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PROBLEMS FOR JUNIORS

JP.301. Prove that in any ABC triangle the following inequality holds:

$$\tan^2\frac{A}{2}\cot\frac{B}{2}+\tan^2\frac{B}{2}\cot\frac{C}{2}+\tan^2\frac{C}{2}\cot\frac{A}{2}\geq\sqrt{3}$$

Proposed by Marian Ursărescu - Romania

JP.302. Prove that in any ABC triangle the following inequality holds:

$$a^2r_b+b^2r_c+c^2r_a\geq 54Rr^2$$

Proposed by Marian Ursărescu - Romania

JP.303. Let be $x,y,z\geq 1$ such that $x^2+y^2+z^2-2xyz=1$ and $n\geq 0$. Prove that:

$$2(n+x)(n+y)(n+z)\leq(n+1)^3(1+xyz)$$

Proposed by Marin Chirciu - Romania

JP.304. Solve the equation in real numbers:

$$3\sqrt[3]{x^2-x+1}+\sqrt[4]{\frac{x^8+1}{2}}=2(x^4-3x+4)$$

Proposed by Hoang Le Nhat Tung - Vietnam

JP.305. Solve the following equation:

$$\sqrt{2(x^4+1)}+2\sqrt{3x-2x^4}=7-3x$$

Proposed by Hoang Le Nhat Tung - Vietnam

JP.306. If $a,b,c>0$ then:

$$(a+2c)\sqrt{a}+(b+2a)\sqrt{b}+(c+2b)\sqrt{c}\leq(a+b+c)\sqrt{3(a+b+c)}$$

Proposed by Daniel Sitaru - Romania

JP.307. Solve the following equation in real numbers:

$$\sqrt{x^3-2x^2+2x}+3\sqrt[3]{x^2-x+1}+2\sqrt[4]{4x-3x^4}=\frac{x^4-3x^3}{2}+7$$

Proposed by Hoang Le Nhat Tung - Vietnam

JP.308. Let $a, b, c \in [1,3]$ such that $a+b+c=6$. Find the maximum value of the expression:

$$P = a^4 + b^4 + c^4$$

Proposed by Hoang Le Nhat Tung - Vietnam

JP.309. If $m \in \mathbb{N}, h_A, h_B, h_C, h_D$ are the heights lengths of a tetrahedron $[ABCD]$ having the radius of the inscribed sphere r, then:

$$m + \frac{1}{4}\left(\left(\frac{h_A - 3r}{h_A + 3r}\right)^{m+1} + \left(\frac{h_B - 3r}{h_B + 3r}\right)^{m+1} + \left(\frac{h_C - 3r}{h_C + 3r}\right)^{m+1} + \right.$$
$$\left. + \left(\frac{h_D - 3r}{h_D + 3r}\right)^{m+1}\right) \geq \frac{m+1}{7}$$

Proposed by D.M. Bătineţu - Giurgiu, Daniel Sitaru - Romania

JP.310. If $a, b, c > 0; abc = 1$ then:

$$\sum_{cyc} \frac{c(a^2+b^2)+1}{a+b} \geq \frac{3}{2}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$$

Proposed by Daniel Sitaru - Romania

JP.311. If $x, y, z > 0; \sqrt{x} + \sqrt{y} + \sqrt{z} = 3$ then:

$$x + y + z \geq \sqrt[3]{x} + \sqrt[3]{y} + \sqrt[3]{z}$$

Proposed by Daniel Sitaru - Romania

JP.312. If $a, b, c > 0$

$$\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right)\left(\frac{a}{a+\lambda b} + \frac{b}{b+\lambda c} + \frac{c}{c+\lambda a}\right) \geq \frac{9}{\lambda+1}, \text{ where } \lambda \geq 0.$$

Proposed by Marin Chirciu - Romania

JP.313. Solve the system of equations:

$$\begin{cases} 4(x+y) = \sqrt[4]{8(x^4+y^4)} + 6\sqrt{xy} \\ 16x^5 - 20x^3 = \sqrt{1-y^2} - 5y \end{cases} \quad \forall x, y \in \mathbb{R}$$

Proposed by Hoang Le Nhat Tung - Vietnam

JP.314. Solve the following system of equations:

$$\begin{cases} a^2+b^2+c^2=a^3+b^3+c^3 \\ a^3b+b^3c+c^3a=3 \end{cases}$$

Proposed by Hoang Le Nhat Tung - Vietnam

JP.315. If s is the semiperimeter of ABC triangle and r_a, r_b, r_c the radii of excircles, then:

$$\frac{s^2-r_ar_b}{s^2+r_ar_b}+\frac{s^2-r_br_c}{s^2+r_br_c}+\frac{s^2-r_cr_a}{s^2+r_cr_a}\geq\frac{3}{2}$$

Proposed by D.M. Bătineţu - Giurgiu, Daniel Sitaru - Romania

PROBLEMS FOR SENIORS

SP.301. Let a, b, c be positive real numbers such that $a+b+c=3$. Find the minimum value of:

$$T=\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{1}{a^3+b^3+abc}+\frac{1}{b^3+c^3+abc}+\frac{1}{c^3+a^3+abc}$$

Proposed by Hoang Le Nhat Tung - Vietnam

SP.302. In ABC acute-angled triangle let r_1, r_2, r_3 be the inradii in $\Delta BOC, \Delta COA, \Delta AOB$, where O is the circumcenter of ΔABC and H the orthocenter. Prove that the following inequality holds:

$$\left(\frac{r_1}{AH}+\frac{r_2}{BH}+\frac{r_3}{CH}\right)\left(\sum_{cyc}\frac{A}{a}\right)<\frac{\sqrt{3}s\cdot\pi}{12Rr}$$

with the usual notations in triangle.

Proposed by Radu Diaconu - Romania

SP.303. Let x, y, z be positive numbers such that $x+y+z=3$. Find the minimum value of:

$$P=\frac{x^3}{y\sqrt{x^3+8}}+\frac{y^3}{z\sqrt{y^3+8}}+\frac{z^3}{x\sqrt{z^3+8}}$$

Proposed by Hoang Le Nhat Tung - Vietnam

SP.304. Let a, b, c be positive real numbers such that $(a+b)(b+c)(c+a)=1$. Find the minimum value of the expression:

$$P=\frac{a}{b(b+2c)(a+3c)^2}+\frac{b}{c(c+2a)(b+3a)^2}+\frac{c}{a(a+2b)(c+3b)^2}$$

Proposed by Hoang Le Nhat Tung - Vietnam

SP.305. Let a, b, c be positive real numbers such that $abc = 1$. Find the maximum value of the expression:

$$P = \frac{1}{\sqrt[3]{3a^4 - 4a + 2b^2 + 11}} + \frac{1}{\sqrt[3]{3b^4 - 4b + 2c^2 + 11}} + \frac{1}{\sqrt[3]{3c^4 - 4c + 2a^2 + 11}}$$

Proposed by Hoang Le Nhat Tung - Vietnam

SP.306. In ΔABC the following relationship holds:

$$\frac{16}{9} \cdot \frac{4R + r}{3R - 2r} \le \frac{a^2}{m_a^2} + \frac{b^2}{m_b^2} + \frac{c^2}{m_c^2} \le 4\left(\frac{R}{r} - 1\right)$$

Proposed by Marin Chirciu - Romania

SP.307. In acute ΔABC the following relationship holds:

$$\sum_{cyc}\left(2 + \frac{\sqrt{h_b h_c}}{a} - \frac{2(s-a)^2}{bc}\right) \le \sum_{cyc}(1 + \csc A)^{\frac{1}{1+\cot A}} \cdot (1 + \sec A)^{\frac{1}{1+\tan A}}$$

Proposed by Florică Anastase - Romania

SP.308. Let $(x_n)_{n\ge1}, x_1 = 0, x_n = \frac{(1-n)x_{n-1} + 1 - 2n}{nx_{n-1} + 2n}$. Find:

$$\Omega = \lim_{n\to\infty} \prod_{k=1}^{n} (2 + x_k)$$

Proposed by Florică Anastase - Romania

SP.309. In any ΔABC the following relationship holds:

$$\frac{r}{4R} \le \sin\left(\frac{\pi - A}{4}\right) \sin\left(\frac{\pi - B}{4}\right) \sin\left(\frac{\pi - C}{4}\right) \le \frac{1}{8}$$

Proposed by Marian Ursărescu - Romania

SP.310. In $\Delta ABC, B' \in (AC)$ the contact point of the external circumscription circle of side AC and C' the contact point of the external circumscription circle of side AB. Prove that: $B'C'$ is tangent of the inscribed circle in ABC if and only if:

$$(s - b)^2 + (s - c)^2 = (s - a)^2$$

Proposed by Marian Ursărescu - Romania

SP.311. If $A \in M_2(\mathbb{R}$ such that $\det(A^4 + 4I_2) = 0$. Prove that:

$$(\det A)^2 = (\operatorname{Tr} A)^2$$

Proposed by Marian Ursărescu - Romania

SP.312. In ΔABC let the point $A' \in (BC)$ such that the incircle in $\Delta AA'B$ and $\Delta AA'C$ have same radius. Prove that:

$$\sqrt[3]{AA' \cdot BB' \cdot CC'} \geq 3r$$

Proposed by Marian Ursărescu - Romania

SP.313. Let x, y, z be positive real numbers such that $xyz = 1$. Find the minimum value of the expression:

$$P = 2(x+y+z) + \frac{x}{y^3+z^3+1} + \frac{y}{z^3+x^3+1} + \frac{z}{x^3+y^3+1}$$

Proposed by Hoang Le Nhat Tung - Vietnam

SP.314. Let a, b, c be positive real numbers such that $a+b+c = 3$. Prove that:

$$\frac{a^2}{b^4c\sqrt[3]{4(b^6+1)}} + \frac{b^2}{c^4a\sqrt[3]{4(c^6+1)}} + \frac{c^2}{a^4b\sqrt[3]{4(a^6+1)}} \geq \frac{3}{2}$$

Proposed by Hoang Le Nhat Tung - Vietnam

SP.315. Find:

$$\Omega = \cos^3\frac{2\pi}{7}\sin^6\frac{2\pi}{7}\sin^6\frac{3\pi}{7} - \cos^3\frac{3\pi}{7}\sin^6\frac{3\pi}{7}\sin^6\frac{\pi}{7} - \cos^3\frac{\pi}{7}\sin^6\frac{\pi}{7}\sin^6\frac{2\pi}{7}.$$

Proposed by Pedro Pantoja - Brazil

UNDERGRADUATE PROBLEMS

UP.301. If $S_n = \sum_{k=1}^{n} 3^{k-1} \cdot \sin^3 \frac{\pi}{3^{k+1}}$ and $I = \pi \int_{\frac{1}{\sqrt{3}}}^{1} \frac{x}{\tan^{-1} x} dx$ then find:

$$\Omega = \lim_{n\to\infty} ([I] \cdot S_n)^n; [*] \text{ - GIF}$$

Proposed by Florică Anastase - Romania

UP.302. Let x, y, z be positive real numbers such that $x + y + z = \frac{3}{xyz}$. Find the minimum value of:

$$Q = (2x^2 - xy + 2y^2)(2y^2 - yz + 2z^2)(2z^2 - zx + 2x^2)$$

Proposed by Hoang Le Nhat Tung - Vietnam

UP.303. Let be $(I_n)_{n\geq 1}, I_n = \int_1^{a^2} \frac{dx}{x(1+\sqrt{x})^n}; a \in \mathbb{R}, a \geq 2;$ $\Omega(a) = \lim_{n\to\infty}(1+I_n)\cdot\sum_{k=1}^{n}\frac{a^k-2^k}{k\cdot(2a)^k}$. Then prove:

$$\frac{a-1}{2a} \leq \Omega(a) \leq \frac{a-1}{a+1}$$

Proposed by Florică Anastase - Romania

UP.304.

$$x_n = \prod_{k=1}^{n}\left(2\sin\frac{k\pi}{2n}\right), n \in \mathbb{N}^*. \text{ Find:}$$

$$\Omega = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{2-nx_n\cdot x_k}{x_k}\right)$$

Proposed by Florică Anastase - Romania

UP.305. Let $(a_n)_{n>1}, a_1 = e, a_n = e^n a_{n-1}^n$ and $(b_n)_{n>1}$ such that:

$$\left(1+\frac{1}{n}\right)^{n+b_n} = \prod_{k=1}^{n}\left(1+\frac{1}{\log a_k}\right). \text{ Find:}$$

$$\Omega = \lim_{n\to\infty} b_n$$

Proposed by Florică Anastase - Romania

UP.306. Let $(x_n)_{n>0}, x_0 = 2, x_n = n(x_{n-1} - (n-1)! - 2) - 2$. Find:

$$\Omega = \lim_{n\to\infty}\frac{\sqrt[n]{1+2n-\sum_{k=1}^{n}x_n}}{n}$$

Proposed by Florică Anastase - Romania

UP.307. Let $(a_n)_{n\geq 1}, a_n \in (0,\infty)$ be a sequence of real numbers such that:

$$a_1 = \sqrt{a}, a > 0, a_{n+1}^2 = n\cdot a_n + 1 \text{ then find:}$$

$$\Omega = \lim_{n\to\infty}\frac{a_n}{n^3}\int_0^1 \sqrt[n]{\frac{x^{2n}+1}{x^n+1}}dx, n \in \mathbb{N}, n \geq 2.$$

Proposed by Florică Anastase - Romania

UP.308. Find:

$$\Omega = \lim_{n\to\infty}\frac{\log\left(\sum_{k=0}^{n}(n+k)\binom{n+k}{k}\right)}{\sqrt[n]{n!}}$$

Proposed by Marian Ursărescu - Romania

UP.309. In acute ΔABC the altitudes AA', BB', CC' intersect at all second times the determined circle by the points A', B', C' inA'', B'', C''. Prove that:

$$(2r)^{2s} \geq (A'A'')^{a} \cdot (B'B'')^{b} \cdot (C'C'')^{c}$$

Proposed by Marian Ursărescu - Romania

UP.310. Let x, y, z be positive real numbers such that $x+y+z = 3$. Prove that:

$$3(\sqrt[3]{x} + \sqrt[3]{y} + \sqrt[3]{z} + 1) \geq 4(xy + yz + zx)$$

Hence, find the minimum value of the expression:

$$P = \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} + \frac{3(\sqrt[3]{x} + \sqrt[3]{y} + \sqrt[3]{z})}{8}$$

Proposed by Hoang Le Nhat Tung - Vietnam

UP.311. Find:

$$\lim_{n\to\infty} \sqrt[3]{n^2}\left(\frac{\sqrt[3]{(n+1)^2}}{\sqrt[3n+3]{(n+1)!}} - \frac{\sqrt[3]{n^2}}{\sqrt[3n]{n!}}\right)$$

Proposed by D.M. Bătineţu - Giurgiu, Neculai Stanciu - Romania

UP.312. If $m \in \mathbb{R}_+ = [0, \infty)$ and $a, b, x, y, z \in \mathbb{R}_+^* = (0, \infty)$, then:

$$\frac{x^{m+1}}{(ay+bz)^{m+1} \cdot \sec^{2m}\frac{\pi}{18}} + \frac{y^{m+1}}{(az+bx)^{m+1} \cdot \csc^{2m}\frac{\pi}{9}} +$$
$$+ \frac{z^{m+1}}{(ax+by)^{m+1} \cdot \csc^{2m}\frac{2\pi}{9}} \geq \frac{3}{4^m \cdot (a+b)^{m+1}}$$

Proposed by D.M. Bătineţu - Giurgiu, Neculai Stanciu - Romania

UP.313. Let be $(a_n)_{n\geq 1}, (b_n)_{n\geq 1}, a_n, b_n \in \mathbb{R}_+^* = (0, \infty), \forall n \in \mathbb{N}^*$,

$$\lim_{n\to\infty} \frac{a_n}{\sqrt[n]{n!}} = a \in \mathbb{R}_+^*, b_n = \prod_{n=1}^{n} a_n. \text{ Find:}$$

$$\lim_{n\to\infty} (\sqrt[n+1]{b_{n+1}} - \sqrt[n]{b_n})$$

Proposed by D.M. Bătineţu - Giurgiu, Neculai Stanciu - Romania

UP.314. Let $(a_n)_{n\geq 1}, (b_n)_{n\geq 1}$ be sequences of real strictly positive numbers such that:

$$\lim_{n\to\infty}\frac{a_n}{(2n-1)!!}=a\in\mathbb{R}_+^*=(0,\infty);\ \lim_{n\to\infty}\frac{b_n}{(2n-1)!!}=b\in\mathbb{R}_+^*.\ \textbf{Find:}$$

$$\lim_{n\to\infty}\left(\sqrt[n+1]{(2n+1)!!}-\sqrt[2n]{a_n\cdot b_n}\right)$$

Proposed by D.M. Bătineţu - Giurgiu, Daniel Sitaru - Romania

UP.315. Let $(a_n)_{n\geq 1}, (b_n)_{n\geq 1}$ be sequences of real strictly positive numbers such that:

$$\lim_{n\to\infty}\frac{a_n}{n!}=a\in\mathbb{R}_+^*,\ \lim_{n\to\infty}\frac{b_n}{(2n-1)!!}=b\in\mathbb{R}_+^*=(0,\infty).\ \textbf{Find:}$$

$$\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!\cdot(2n+1)!!}-\sqrt[n]{a_nb_n}\right)^{\frac{1}{n}}.$$

Proposed by D.M. Bătineţu - Giurgiu - Romania

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