2x+z^2x.x^2y$$

$$= xyz\sum xy^2 \text{ and lastly,}$$

$$\sum x^4y^2 \underbrace{\overset{A-G}{\geq}}_{(v)} 3x^2y^2z^2 \therefore (i)+(ii)+(iii)+(iv)+(v) \Rightarrow (1) \text{ is true}$$

$$\therefore \frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}+\frac{27r^2}{s^2} \geq 4 \text{ (Proved)}$$