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# R M M

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# SOLUTIONS

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**JP.256. If** $a, b, c, d > 0; a + b + c + d = 4; 0 \le n \le 3$ **then:**

$$\frac{a}{b^3 + b^2 + b + n} + \frac{b}{c^3 + c^2 + c + n} + \frac{c}{d^3 + d^2 + d + n} + \frac{d}{a^3 + a^2 + a + n} \ge \frac{4}{n + 3}$$

***Proposed by Marin Chirciu – Romania***

***Solution by Tran Hong-Dong Thap-Vietnam***

*Let* $f(x) = \frac{1}{x^3+x^2+x+n}$ $(0 < x < 4; 0 \le n \le 3) \Rightarrow f'(x) = -\frac{3x^2+2x+1}{(x^3+x^2+x+n)^2}$

$$\therefore \frac{1}{x^3 + x^2 + x + n} \ge -\frac{6}{(3 + n^2)}(x - 1) + \frac{1}{3 + n} = g(x)$$

$$\Leftrightarrow \frac{1}{x^3+x^2+x+n} \ge -\frac{6x}{(3+n)^2} + \frac{9+n}{(3+n)^2} \quad \textit{(1)}$$

$$\Leftrightarrow (3 + n)^2 \ge (x^3 + x^2 + x + n)(9 + n - 6x)$$

$$\Leftrightarrow (x - 1)^2[6x^2 + 9x + 9 - n(n + 3)] \ge 0 \quad (*)$$

$$\because 0 \le n \le 3, 0 < x < 4 \Rightarrow -n(x + 3) \ge -3x - 9$$

$\Rightarrow 6x^2 + 9x + 9 - 3x - 9 = 6x^2 + 3x > 0\ (0 < x < 4) \Rightarrow$ ***(*) true*** $\Rightarrow$ ***(1) true.***

$$u = ab + bc + cd + da \le \frac{(a + b + c + d)^2}{4} = 4$$

$$af(b) + bf(c) + cf(a) + af(d) \ge a \cdot g(b) + b \cdot g(c) + c \cdot g(d) + d \cdot g(a)$$

$$= \frac{9 + n}{(3 + n)^2}[a + b + c + d] - \frac{6}{(3 + n)^2}[ab + bc + cd + da]$$

$$= \frac{9 + n}{(3 + n)^2} \cdot 4 - \frac{6}{(3 + n)^2}[ab + bc + cd + da]$$

$$\ge \frac{4(9 + n)}{(3 + n)^2} - \frac{6 \cdot 4}{(3 + n)^2} = \frac{4(3 + n)}{(3 + n)^2} = \frac{4}{3 + n}$$

***Proved. Equality*** $\Leftrightarrow a = b = c = d = 1.$

**JP.257. Let be:**

$$\Omega = \left\{y | y = \frac{\sin^2 x}{\cos^2 x + \tan^2 x} + \frac{\cos^2 x}{\sin^2 x + \cot^2 x}; x \in \left(0, \frac{\pi}{2}\right)\right\}$$

**Find:** $\Omega_1 = \inf \Omega;\ \Omega_2 = \sup \Omega$

***Proposed by Marin Chirciu – Romania***

**Solutions 1 by Sergio Esteban-Argentina**

$$y = \frac{\sin^2 x \cdot \cos^2 x}{\cos^4 x + \sin^2 x} + \frac{\sin^2 x \cdot \cos^2 x}{\sin^4 x + \cos^2 x}$$

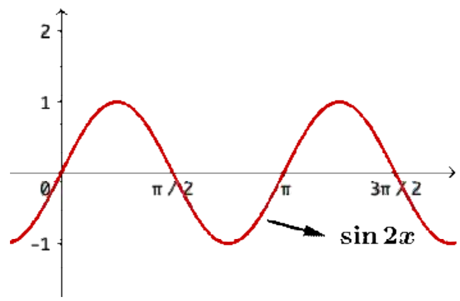
***Sabemos,*** $\sin^4 x + \cos^2 x = \sin^4 x + 1 - \sin^2 x = 1 - \sin^2 x\,(1 - \sin^2 x)$

$$= 1 - \sin^2 x \cdot \cos^2 x = 1 - \frac{1}{4}\sin^2 2x$$

***y*** $\cos^4 x + \sin^2 x = 1 - \cos^2 x + \cos^4 x = 1 - \cos^2 x\,(1 - \cos^2 x)$

$$y = \frac{1}{4}\sin^2 2x\left[\frac{2}{1 - \frac{1}{4}\sin^2 2x}\right]$$

***Sabemos que*** $\sin Bx$ ***tiene periodo igual a*** $\frac{2\pi}{|B|}$ ⇒***Entonces el periodo de*** $\sin 2x$ ***es*** $\pi$

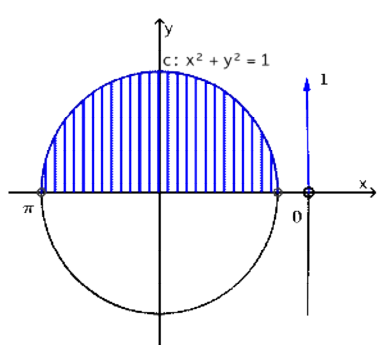



***Entonces*** $0 < \sin 2x \leq 1;\ 0 < \sin^2 2x \leq 1$

$$\Rightarrow 1 < \frac{2}{1 - \frac{1}{4}\sin^2 2x} \leq \frac{2}{1 - \frac{1}{4}} \text{ y } 0 < \frac{1}{4}\sin^2 2x \leq \frac{1}{4}$$

$$\Rightarrow 1 \cdot 0 < y \leq \frac{1}{4}\left(\frac{2}{1 - \frac{1}{4}}\right) = \frac{2}{3}$$

***Solucion 2:***



***Sea*** $y = \frac{1}{4}\sin^2 2x\left[\frac{2}{1-\frac{1}{4}\sin^2 2x}\right]; 0 < x < \frac{\pi}{2} \Rightarrow 0 < 2x < \pi$

***Entonces:*** $0 < \sin 2x \leq 1 \to 0 < \sin^2 2x \leq 1$. ***Dando*** $T = \sin^2 2x$

$$y = f(T) = \frac{T}{2-\frac{1}{2}T}, T \in (0,1]. \textit{ Pero } f'(T) = \frac{8}{(4-T)^2} > 0$$

***Por lo que*** $f(T)$ ***es estrateamente crecoente***

$$\therefore \Omega_1 = \inf\Omega = f(0) = 0 \; y \; \Omega_2 = \sup\Omega = f(1) = \frac{1}{2-\frac{1}{2}} = \frac{2}{3}$$

***Solution 2 by Gabriel Ruddy Cruz Mendez-Lima-Peru***

$$f_{(x)} = \frac{\sin^2 x}{\cos^2 x + \tan^2 x} + \frac{\cos^2 x}{\sin^2 x + \cot^2 x} \to$$

$$\to f_{(x)} = \frac{\frac{\sin^2 x}{\sin^2 x}}{\frac{\cos^2 x}{\sin^2 x} + \frac{\sin^2 x}{\cos^2 x}\frac{1}{\sin^2 x}} + \frac{\frac{\cos^2 x}{\cos^2 x}}{\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\sin^2 x}\frac{1}{\cos^2 x}}$$

$$\to f_{(x)} = \frac{1}{\cot^2 x + \sec^2 x} + \frac{1}{\tan^2 x + \csc^2 x} \to$$

$$\to f_{(x)} = \frac{1}{\cot^2 x + 1 + \tan^2 x} + \frac{1}{\tan^2 x + 1 + \cot^2 x}$$

$$f_{(x)} = \frac{2}{1 + \tan^2 x + \cot^2 x}; x \in < 0, \frac{\pi}{2} > \to \tan^2 x + \cot^2 x \in [2; +\infty[ \to$$

$$\to 1 + \tan^2 x + \cot^2 x \in [3; \; +\infty[$$

$$\frac{1}{1 + \tan^2 x + \cot^2 x} \in \left]0; \frac{1}{3}\right] \to f_{(x)} \in \left]0; \frac{2}{3}\right]$$


$$\inf\Omega = 0, \sup\Omega = \frac{3}{2}$$

**JP.258. If** $a, b, c > 0; abc = 1; n \in \mathbb{N}$ **then:**

$$\frac{1}{b^{n+3} + c^{n+3} + a^n} + \frac{1}{c^{n+3} + a^{n+3} + b^n} + \frac{1}{a^{n+3} + b^{n+3} + c^n} \leq 1$$

***Proposed by Marin Chirciu – Romania***

*Solution by Rahim Shahbazov-Baku-Azerbaijan*

$b^{n+3} + c^{n+3} \geq bc(b^{n+1} + c^{n+1})$ ***we have:***

$$\frac{1}{a^n + b^{n+3} + c^{n+3}} \leq \frac{1}{a^n + bc(b^{n+1} + c^{n+1})} = \frac{a}{a^{n+1} + b^{n+1} + c^{n+1}}$$

***We have:*** $LHS \leq \frac{a+b+c}{a^{n+1}+b^{n+1}+c^{n+1}} \leq 1 \Rightarrow a^{n+1} + b^{n+1} + c^{n+1} \geq a + b + c$

$$\left.\begin{matrix} a^{n+1} + n \geq (n+1)a \\ b^{n+1} + n \geq (n+1)b \\ c^{n+1} + n \geq (n+1)c \end{matrix}\right\} \Rightarrow a^{n+1} + b^{n+1} + c^{n+1} + 3n \geq$$

$$\geq a + b + c + n(a + b + c) \geq 3n + a + b + c$$

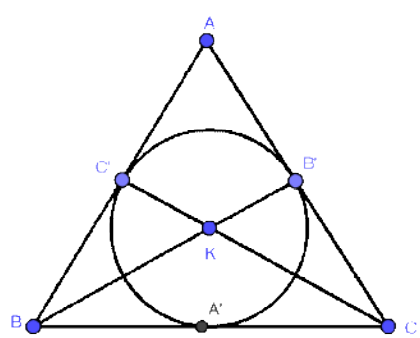
$$\Rightarrow a^{n+1} + b^{n+1} + c^{n+1} \geq a + b + c$$

**JP.259. Let be $\Delta ABC$ and $BB', CC'$ the symedians from $B, C$. If the circumcircle $\Delta AB'C'$ is tangent to $BC$ then the following inequality holds:**

$$\frac{b^2}{a^2 + c^2} + \frac{c^2}{c^2 + b^2} > \frac{1}{2}$$

***Proposed by Marian Ursărescu – Romania***

*Solution by proposer*




$$\begin{matrix} \rho(B) = BA'^2 = BC' \cdot BA \\ \rho(C) = CA'^2 = B'C \cdot CA \end{matrix} \Rightarrow$$

$$a = BA' + A'C = \sqrt{C \cdot BC'} + \sqrt{b \cdot B'C} \quad \textbf{(1)}$$

$$\frac{BC'}{C'A} = \frac{a^2}{b^2} \Rightarrow \frac{BC'}{c} = \frac{a^2}{a^2+b^2} \Rightarrow BC' = \frac{a^2c}{a^2+b^2} \quad \textbf{(2)}$$

$$\frac{B'C}{B'A} = \frac{a^2}{c^2} \Rightarrow \frac{B'C}{b} = \frac{a^2}{a^2+c^2} \Rightarrow B'C = \frac{a^2b}{a^2+c^2} \quad \textbf{(3)}$$

***From (1)+(2)+(3)***$\Rightarrow a = \sqrt{\frac{a^2c^2}{a^2+b^2}} + \sqrt{\frac{a^2b^2}{a^2+c^2}} \Leftrightarrow \frac{b}{\sqrt{a^2+c^2}} + \frac{c}{\sqrt{a^2+b^2}} = 1$ ***(4)***

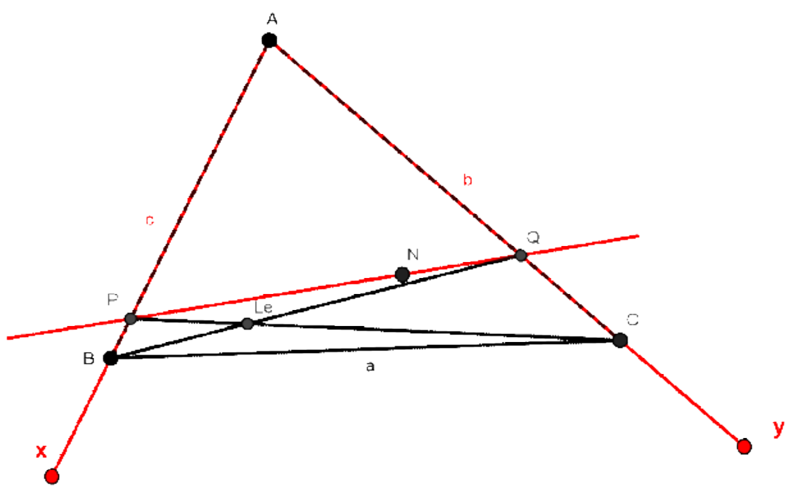
***But*** $2\left(\frac{b^2}{a^2+c^2} + \frac{c^2}{a^2+b^2}\right) \geq \left(\frac{b}{\sqrt{a^2+c^2}} + \frac{c}{\sqrt{a^2+b^2}}\right)^2 \Rightarrow \frac{b^2}{a^2+c^2} + \frac{c^2}{a^2+b^2} > \frac{1}{2}$

**JP.260. In $\Delta ABC$, $N$ – Nagel's point, $BQ$, $CP$ – symedians. Prove that:**

$$P, N, Q \text{ – collinears} \Leftrightarrow \frac{1}{b^2 r_b} + \frac{1}{c^2 r_c} = \frac{1}{a^2 r_a}$$

***Proposed by Marian Ursărescu – Romania***

***Solution by Thanasis Gakopoulos-Larisa-Greece***




***PLAGIOGONAL system:*** $AB \equiv Ax, AC \equiv Ay, A(0,0), B(c,0), C(0,b)$

$$P\left(\frac{b^2c}{a^2+b^2}, 0\right), Q\left(0, \frac{bc^2}{a^2+c^2}\right), N\left(\frac{a-b+c}{a+b+c}c, \frac{a+b-c}{a+b+c}b\right)$$

$$P, N, Q \text{ collinears} \Leftrightarrow \begin{vmatrix} 1 & 1 & 1 \\ \frac{b^2c}{a^2+b^2} & 0 & \frac{a-b+c}{a+b+c}c \\ 0 & \frac{bc^2}{a^2+c^2} & \frac{a+b-c}{a+b+c}b \end{vmatrix} = 0 \Leftrightarrow$$

$\Leftrightarrow a^3b^2 + a^3c^2 + a^2b^3 - a^2b^2c - a^2bc^2 + ab^2c^2 + a^2c^3 - b^3c^2 - b^3c^3 = 0$ ***(1)***

$$\frac{1}{b^2\sqrt{b}} + \frac{1}{c^2\sqrt{c}} - \frac{1}{a^2\sqrt{a}} = 0 \Leftrightarrow \frac{s-b}{b^2 \cdot S} + \frac{s-c}{c^2 \cdot S} - \frac{s-a}{a^2 \cdot S} = 0 \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{2S}\left(\frac{a-b+c}{b^2}+\frac{a+b-c}{c^2}-\frac{a+b-c}{a^2}\right)=0 \Leftrightarrow$$

$$\Leftrightarrow a^3b^2+a^3c^2+a^2b^2-a^2b^2c-a^2bc^2+ab^2c^2+a^2c^3-b^3c^2-b^2c^3=0 \text{ (2)}$$

$$\text{(1), (2)} \to P, N, Q \text{ collinears} \Leftrightarrow \frac{1}{b^2r_b}+\frac{1}{c^2r_c}=\frac{1}{a^2r_a}$$

**JP.261. If $a,b,c>0;\ a+b+c=1;\ n\ge 0$ then:**

$$\frac{a}{(n+bc)^2}+\frac{b}{(n+ca)^2}+\frac{c}{(n+ab)^2}\ge\frac{81}{(9n+1)^2}$$

***Proposed by Marin Chirciu – Romania***

***Solution 1 by Remus Florin Stanca-Romania***

$$\text{Let } f:\mathbb{R}_+^*\to\mathbb{R}_+^* \text{ such that } f(x)=\frac{1}{(n+x)^2},\ n\ge 0\Rightarrow f'(x)=-\frac{2}{(n+x)^3}\Rightarrow$$

$$\Rightarrow f''(x)=\frac{6}{(n+x)^4}>0\Rightarrow f \text{ is convex and}$$

$$a+b+c=1 \overset{Jensen}{\Rightarrow} af(bc)+bf(ac)+cf(ab)\ge f(3abc)\Rightarrow$$

$$\Rightarrow \frac{a}{(n+bc)^2}+\frac{b}{(n+ac)^2}+\frac{c}{(n+ab)^2}\ge\frac{1}{(n+3abc)^2} \text{ (1)}$$

$$a+b+c\ge 3\sqrt[3]{abc}\Rightarrow 3abc\le\frac{1}{9}\Rightarrow 3abc+n\le\frac{9n+1}{9}\Rightarrow(3abc+n)^2\le\frac{(9n+1)^2}{81}\Rightarrow$$

$$\Rightarrow\frac{1}{(3abc+n)^2}\ge\frac{81}{(9n+1)^2}\overset{(1)}{\Rightarrow}\frac{a}{(n+bc)^2}+\frac{b}{(n+ac)^2}+\frac{c}{(n+ab)^2}\ge\frac{81}{(9n+1)^2} \text{ (Q.E.D.)}$$

***Solution 2 by Soumava Chakraborty-Kolkata-India***

$$LHS=\sum\frac{a^3}{(na+abc)^2}\overset{Radon}{\ge}\frac{(\sum a)^3}{(n\sum a+3abc)^2}\overset{\because 1=\sum a}{=}\frac{(\sum a)^4}{(n\sum a+3abc)^2}\overset{?}{\ge}\frac{81}{(9n+1)^2}\Leftrightarrow$$

$$\Leftrightarrow\frac{(\sum a)^2}{n\sum a+3abc}\overset{?}{\ge}\frac{9}{9n+1}\overset{\because 1=\sum a}{\Leftrightarrow}\frac{(\sum a)^3}{n(\sum a)^3+3abc}\overset{?}{\ge}\frac{9}{9n+1}\Leftrightarrow$$

$$\Leftrightarrow(9n+1)(\sum a)^3\overset{?}{\ge}9n(\sum a)^3+27abc\Leftrightarrow(\sum a)^3\overset{?}{\ge}27abc\to \text{true by AM-GM}$$

$$\therefore\frac{a}{(n+bc)^2}+\frac{b}{(n+ca)^2}+\frac{c}{(n+ab)^2}\ge\frac{81}{(9n+1)^2}$$

**JP.262. If $a,b,c>0$; $a^{3n-1}+b^{3n-1}+c^{3n-1}=1$; $n\in\mathbb{N}$; $n\geq 1$ then:**

$$\frac{a^2}{b^{3n}}+\frac{b^2}{c^{3n}}+\frac{c^2}{a^{3n}}\geq (a^n+b^n+c^n)^3$$

***Proposed by Marin Chirciu – Romania***

***Solution 1 by Tran Hong-Dong Thap-Vietnam***

$$1=a^{3n-1}+b^{3n-1}+c^{3n-1}=\frac{a^{3n}}{a}+\frac{b^{3n}}{b}+\frac{c^{3n}}{c}\overset{Holder}{\geq}\frac{(a^n+b^n+c^n)^3}{3(a+b+c)}$$

$$\Rightarrow 3(a+b+c)\geq (a^n+b^n+c^n)^3$$

$$1=a^{3n-1}+b^{3n-1}+c^{3n-1}\geq\frac{(a+b+c)^{3n-1}}{3^{3n-2}}$$

$$\Rightarrow (a+b+c)^{3n-1}\leq 3^{3n-2}\Rightarrow (a+b+c)\leq\sqrt[3n-1]{3^{3n-2}}$$

$$\Rightarrow (a^n+b^n+c^n)^3\leq 3\cdot\sqrt[3n-1]{3^{3n-2}}\quad (**)$$

$$1=a^{3n-1}+b^{3n-1}+c^{3n-1}\geq 3\sqrt[3]{(abc)^{3n-1}}$$

$$\Rightarrow (abc)^{3n-1}\leq\frac{1}{3^3}\Rightarrow abc\leq\frac{1}{\sqrt[3n-1]{3^3}}\quad (*)$$

$$\frac{a^2}{b^{3n}}+\frac{b^2}{c^{3n}}+\frac{c^2}{a^{3n}}\overset{AM-GM}{\geq}3\sqrt[3]{\frac{(abc)^2}{(abc)^{3n}}}=3\cdot\frac{1}{\sqrt[3]{(abc)^{3n-2}}}$$

$$\overset{(*)}{\geq}3\sqrt[3]{\left(\sqrt[3n-1]{3^3}\right)^{3n-2}}=3\cdot\sqrt[3n-1]{3^{3n-2}}\overset{(**)}{\geq}(a^n+b^n+c^n)^3$$

***Solution 2 by Sanong Huayrerai -Nakon Pathom-Thailand***

***For*** $a,b,c>0, n\geq 1$ ***and*** $a^{3n-1}+b^{3n-1}+c^{3n-1}=1$

***We have:*** $1=3^{n-1}+b^{3n-1}+c^{3n-1}=\frac{a^{3n}}{a}+\frac{b^{3n}}{b}+\frac{c^{3n}}{c}\geq\frac{(a^n+b^n+c^n)^3}{3(a+b+c)}$

***Consider*** $\frac{a^2}{b^{3n}}+\frac{b^2}{c^{3m}}+\frac{c^2}{a^{3m}}=\frac{\frac{a^2}{b}}{b^{3n-1}}+\frac{\frac{b^2}{c}}{c^{3n-1}}+\frac{\frac{c^2}{a}}{a^{3n-1}}\geq\left(\frac{a}{\sqrt{b}}+\frac{b}{\sqrt{c}}+\frac{c}{\sqrt{a}}\right)^2\geq 3(a+b+c)$

***Iff*** $\left(\frac{(a+b+c)^2}{a\sqrt{b}+b\sqrt{c}+c\sqrt{a}}\right)^2\geq 3(a+b+c)$

***Iff*** $(a+b+c)^3\geq 3\left(a\sqrt{b}+b\sqrt{c}+c\sqrt{a}\right)^2$

***Iff*** $(a+b+c)(ab+bc+ca)\geq\left(a\sqrt{b}+b\sqrt{c}+c\sqrt{a}\right)^2$

*Iff* $a^2b+b^2c+c^2a+a^2c+c^2b+b^2a+3abc\geq$

$\geq a^2b+b^2c+c^2a+2\left(ab\sqrt{bc}+bc\sqrt{ac}+ac\sqrt{ab}\right)$. *Ok*

*Because* $a^2c+c^2b+b^2a+3abc\geq 2\left(ab\sqrt{bc}+bc\sqrt{ac}+ac\sqrt{ab}\right)$

*Therefore, it is to be true.*

**JP.263. If $m\geq 1; a,b,c,x,y,z>0$ then:**

$$\frac{x^{3m}}{(yz)^m(ay^m+bz^m)}+\frac{y^{3m}}{(zx)^m(az^m+bx^m)}+\frac{z^{3m}}{(xy)^m(ax^m+by^m)}\geq\frac{3}{a+b}$$

*Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania*

*Solution 1 by Tran Hong-Dong Thap-Vietnam*

*Let* $u=x^m; v=y^m; w=z^m$ $(u,v,w>0)$

*Inequality* $\Leftrightarrow \frac{u^3}{vw(av+bw)}+\frac{v^3}{uw(aw+bu)}+\frac{w^3}{uv(au+bv)}\geq\frac{3}{a+b}$

$$\Leftrightarrow \frac{\left(\frac{u}{\sqrt[3]{vw}}\right)^3}{av+bw}+\frac{\left(\frac{v}{\sqrt[3]{uw}}\right)^3}{aw+bu}+\frac{\left(\frac{w}{\sqrt[3]{uv}}\right)^3}{au+bv}\geq\frac{3}{a+b}\quad (*)$$

$$\because LHS_{(*)}\overset{Holder}{\geq}\frac{\left(\frac{u}{\sqrt[3]{vw}}+\frac{v}{\sqrt[3]{uw}}+\frac{w}{\sqrt[3]{uv}}\right)^3}{3(a+b)(u+v+w)}$$

*We must show that:* $\left(\frac{u}{\sqrt[3]{vw}}+\frac{v}{\sqrt[3]{uw}}+\frac{w}{\sqrt[3]{uv}}\right)^3\geq 9(u+v+w)$

$$\Leftrightarrow\left(\frac{u\sqrt[3]{u}+v\sqrt[3]{v}+w\sqrt[3]{w}}{\sqrt[3]{uvw}}\right)^3\geq 9(u+v+w)$$

$$\Leftrightarrow\left(u\sqrt[3]{u}+u\sqrt[3]{u}+w\sqrt[3]{w}\right)^3\geq 9uvw(u+v+w)$$

$\therefore$ *Let* $X=\sqrt[3]{u}; Y=\sqrt[3]{v}; Z=\sqrt[3]{w}$

$$\Leftrightarrow (X^4+Y^4+Z^4)^3\geq 9(XYZ)^3(X^3+Y^3+Z^3)$$

$$\because X^4+Y^4+Z^4\overset{Chebyshev}{\geq}\frac{1}{3}(X+Y+Z)(X^3+Y^3+Z^3)$$

*Hence, we must show that:* $\frac{(X+Y+Z)^3}{27}(X^3+Y^3+Z^3)^3\geq 9(XYZ)^3(X^3+Y^3+Z^3)$ ***(**)***

$$\therefore\frac{(X+Y+Z)^3}{27}\overset{AM-GM}{\geq}XYZ\therefore (X^3+Y^3+Z^3)^2\overset{AM-GM}{\geq}\left(3\sqrt[3]{X^3Y^3Z^3}\right)^2=9(XYZ)^2$$

$\Rightarrow LHS_{(**)}\geq 9(XYZ)^3(X^3+Y^3+Z^3)\Rightarrow$ ***(*) true. Proved.***

**Solution 2 by Avishek Mitra-West Bengal-India**

$$\Leftrightarrow \Omega = \sum \frac{x^{3m}}{(yz)^m(ay^m + bz^m)} = \sum \frac{(x^{2m})^2}{(xyz)^m(ay^m + bz^m)} \overset{Bergstrom}{\geq}$$

$$\geq \frac{(\sum x^{2m})^2}{(xyz)^m(a\sum x^m + b\sum x^m)} = \frac{(x^{2m} + y^{2m} + y^{2m})^2}{(xyz)^m(x^m + y^m + z^m)(a + b)}$$

$\Leftrightarrow$ ***Need to show*** $\Rightarrow (\sum x^{2m})^2 \geq 3(xyz)^m \sum x^m$

$$\Leftrightarrow (1^2 + 1^2 + 1^2)(x^{2m} + y^{2m} + z^{2m}) \overset{CBS}{\geq} (x^m + y^m + z^m)^2$$

$$\Rightarrow (\sum x^{2m}) \geq \frac{(\sum x^m)^2}{3} \quad (i)$$

$$\Leftrightarrow \sum x^{2m} \overset{AM-GM}{\geq} 3(xyz)^{\frac{2m}{3}} \quad (ii)$$

$$\Leftrightarrow \sum x^m \overset{AM-GM}{\geq} 3(xyz)^{\frac{m}{3}} \quad (iii)$$

$$\Rightarrow i \times ii \times iii \ldots \rightarrow$$

$$\Leftrightarrow \left(\sum x^{2m}\right)^2 \left(\sum x^m\right) \geq 3(xyz)^{\frac{2m}{3}+\frac{m}{3}}\left(\sum x^{2m}\right)^2$$

$\Rightarrow (\sum x^{2m})^2 \geq 3(xyz)^m(\sum x^m)$ ***(*true). proved***

**Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand**

***For*** $m \geq 1, a, b, x, y, z > 0$ ***we have***

$$\frac{x^{3m}}{(yz)^m(ay^m + bz^m)} + \frac{y^{3m}}{(zx)^m(az^m + bx^m)} + \frac{z^{3m}}{(xy)^m(zx^m + by)^m} \geq$$

$$\geq \frac{(x^{2m} + y^{2m} + z^{2m})^2}{(xzy)^m(a + b)(x^m + y^m + z^m)} \geq \frac{\left[\left(\frac{x^m + y^m + z^m}{3}\right)^2\right]^2}{(x + y)^m(a + b)(x^m + y^m + z^m)} \geq \frac{3}{(a + b)}$$

***Iff*** $\frac{(x^m+y^m+z^m)^3}{(xyz)^m} \geq 2y$. ***And it is true. Therefore, it is true.***

**JP.264. If $x, y, z \in \left(0, \frac{\pi}{2}\right)$ then in $\Delta ABC$ the following relationship holds:**

$$a^2\tan x + b^2\tan y + c^2\tan z > 4F\sqrt{xy + yz + zx}$$

***Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru-Romania***

***Solution by Marian Ursărescu-Romania***

***We use Oppenheimer-Klamkin's inequality:***

$\forall x,y,z>0$ ***then*** $a^2m+b^2n+c^2p\geq 4F\sqrt{mn+np+pq}$ ***(1)***

***In (1) let:*** $m=\tan x; n=\tan y; p=\tan z$

$$a^2\tan x+b^2\tan y+c^2\tan z\geq 4F\sqrt{\tan x\cdot\tan y+\tan y\cdot\tan z+\tan z\cdot\tan x}\quad (2)$$

***But:*** $\tan x>x; \tan y>y; \tan z>z, x,y,z\in\left(0,\frac{\pi}{2}\right)$ ***(3)***

***From (2)+(3) we have:*** $a^2\tan x+b^2\tan y+c^2\tan z>4F\sqrt{xy+yz+zx}$

**JP.265. In $\Delta ABC$ the following relationship holds:**

$$\frac{6}{5}-\frac{9r}{10R}\leq\left(\frac{a}{b+c}\right)^2+\left(\frac{b}{c+a}\right)^2+\left(\frac{c}{a+b}\right)^2\leq\frac{3R}{8r}$$

***By Marin Chirciu – Romania***

***Solution 1 by proposer***

***We prove the following lemma:***

***In $\Delta ABC$ the following relationship holds:***

$$\sum\frac{a^2}{(b+c)^2}=\frac{2[s^4-s^2(4Rr+6r^2)+r^2(6R^2+4Rr+r^2)]}{(s^2+r^2+2Rr)^2}$$

***Demonstration.***

***We have*** $\sum\left(\frac{a}{b+c}\right)^2=\frac{\sum a^2(a+b)^2(a+c)^2}{\prod(b+c)^2}$.

***Using*** $\sum a^2(a+b)^2(a+c)^2=8s^2[s^4-s^2(4Rr+6r^2)+r^2(6R^2+Rr+r^2)]$ ***and*** $\prod(b+c)=2s(s^2+r^2+2Rr)$ ***we obtain:***

$$\sum\left(\frac{a}{b+c}\right)^2=\frac{2[s^4-s^2(4Rr+6r^2)+r^2(6R^2+4Rr+r^2)]}{(s^2+r^2+2Rr)^2}$$

***The left hand inequality. Using the Lemma the inequality can be written:***

$$\frac{2[s^4-s^2(4Rr+6r^2)+r^2(6R^2+4Rr+r^2)]}{(s^2+r^2+2Rr)^2}\geq\frac{6}{5}-\frac{9r}{10R}\Leftrightarrow$$

$$\Leftrightarrow 20R[s^4-s^2(4Rr+6r^2)+r^2(6R^2+4Rr+r^2)]\geq(12R-9r)(s^2+r^2+2Rr)^2\Leftrightarrow$$

$$\Leftrightarrow s^2[(8R+9r)s^2-r(128R^2+108Rr-18r^2)]+r^2(72R^3+68R^2r+44Rr^2+9r^3)\geq 0$$

***We distinguish the following cases:***

***Case 1). If*** $[(8R+9r)s^2-r(128R^2+108Rr-18r^2)]\geq 0$***, the inequality is obviously.***

***Case 2). If*** $[(8R+9r)s^2-r(128R^2+108Rr-18r^2)]<0$***, the inequality can be written:***

$$r^2(72R^3+68R^2r+44Rr^2+9r^3)\geq s^2[r(128R^2+108Rr-18r^2)-(8R+9r)s^2],$$

***which follows from Gerretsen's inequality*** $16Rr-5r^2\leq s^2\leq 4R^2+4Rr+3r^2$.

***It remains to prove that:*** $r^2(72R^3+68R^2r+44Rr^2+9r^3)\geq$

$$\geq(4R^2+4Rr+3r^2)[r(128R^2+108Rr-18r^2)-(8R+9r)(16Rr-5r^2)]$$

$$\Leftrightarrow 72R^3+68R^2r+44Rr^2+9r^3\geq(4R^2+4Rr+3r^2)(4R+27r)\Leftrightarrow$$

$$\Leftrightarrow 14R^3-14R^2r-19Rr^2-18r^3\geq 0\Leftrightarrow(R-2r)(14R^2+14Rr+9r^2)\geq 0$$

***Obviously from Euler's inequality*** $R\geq 2r$.

***The right hand inequality. Using Lemma the inequality can be written:***

$$\frac{2[s^4-s^2(4Rr+6r^2)+r^2(6R^2+4Rr+r^2)]}{(s^2+r^2+2Rr)^2}\leq\frac{3R}{8r}\Leftrightarrow$$

$$s^2[s^2(3R-16r)+r(12R^2+70Rr+96r^2)]+$$

$$+r^2(12R^3-84R^2r-61Rr^2-16r^3)\geq 0$$

***We distinguish the following cases:***

***Case 1). If*** $(3R-16r)\geq 0$***, using Gerretsen's inequality*** $s^2\geq 16Rr-5r^2$.

***It remains to prove that:***

$$(16Rr-5r^2)[(16Rr-5r^2)(3R-16r)+r(12R^2+70Rr+96r^2)]+$$

$$+r^2(12R^3-84R^2r-61Rr^2-16r^3)\geq 0$$

$$\Leftrightarrow 243R^3-900R^2r+940Rr^2-244r^3\geq 0\Leftrightarrow$$

$$\Leftrightarrow(R-2r)(243R^2-414Rr+112r^2)\geq 0$$

***Obviously from Euler's inequality*** $R\geq 2r$

***Case 2) If*** $(3R-16r)<0$ ***the inequality can be written:***

$$r^2(12R^3-84R^2r-61Rr^2-16r^3)\geq s^2[s^2(16r-3R)-r(12R^2+70Rr+96r^2)]$$

***Which follows from Gerretsen's inequality*** $s^2\leq 4R^2+4Rr+3r^2$

***It remains to prove that:*** $r^2(12R^3-84R^2r-61Rr^2-16r^3)\geq$

$$\geq(4R^2+4Rr+3r^2)[(4R^2+4Rr+3r^2)(16r-3r)-r(12R^2+70Rr+96r^2)]\Leftrightarrow$$

$$12R^5-28R^4r-22R^3r^2+21R^2r^3+44Rr^4+32r^5\ge 0\Leftrightarrow$$

$\Leftrightarrow (R-2r)^2(12R^3+20R^2r+19Rr^2+8r^3)\ge 0$, ***obviously with equality if*** $R=2r$.

***Equality holds if and only if the triangle is equilateral.***

***Solution 2 by Soumava Chakraborty-Kolkata-India***

$$\sum\frac{a^2}{(b+c)^2}=\sum\frac{\left(2s-(b+c)\right)^2}{(b+c)^2}=\sum\frac{4s^2-4s(b+c)+(b+c)^2}{(b+c)^2}\overset{(i)}{=}$$

$$4s^2\left[\frac{\sum\{(c+a)^2(a+b)^2\}}{\{\prod(b+c)\}^2}\right]-4s\left[\frac{\sum(c+a)(a+b)}{\prod(b+c)}\right]+3$$

$$\sum\{(c+a)^2(a+b)^2\}=\sum\left(\sum ab+a^2\right)^2=\sum\left\{\left(\sum ab\right)^2+2a^2\sum ab+a^4\right\}=$$

$$=3\left(\sum ab\right)^2+2\left(\sum ab\right)(\sum a^2)+(\sum a^2)^2-2\sum a^2b^2$$

$$=\left(\sum ab\right)^2+2\left(\sum ab\right)(\sum a^2)+(\sum a^2)^2+2\sum a^2b^2+4abc(2s)-2\sum a^2b^2=$$

$$=\left(\sum ab+\sum a^2\right)^2+32Rrs^2=(3s^2-4Rr-r^2)^2+32Rrs^2$$

$$\therefore\sum\{(c+a)^2(a+b)^2\}\overset{(ii)}{=}(3s^2-4Rr-r^2)^2+32Rrs^2$$

$$Again,\sum(c+a)(a+b)=\sum\left(\sum ab+a^2\right)=$$

$$=3\sum ab+\sum a^2=\sum a^2+2\sum ab+\sum ab=4s^2+s^2+4Rr+r^2$$

$$\therefore\sum(c+a)(a+b)\overset{(iii)}{=}5s^2+4Rr+r^2$$

$$\because\prod(b+c)=s^2+2Rr+r^2\ \therefore (i),(ii),(iii)\Rightarrow$$

$$\Rightarrow\sum\frac{a^2}{(b+c)^2}=\frac{4s^2\{(3s^2-4Rr-r^2)^2+32Rrs^2\}}{4s^2(s^2+2Rr+r^2)^2}-\frac{4s(5s^2+4Rr+r^2)}{2s(s^2+2Rr+r^2)}+3$$

$$=\frac{(3s^2-4Rr-r^2)^2+32Rrs^2-2(5s^2+4Rr+r^2)(s^2+2Rr+r^2)+3(s^2+2Rr+r^2)^2}{(s^2+2Rr+r^2)^2}$$

$$\overset{(p)}{=}\frac{2s^4-s^2(8Rr+12r^2)+12R^2r^2+8Rr^3+2r^4}{(s^2+2Rr+r^2)^2}\le\frac{3R}{8r}$$

$$\Leftrightarrow 3R(s^2+2Rr+r^2)^2-8r\{2s^4-s^2(8Rr+12r^2)+12R^2r^2+8Rr^3+2r^4\}\ge 0$$

$$\Leftrightarrow (3R-16r)s^4+s^2(12R^2r+70Rr^2+96r^3)+12R^3r^2-84R^2r^3-61Rr^4-16r^5 \overset{(1)}{\gtrless} 0$$

**Case 1** $3R-16r\geq 0$

$$\therefore LHS\ of\ (1) \overset{Gerretsen}{\gtrless} (3R-16r)(16Rr-5r^2)s^2+$$

$$+s^2(12R^2r+70Rr^2+96r^3)+12R^3r^2-84R^2r^3-61Rr^4-16r^5$$

$$=rs^2(60R^2-201Rr+176r^2)+12R^3r^2-84R^2r^3-61Rr^4-16r^5 \overset{?}{>} 0$$

$$\Leftrightarrow s^2(60R^2-201Rr+176r^2)+12R^3r-84R^2r^2-61Rr^3-16r^4 \underset{(2)}{\overset{?}{>}} 0$$

$$\because 60R^2-201Rr+176r^2>0 \therefore LHS\ of\ (2) \overset{Gerretsen}{\gtrless}$$

$$\geq (16Rr-5r^2)(60R^2-201Rr+176r^2)+12R^3r-84R^2r^2-61Rr^3-16r^4 \overset{?}{>} 0$$

$$\Leftrightarrow 243t^3-900t^2+940t-224 \overset{?}{>} 0 \Leftrightarrow$$

$$\Leftrightarrow (t-2)\{(t-2)(243t+72)+256\} \overset{?}{>} 0 \rightarrow true \because t\geq \frac{16}{3}>2$$

$$\Rightarrow (2)\Rightarrow (1) is\ true\ (strict\ inequality) \therefore \left(\frac{a}{b+c}\right)^2+\left(\frac{b}{c+a}\right)^2+\left(\frac{c}{a+b}\right)^2<\frac{3R}{8r}$$

**Case 2** $3R-16r<0$

$$Now, (1)\Leftrightarrow (16r-3R)s^4-s^2(12R^2r+70Rr^2+96r^3)-12R^3r^2+$$

$$+84R^2r^3+61Rr^4+16r^5 \overset{(1a)}{\lesseqgtr} 0$$

$$Now, Rouche \Rightarrow s^2-(m-n)\geq 0\ and\ s^2-(m+n)\leq 0,$$

$$where\ m=2R^2+10Rr-r^2\ and\ n=2(R-2r)\sqrt{R^2-2Rr}$$

$$\therefore \left(s^2-(m+n)\right)\left(s^2-(m-n)\right)\leq 0 \Rightarrow s^4-s^2(2m)+m^2-n^2\leq 0$$

$$\Rightarrow s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3\leq 0 \Rightarrow$$

$$\Rightarrow (16r-3R)s^4-s^2(16r-3R)(4R^2+20Rr-2r^2)+r(16r-3R)(4R+r)^3 \overset{(iv)}{\lesseqgtr} 0$$

$$(iv)\Rightarrow in\ order\ to\ prove\ (1a), it\ suffices\ to\ prove:$$

$$(16r-3R)s^4-s^2(12R^2r+70Rr^2+96r^3)-12R^3r^2+84R^2r^3+61Rr^4+16r^5$$

$$\leq (16r-3R)s^4-s^2(16r-3R)(4R^2+20Rr-2r^2)+r(16r-3R)(4R+r)^3$$

$$\Leftrightarrow s^2[(3R-16r)(4R^2+20Rr-2r^2)+12R^2r+70Rr^2+96r^3]+$$

$$+12R^3r^2-84R^2r^3-61Rr^4-16r^5-r(3R-16r)(4R+r)^3\geq 0$$

$$\Leftrightarrow s^2(12R^3+8R^2r-256Rr^2+128r^3)+12R^3r^2-84R^2r^3-$$

$$-61Rr^4-16r^5-r(3R-16r)(4R+r)^3\geq 0$$

$$\Leftrightarrow s^2(12R^3+8R^2r-176Rr^2+224r^3)+12R^3r^2-84R^2r^3-61Rr^4-16r^5-$$

$$-r(3R-16r)(4R+r)^3 \overset{(3)}{\overset{?}{\geq}} (80Rr^2+96r^3)s^2$$

$$Now, LHS\ of\ (3) \underbrace{\overset{Gerretsen}{\geq}}_{(a)}$$

$$(16Rr-5r^2)(12R^3+8R^2r-176Rr^2+224r^3)+12R^3r^2-84R^2r^3-61Rr^4-$$

$$-16r^5-r(3R-16r)(4R+r)^3$$

$$and\ RHS\ of\ (3) \underbrace{\overset{Gerretsen}{\leq}}_{(b)} (80Rr^2+96r^3)(4R^2+4Rr+3r^2)$$

$$(a),(b)\Rightarrow in\ order\ to\ prove\ (3), it\ suffices\ to\ prove:$$

$$(16Rr-5r^2)(12R^3+8R^2r-176Rr^2+224r^3)+12R^3r^2-84R^2r^3-$$

$$-61Rr^4-16r^5-r(3R-16r)(4R+r)^3$$

$$\geq (80Rr^2+96r^3)(4R^2+4Rr+3r^2)\Leftrightarrow 20t^3-91t^2+124t-44\geq 0\Leftrightarrow$$

$$(20t-11)(t-2)^2\geq 0\rightarrow true \because 2 \overset{Euler}{\leq} t<\frac{16}{3}$$

$$\Rightarrow (3)\Rightarrow (1a)\Rightarrow (1) is\ true \therefore \left(\frac{a}{b+c}\right)^2+\left(\frac{b}{c+a}\right)^2+\left(\frac{c}{a+b}\right)^2\leq\frac{3R}{8r}$$

$$Combining\ both\ cases, in\ any\ \Delta ABC, \left(\frac{a}{b+c}\right)^2+\left(\frac{b}{c+a}\right)^2+\left(\frac{c}{a+b}\right)^2 \overset{(m)}{\leq}\frac{3R}{8r}$$

$$Again,(p)\Rightarrow\sum\frac{a^2}{(b+c)^2}=\frac{2s^4-s^2(8Rr+12r^2)+12R^2r^2+8Rr^3+2r^4}{(s^2+2Rr+r^2)^2}\overset{?}{\geq}$$

$$\geq \frac{6}{5}-\frac{9r}{10R}=\frac{12R-9r}{10R}$$

$$\Leftrightarrow 10R[2s^4-s^2(8Rr+12r^2)+12R^2r^2+8Rr^3+2r^4]\overset{?}{\geq}$$

$$\geq (12R-9r)(s^2+2Rr+r^2)^2$$

$$\Leftrightarrow s^4(8R+9r)-s^2(128R^2r+108Rr^2-18r^3)+72R^3r^2+68R^2r^3+$$

$$+44Rr^4+9r^5 \underbrace{\overset{?}{\geq}}_{(4)} 0$$

$$Now, LHS\ of\ (4) \overset{Gerretsen}{\geq} s^2(8R+9r)(16Rr-5r^2)-$$

$$-s^2(128R^2r+108Rr^2-18r^3)+72R^3r^2+68R^2r^3+44Rr^4+9r^5 \overset{?}{\geq} 0$$

$$\Leftrightarrow 72R^3+68R^2r+44Rr^2+9r^3 \underbrace{\overset{?}{\geq}}_{(5)} (4R+27r)s^2$$

$$Again, RHS\ of\ (5) \overset{Gerretsen}{\leq} (4R+27r)(4R^2+4Rr+3r^2) \overset{?}{\leq}$$

$$\leq 72R^3+68R^2r+44Rr^2+9r^3 \Leftrightarrow 14t^3-14t^2+124t-44 \overset{?}{\geq} 0$$

$$\Leftrightarrow (t-2)(14t^2+14t+9) \overset{?}{\geq} 0 \to true \because t \overset{Euler}{\geq} 2 \Rightarrow$$

$$\Rightarrow (5) \Rightarrow (4) is\ true \therefore \sum \frac{a^2}{(b+c)^2} \overset{(n)}{\geq} \frac{6}{5}-\frac{9r}{10R}$$

$$(m),(n) \Rightarrow \frac{6}{5}-\frac{9r}{10R} \leq \left(\frac{a}{b+c}\right)^2+\left(\frac{b}{c+a}\right)^2+\left(\frac{c}{a+b}\right)^2 \leq \frac{3R}{8r}\ (Proved)$$

**JP.266. If $a, b, c > 0$ then:**

$$a\sqrt{a^2+2(b+c)^2}+b\sqrt{b^2+2(c+a)^2}+c\sqrt{c^2+2(a+b)^2} \leq (a+b+c)^2$$

***Proposed by Nguyen Viet Hung – Hanoi – Vietnam***

***Solution 1 by Rahim Shahbazov-Baku-Azerbaijan***

$$\sqrt{ax}+\sqrt{by}+\sqrt{cz} \leq \sqrt{(a+b+c)(x+y+z)} \quad \textit{AM-GM}$$

***If we use:*** $LHS=\sum a\sqrt{a^2+2(b+c)^2}=\sum \sqrt{a}\cdot\sqrt{a^3+2a(b+c)^2}$

$$\leq \sqrt{(a+b+c)\cdot\left(a^3+b^3+c^3+2\sum a(b+c)^2\right)} \leq (a+b+c)^2$$

$$\Rightarrow (a+b+c)^3 \geq a^3+b^3+c^3+2\sum a(b+c)^2 \Rightarrow (a+b)(b+c)(a+c) \geq 8abc$$

$$a+b \geq 2\sqrt{ab}$$

***Solution 2 by Michael Sterghiou-Greece***

$$\sum_{cyc} a\sqrt{a^2+2(b+c)^2} \leq (a+b+c)^2 \quad \textit{(1)}$$

*$f(t)=\sqrt{t}$ is concave on $(0,+\infty)$ so by generalized Jensen*

*with "weights" $(a,b,c)$ we have $LHS\ (1)\le\sum_{cyc}a\cdot\sqrt{\frac{\sum_{cyc}a(a^2+2(b+c)^2)}{\sum_{cyc}a}}$*

*which suffices to $bc\le(a+b+c)^2$. Some computation gives*

$$2\sum_{cyc}(ab^2+ac^2+2abc)\le\left(\sum_{cyc}a\right)^3-\sum_{cyc}a^3 \text{ or } \sum a^2b+a^2c\ge 6abc\quad (2)$$

*considering that $\sum_{cyc}(a^2b+a^2c)=\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)-3abc$. But (2)*

*is obvious by AM-GM $\left[LHS\ (2)\ge 6\sqrt[6]{a^6b^6c^6}=6abc\right]$. Done!*

**JP.267. In $\Delta ABC$ the following relationship holds:**

$$\frac{a^2}{m_b^2+m_c^2}+\frac{b^2}{m_c^2+m_a^2}+\frac{c^2}{m_a^2+m_b^2}\le 2$$

***Proposed by Nguyen Viet Hung-Hanoi-Vietnam***

***Solution 1 by Daniel Văcaru-Romania***

*We know $m_a\ge\sqrt{s(s-a)}\Rightarrow m_a^2\ge s(s-a)\Rightarrow m_a^2+m_b^2\ge s(2s-a-b)=s\cdot c$*

$$\frac{a^2}{m_b^2+m_c^2}\le\frac{a^2}{sa}=\frac{a}{s}\Rightarrow\sum_{cyc}\frac{a^2}{m_b^2+m_c^2}\le\frac{a+b+c}{s}=2$$

***Solution 2 by Tran Hong-Dong Thap-Vietnam***

$$m_a\ge\sqrt{s(s-a)}\Rightarrow m_a^2\ge s(s-a)$$

$$m_b\ge\sqrt{s(s-b)}\Rightarrow m_b^2\ge s(s-b)$$

$$m_c\ge\sqrt{s(s-c)}\Rightarrow m_c^2\ge s(s-c)$$

$$\Rightarrow\frac{a^2}{m_b^2+m_c^2}+\frac{b^2}{m_c^2+m_a^2}+\frac{c^2}{m_a^2+m_b^2}\le\frac{a^2}{s(2s-(b+c))}+\frac{b^2}{s(2s-(c+a))}+\frac{c^2}{s(2s-(a+b))}$$

$$=\frac{a^2}{s\cdot a}+\frac{b^2}{s\cdot b}+\frac{c^2}{s\cdot c}=\frac{a+b+c}{s}=\frac{2s}{s}=2$$

**JP.268. Let $A'B'C'$ be the intouch triangle of $\Delta ABC$. Prove that:**

$$A'B'+B'C'+C'A'\le\frac{\sqrt{ab}+\sqrt{bc}+\sqrt{ca}}{2}<s$$

***Proposed by Marian Ursărescu-Romania***

***Solution by Tran Hong-Dong Thap-Vietnam***

In $\Delta A'B'C'$: $B'C'^2 = (s-a)^2 + (s-a)^2 - 2(s-a)^2 cosA$

$$= 2(s-a)^2 - 2(s-a)^2 cosA$$

$= 2(s-a)^2(1-cosA) = 4(s-a)^2 sin^2\frac{A}{2}$ then $B'C' = 2(s-a)sin\frac{A}{2}$

Similary: $A'B' = 2(s-c)sin\frac{C}{2}$; $A'C' = 2(s-b)sin\frac{B}{2}$ then

$$A'B' + B'C' + C'A' = 2\left((s-a)sin\frac{A}{2} + (s-c)sin\frac{C}{2} + (s-b)sin\frac{B}{2}\right)$$

$$= 2\left(\frac{r}{tan\frac{A}{2}}\cdot sin\frac{A}{2} + \frac{r}{tan\frac{B}{2}}\cdot sin\frac{B}{2} + \frac{r}{tan\frac{C}{2}}\cdot sin\frac{C}{2}\right)$$

$$= 2r\left(cos\frac{A}{2} + cos\frac{B}{2} + cos\frac{C}{2}\right) \overset{Jensen}{\leq} 2r\cdot\frac{3\sqrt{3}}{2} = 3\sqrt{3}\cdot r$$

$$\frac{\sqrt{ab}+\sqrt{bc}+\sqrt{ca}}{2} \overset{Am-G}{\geq} \frac{3\sqrt[3]{3}}{2} = \frac{3\sqrt[3]{4Rrs}}{2}$$

So, we need to prove: $\frac{3\sqrt[3]{4Rrs}}{2} \geq 3\sqrt{3}\cdot r \Leftrightarrow \sqrt[3]{4Rrs} \geq \sqrt{3}\cdot r \Leftrightarrow Rs \geq 6\sqrt{3}\cdot r^2$

Which is true because $\begin{cases} R \geq 2r \\ s \geq 3\sqrt{3}\cdot r \end{cases} \Rightarrow Rs \geq 6\sqrt{3}\cdot r^2$

Hence $A'B' + B'C' + C'A' \leq \frac{\sqrt{ab}+\sqrt{bc}+\sqrt{ca}}{2}$

Lastly, using inequality: $X^2 + Y^2 + Z^2 \geq XY + YZ + ZX$

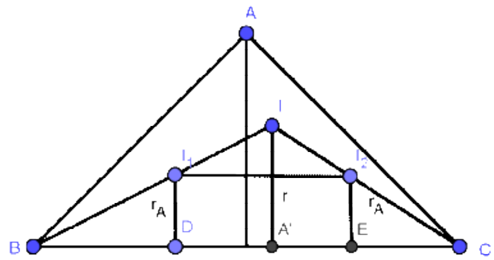
Choose $X = \sqrt{a}, Y = \sqrt{b}, Z = \sqrt{c} \Rightarrow \frac{\sqrt{ab}+\sqrt{bc}+\sqrt{ca}}{2} \leq \frac{a+b+c}{2} = \frac{2s}{s} = s$.Proved.

**JP.269 Let $\Delta ABC$ be a triangle. Let be $A' \in (BC)$ such that the incircle in $\Delta AA'B \wedge \Delta AA'C$ have the same radius. Analogous we obtain the points $B' \in (AC), C' \in (AB)$. Prove that: $AA' + BB' + CC' \geq 9r$.**

*Proposed by Marian Ursărescu – Romania*

*Solution by proposer*

Let be $r_a$ = the radixes of incircles with $\Delta ABA', ACA'$.




$$S = S_{ABA'} + S_{ACA'} = s_{ABA'} \cdot r_A + s_{ACA'} \cdot r_A = r_A(s_{ABA'} + s_{ACA'}) = r_A(s + AA') \text{ (1)}$$

$$\Delta II_1I_2 \sim \Delta IBC \Rightarrow \frac{I_1I_2}{BC} = \frac{r - r_A}{r} \Rightarrow 1 - \frac{r_A}{r} = \frac{I_1I_2}{BC} \Rightarrow \frac{r_A}{r} = 1 - \frac{I_1I_2}{a} \text{ (2)}$$

***$I_1I_2EO$ rectangle***

$$\Rightarrow I_1I_2 = DE = DA' + A'E = s_{ACA'} - c + s_{ACA'} - b = s - b - c + AA' \text{ (3)}$$

**From (2)+(3)**$\Rightarrow \frac{r_A}{r} = 1 - \frac{s-b-c+AA'}{a} = \frac{s-AA'}{a} \Rightarrow r_a = \frac{r}{a}(s - AA')$ **(4)**

**From (1)+(4)**$\Rightarrow \frac{r}{a}(s - AA')(s + AA') = S \Rightarrow s^2 - AA'^2 = as \Rightarrow$

$AA'^2 = s^2 - sa \Rightarrow AA' = \sqrt{s(s-a)}$. **Analogous** $BB' = \sqrt{s(s-b)}; CC' = \sqrt{s(s-b)}$

$$\Rightarrow AA' + BB' + CC' = \sqrt{s}\left(\sqrt{s-a} + \sqrt{s-b} + \sqrt{s-c}\right)$$

$$AA' + BB' + CC' \geq 3\sqrt[3]{AA' \cdot BB' \cdot CC'} = 3\sqrt[3]{sS} = 3\sqrt[3]{s^2 r} \geq 3\sqrt[3]{2 + r^3} = 9r.$$

**JP.270. If $m, n > 0$ then in $\Delta ABC$ the following relationship holds:**

$$\frac{r_a^2 + r_b r_c}{nr_b + mr_c} + \frac{r_b^2 + r_c r_a}{nr_c + mr_a} + \frac{r_c^2 + r_a r_b}{nr_a + mr_b} \geq \frac{18r}{m+n}$$

***Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania***

***Solution 1 by Adrian Popa-Romania***

$$\frac{r_a^2 + r_b r_c}{nr_b + mr_c} + \frac{r_b^2 + r_c r_a}{nr_c + mr_a} + \frac{r_c^2 + r_a r_b}{nr_a + mr_b} \geq \frac{18r}{m+n}$$

$$\therefore \frac{r_a^2}{nr_b + mr_c} + \frac{r_b^2}{nr_c + mr_a} + \frac{r_c^2}{nr_a + mr_b} \overset{J.Bergstrom}{\geq} \frac{(r_a + r_b + r_c)^2}{(r_a + r_b + r_c)(m+n)} =$$

$$= \frac{r_a + r_b + r_c}{m+n} = \frac{4R + r}{m+n} \therefore \text{(1)}$$

$$\therefore \frac{r_b r_c}{nr_b + mr_c} + \frac{r_a r_c}{nr_c + mr_a} + \frac{r_a r_b}{nr_a + mr_b} = \frac{r_a r_b r_c}{nr_a r_b + mr_a r_c} + \frac{r_a r_b r_c}{nr_c r_b + mr_a r_b} + \frac{r_a r_b r_c}{mr_c r_a + mr_c r_c} =$$

$$= r_ar_br_c\left(\frac{1}{nr_ar_b+mr_ar_c}+\frac{1}{nr_br_c+mr_br_a}+\frac{1}{nr_cr_a+mr_cr_a}\right)\overset{J.Bergstrom}{\geq}$$

$$\geq r_ar_br_c\cdot\frac{9}{(m+n)(r_ar_b+r_br_c+r_cr_a)}=\frac{9s^2r}{(m+n)\cdot s^2}=\frac{9r}{m+n}\ \therefore\ \ (2)$$

$$(1)+(2):\frac{4R+r}{m+n}+\frac{9r}{m+n}\overset{J.Euler}{\geq}\frac{9r+r}{m+n}+\frac{9r}{m+n}=\frac{18r}{m+n}$$

## Solution 2 by Tran Hong-Dong Thap-Vietnam

$$\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r}\Leftrightarrow r=\frac{r_ar_br_c}{r_ar_b+r_br_c+r_cr_a}$$

Let $x=r_a; y=r_b; z=r_c\ \ (x,y,z>0)$

We need to prove: $\frac{x^2+yz}{ny+mz}+\frac{y^2+xz}{nz+mx}+\frac{z^2+xy}{nx+my}\geq\frac{18xyz}{(m+n)(xy+yz+zx)}$

$$\frac{x^3+xyz}{nxy+mxz}+\frac{y^3+xyz}{nyz+mxy}+\frac{z^3+xyz}{nxz+myz}\geq\frac{18xyz}{(m+n)(xy+yz+zx)}\ \ (*)$$

$$\frac{x^3}{nxy+mxz}+\frac{y^3}{nyz+mxy}+\frac{z^3}{nxz+myz}\overset{AM-GM}{\geq}3\cdot\sqrt[3]{\frac{(xyz)^3}{(nxy+mxz)(nyz+mxy)(nxz+myz)}}$$

$$=\frac{3xyz}{\sqrt[3]{(nxy+mxz)(nyz+mxy)(nxz+myz)}}\overset{AM-GM}{\geq}3xyz\cdot\frac{3}{(m+n)(xy+yz+zx)}=\frac{9xyz}{(m+n)(xy+yz+zx)}\ \ (1)$$

$$xyz\left(\frac{1}{nxy+mxz}+\frac{1}{nyz+mxy}+\frac{1}{nxz+myz}\right)\overset{AM-GM}{\geq}$$

$$\geq xyz\cdot\frac{3}{\sqrt[3]{(nxy+mxz)(nyz+mxy)(nxz+myz)}}\overset{AM-GM}{\geq}$$

$$3xyz\cdot\frac{3}{(m+n)(xy+yz+zx)}=\frac{9xyz}{(m+n)(xy+yz+zx)}\ \ (2)$$

$$\overset{(1)+(2)}{\Rightarrow}LHS_{(*)}\geq\frac{9xyz+9xyz}{(m+n)(xy+yz+zx)}=\frac{18xyz}{(m+n)(xy+yz+zx)}.$$ **Proved.**

## Solution 3 by Avishek Mitra-West Bengal-India

$$\Leftrightarrow\Omega_1=\sum\frac{r_a^2}{nr_b+mr_c}=\sum\frac{r_a^3}{nr_ar_b+mr_ar_c}\overset{Holder}{\geq}\sum\frac{(\sum r_a)^3}{3(n\sum r_ar_b+m\sum r_ar_c)}\geq$$

$$\geq\frac{3(\sum r_ar_b)\cdot(\sum r_a)}{3(m+n)\sum r_ar_b}$$

$$\left[\because \sum x^2 \geq \sum xy \Rightarrow \left(\sum x\right)^2 \geq 3\sum xy \text{, put } x = r_a, y = r_b, z = r_c \Rightarrow \atop \Rightarrow \left(\sum r_a^2\right)^2 \geq 3\sum r_a r_b\right]$$

$$\Leftrightarrow \Omega_1 \geq \frac{(\sum r_a)}{(m+n)} = \frac{4R + r}{(m+n)} \overset{Euler}{\geq} \frac{4\cdot 2r + r}{(m+n)} = \frac{9r}{(m+n)}$$

$$\Leftrightarrow \Omega_2 = \sum \frac{r_b r_c}{m r_b + n r_c} = \sum \frac{\prod r_a}{m r_a r_b + n r_a r_c} = \prod r_a \sum \frac{1}{m r_a r_b + n r_a r_c} \overset{Bergstrom}{\geq}$$

$$\geq \frac{(1+1+1)^2}{n\sum r_a r_b + m\sum r_a r_c}$$

$$= \prod r_a \cdot \frac{9}{(m+n)\sum r_a r_b} = s^2 r \cdot \frac{9}{(m+n)\cdot s^2} = \frac{9r}{(m+n)}$$

$$\Leftrightarrow \Omega = \Omega_1 + \Omega_2 = \sum \frac{r_a^2 + r_b r_c}{n r_b + m r_c} \geq \frac{9r}{(m+n)} + \frac{9r}{(m+n)} = \frac{18r}{(m+n)}$$ ***(proved)***

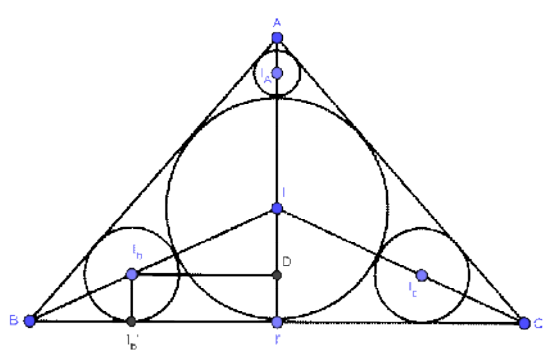
**SP.256. Let $ABC$ be a triangle and $(I, r)$ its incircle. The circle $(I_A, r_A)$ is externally tangent to the circle $(I, r)$ and internally tangent to the sides $AB$ and $AC$ of the triangle. The circles $(I_B, r_B)$ and $(I_C, r_C)$ are defined similarly. Prove that:**

**1.** $r_A + r_B + r_C \geq r$

**2.** $\frac{(r-r_A)^2}{r r_A} + \frac{(r-r_B)^2}{r r_B} + \frac{(r-r_C)^2}{r r_C} \geq 4$

***Proposed by George Apostolopoulos – Messolonghi – Greece***

***Solution by Marian Ursărescu – Romania***




***1)*** $I'D \perp BC, I_b I_b' \perp BC$

$$I_bD \perp II'$$

$$\left.\begin{array}{l}\sin II_bD = \frac{ID}{II_b} \\ But\ \frac{B}{2} = II_bD\end{array}\right\} \Rightarrow \sin\frac{B}{2} = \frac{ID}{II_b} = \frac{r-r_B}{r+r_B}$$ ***and similarly (1)***

***But in any*** $\Delta ABC$ ***we have:*** $\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\le\frac{3}{2}$ ***(2)***

$$\sum\frac{r-r_A}{r+r_A}\le\frac{3}{2}\Rightarrow\sum\frac{1-\frac{r_A^2}{r}}{1+\frac{r_A}{r_B}}\le\frac{3}{2}\quad (3)$$

***Let*** $\frac{r_A}{r}=x,\frac{r_B}{r}=y,\frac{r_C}{r}=z$ ***(4)***

***From (3)+(4)*** $\Rightarrow\sum\frac{1-x}{1+x}\le\frac{3}{2}\Leftrightarrow\sum\frac{2}{1+x}-3\le\frac{3}{2}\Leftrightarrow\sum\frac{1}{1+x}\le\frac{9}{4}$

***But*** $(x+1+y+z+1)\left(\frac{1}{x+1}+\frac{1}{y+1}+\frac{1}{z+1}\right)\ge 9\Rightarrow$

$$x+y+z+3\ge\frac{9}{\sum\frac{1}{1+x}}\ge\frac{9}{\frac{9}{4}}=4\Rightarrow x+y+z+3\ge 4\Rightarrow$$

$$x+y+z\ge 1\Leftrightarrow\frac{r_A}{r}+\frac{r_B}{r}+\frac{r_C}{r}\ge 1\Rightarrow r_A+r_B+r_c\ge r$$

***2.*** $\Delta II_BD\Rightarrow I_BD^2 = II_B^2-ID^2=(r+r_B)^2-(r-r_B)^2=4rr_B\Rightarrow$

$\Rightarrow\tan^2\frac{B}{2}=\frac{ID^2}{I_BD^2}=\frac{(r-r_B)^2}{4rr_B}\Rightarrow$ ***we must show:***

$$\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}\ge 1\quad (5)$$

***But in any*** $\Delta ABC$ ***we have*** $\sum\tan\frac{A}{2}\tan\frac{B}{2}=1\Rightarrow$

$\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}\ge\sum\tan\frac{A}{2}\tan\frac{B}{2}=1\Rightarrow$ ***(5) it is true.***

**SP.257. In $\Delta ABC, BB', CC'$ –internal bisectors.If the circumcircle of $\Delta AB'C'$ is tangent to the side $BC$, then:**

$$\frac{2b}{2a+c}+\frac{2c}{2a+b}<1$$

***Proposed by Marian Ursărescu-Romania***

*Solution by proposer*

$$\rho(B) = BA'^2 = BC' \cdot BA;\ \rho(C) = CA'^2 = CB' \cdot CA \Rightarrow$$

$$a = BA' + A'C = \sqrt{c \cdot BC'} + \sqrt{b \cdot B'C} \quad (1)$$

$$\frac{BC'}{C'A} = \frac{a}{b} \Rightarrow \frac{BC'}{c} = \frac{a}{a+b} \Rightarrow BC' = \frac{ac}{a+b} \quad (2)$$

$$\frac{B'C}{B'A} = \frac{a}{c} \Rightarrow \frac{B'A}{b} = \frac{a}{a+c} \Rightarrow B'C = \frac{ab}{a+c} \quad (3)$$

*From (1)+(2)+(3) we have:* $a = \sqrt{\frac{ac^2}{a+b}} + \sqrt{\frac{ab^2}{a+c}} \Rightarrow \sqrt{a} = \frac{c}{\sqrt{a+b}} + \frac{b}{\sqrt{a+c}}$

$$\Rightarrow 1 = \frac{c}{\sqrt{a(a+b)}} + \frac{b}{\sqrt{a(a+c)}} \quad (4)$$

$$\sqrt{a(a+b)} \le \frac{2a+b}{2} \Rightarrow \frac{1}{\sqrt{a(a+b)}} > \frac{2}{2a+b}$$

$$\sqrt{a(a+c)} \le \frac{2a+c}{2} \Rightarrow \frac{1}{\sqrt{a(a+c)}} > \frac{2}{2a+c} \quad (5)$$

*From (4)+(5) we have:* $1 = \frac{c}{\sqrt{a(a+b)}} + \frac{b}{\sqrt{a(a+c)}} > \frac{2c}{2a+b} + \frac{2}{2a+c}$

**SP.258. Let $A'B'C'$ be the circumcevian triangle of symedians in $\Delta ABC$. Prove that:**

$$\frac{[A'B'C']}{[ABC]} \le \left(\frac{R}{2r}\right)^2$$

*Proposed by Marian Ursărescu-Romania*

*Solution by Tran Hong-Dong Thap-Vietnam*

$$\frac{[A'B'C']}{[ABC]} = \frac{A'B' \cdot B'C' \cdot C'A'}{4R} \cdot \frac{4R}{abc} = \frac{A'B' \cdot B'C' \cdot C'A'}{abc}$$

$$\begin{cases} \widehat{BAK} \equiv \widehat{BB'A'} \\ \widehat{ABB'} \equiv \widehat{AA'B'} \end{cases} \Rightarrow \Delta KAB \sim \Delta K'B'A' \Rightarrow \frac{KA}{KB'} = \frac{AB}{B'A'}$$

$$\Rightarrow B'A' = \frac{AB \cdot KB'}{KA} = c \cdot \frac{KB}{KA} \cdot \frac{KB'}{KB} = \frac{c \cdot \rho(K)}{KA \cdot KB}$$

*Similary* $B'C' = \frac{c\cdot\rho(K)}{KB\cdot KC}$ and $A'C' = \frac{c\cdot\rho(K)}{KA\cdot KC}$. So, $\frac{[A'B'C']}{[ABC]} = \frac{abc(\rho(K))^3}{abc(KA\cdot KB\cdot KC)^3} = \frac{(\rho(K))^3}{(KA\cdot KB\cdot KC)^3}$

*But:* $\rho(K) = R^2 - OK^2 = \frac{3(abc)^2}{(a^2+b^2+c^2)^2}$

$$KA = \frac{b^2+c^2}{a^2+b^2+c^2}\cdot s_a = \frac{b^2+c^2}{a^2+b^2+c^2}\cdot\frac{2bc}{b^2+c^2}\cdot m_a = \frac{2bcm_a}{a^2+b^2+c^2} \Rightarrow$$

$$KA^2 = \frac{(2bcm_a)^2}{(a^2+b^2+c^2)^2} \Rightarrow$$

$$\frac{[A'B'C']}{[ABC]} = \frac{\frac{27(abc)^6}{(a^2+b^2+c^2)^6}}{\frac{64(abc)^4}{(a^2+b^2+c^2)^6}\cdot(m_am_bm_c)^2} = \frac{27}{64}\cdot\left(\frac{abc}{m_am_bm_c}\right)^2$$

$$= \frac{27}{64}\cdot\left(\frac{4Rrs}{m_am_bm_c}\right)^2 \overset{m_am_bm_c\geq s^2r}{\leq} \frac{27}{64}\cdot\left(\frac{4Rrs}{s^2r}\right)^2 = \frac{27}{4}\cdot\frac{R^2}{s^2} \overset{s\geq 3\sqrt{3}r}{\leq} \frac{27}{4}\cdot\frac{R^2}{27r^2} = \left(\frac{R}{2r}\right)^2$$

**SP.259. In $\Delta ABC$, $\Gamma$ –is Gergonne point, $BN, CM$ –simedians from $B, C$. Prove that the points $B, \Gamma, N$ –are collinear if only if $\frac{r_b}{b^2} + \frac{r_c}{c^2} = \frac{r_a}{a^2}$.**

***Proposed by Marian Ursărescu-Romania***

***Solution by proposer***

*From tranversal theorem we have:*

$B, \Gamma, N$ –*are collinear if only if* $\frac{MB}{MA}\cdot\frac{1}{s-b} + \frac{NC}{NA}\cdot\frac{1}{s-c} = \frac{1}{s-a}$ (1)

*From Steiner theorem we have:* $\frac{MA}{MB} = \left(\frac{BC}{AC}\right)^2 = \frac{a^2}{b^2}$ and $\frac{NC}{NA} = \left(\frac{BC}{AB}\right)^2 = \frac{a^2}{c^2}$ (2)

*From (1)+(2) we have:* $\frac{a^2}{b^2(s-b)} + \frac{a^2}{c^2(s-c)} = \frac{1}{s-a}$

$$\frac{1}{b^2(s-b)} + \frac{1}{c^2(s-c)} = \frac{1}{a^2(s-a)}$$

*But:* $r_a = \frac{S}{s-a} \Rightarrow s-a = \frac{S}{r_a}$, *and analogs* $s-b = \frac{S}{r_b}$; $s-c = \frac{S}{r_c}$

So, $\frac{r_b}{b^2} + \frac{r_c}{c^2} = \frac{r_a}{a^2}$

**SP.260. If $a_1, a_2, \dots, a_n > 0$; $k \in \mathbb{N}$ $k \geq 1$; $n > 0$ fixed then find the minimum of:**

$$\Omega = 2\left(a_1^3 + a_2^3 \dots + a_k^3\right) - n(a_1a_2 + a_2a_3 + \dots + a_ka_1)$$

***Proposed by Marin Chirciu-Romania***

***Solution 1 by Tran Hong-Dong Thap-Vietnam***

$$a_1^3 + a_2^3 \dots + a_k^3 \geq \frac{(a_1 + a_2 + \dots + a_k)^3}{k^2}; k \geq 1$$

$$a_1a_2 + a_2a_3 + \dots + a_ka_1 \leq \frac{(a_1 + a_2 + \dots + a_k)^3}{k^2}; k \geq 1$$

$$\Omega = 2\left(a_1^3 + a_2^3 \dots + a_k^3\right) - n(a_1a_2 + a_2a_3 + \dots + a_ka_1)$$

$$\geq 2 \cdot \frac{(a_1 + a_2 + \dots + a_k)^3}{k^2} - n \cdot \frac{(a_1 + a_2 + \dots + a_k)^3}{k^2}$$

$$\overset{t=a_1+a_2+\dots+a_k>0}{\triangleq} \frac{2t^3}{k^2} - \frac{nt^2}{k} = \frac{2t^3 - nkt^2}{k^2} = \psi$$

$$\varphi(t) = 2t^3 - nkt^2; (t > 0) \Rightarrow \varphi'(t) = 6t^2 - 2nkt$$

$$\varphi'(t) = 0 \Leftrightarrow 2t(3t - nk) = 0 \overset{t>0}{\Leftrightarrow} t = \frac{nk}{3} \in (0, \infty)$$

$$\varphi'(t) < 0, \forall t \in \left(0, \frac{nk}{3}\right); \ \varphi'(t) > 0, \forall t \in \left(\frac{nk}{3}, \infty\right)$$

$$\varphi(t) \geq \varphi_{min}\left(\frac{nk}{3}\right) = 2 \cdot \frac{(nk)^3}{27} - nk \cdot \frac{(nk)^2}{9} = -\frac{1}{27}(nk)^3 \Rightarrow \Omega \geq \psi = -\frac{1}{27} \cdot kn^3$$

**So,** $\Omega_{min} = -\frac{1}{27} \cdot kn^3 \Leftrightarrow a_1 = a_2 = \dots = a_k = \frac{n}{3}$

***Solution 2 by Sergio Esteban-Argentina***

**Let** $f(a_1, a_2, \dots, a_k) = 2\left(a_1^3 + a_2^3 \dots + a_k^3\right) - n(a_1a_2 + a_2a_3 + \dots + a_ka_1)$

$$\Delta f = \left(6a_1^2 - n(a_2 + a_k), 6a_2^2 - n(a_1 + a_3), \dots, 6a_k^2 - n(a_{k-1} + a_1)\right) = (0, 0, \dots, 0)$$

***It's zero when*** $a_1 = a_2 = \dots = a_k = \frac{n}{3}$***. We calculate the Hessiano of f is*** $(H_f)$:

$$\begin{bmatrix} 12a_1 & -n & 0 & 0 & \dots & -n \\ -n & 12a_2 & -n & 0 & \dots & 0 \\ 0 & -n & 12a_3 & -n & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -n & \dots & \dots & \dots & \dots & 12a_k \end{bmatrix}$$

*So, $H_{f\left(\frac{n}{3},\dots\frac{n}{3}\right)}$ is:*

$$\begin{bmatrix} 4n & -n & 0 & 0 & \dots & -n \\ -n & 4n & -n & 0 & \dots & 0 \\ 0 & -n & 4n & -n & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -n & \dots & \dots & 0 & \dots & 4n \end{bmatrix}$$

*We use the criterion that says that if all the determinants of the main minors of $H_{f\left(\frac{n}{3},\dots\frac{n}{3}\right)}$ are strictly positive, then $H_{f\left(\frac{n}{3},\dots\frac{n}{3}\right)}$ is positively defined, we see that:*

$$4n, \begin{vmatrix} 4n & -n \\ -n & 4n \end{vmatrix}, \begin{vmatrix} 4n & -n & 0 \\ -n & 4n & -n \\ 0 & -n & 4n \end{vmatrix}, \dots, \begin{bmatrix} 4n & -n & 0 & 0 & \dots & -n \\ -n & 4n & -n & 0 & \dots & 0 \\ 0 & -n & 4n & -n & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -n & \dots & \dots & 0 & \dots & 4n \end{bmatrix}$$

*They have the following form $det(k) = \frac{(2+\sqrt{3})^k-(2-\sqrt{3})^k}{2\sqrt{3}} n^{k-1}, k \in \mathbb{N}, k \geq 2$, where it is positive. By method of Gauss Jordan triangulating the matrix we will notice that in the diagonal only positive numbers remain, with which we conclude that $det\left(H_{f\left(\frac{n}{3},\dots\frac{n}{3}\right)}\right) > 0$ and $det(k) > 0$, then f reachers a global minimum in $f\left(\frac{n}{3},\dots\frac{n}{3}\right)$*

$$\Omega_{min} = f\left(\frac{n}{3},\dots\frac{n}{3}\right) = -\frac{1}{27}\cdot kn^3$$

**SP.261. If $x, y, z > 0$ then in $\Delta ABC$ the following relationship holds:**

$$\frac{(x+y)^2}{z^2} + \frac{(y+z)^2}{x^2} + \frac{(z+x)^2}{y^2} + a^4 + b^4 + c^4 \geq 16\sqrt{3}F$$

***Proposed by D.M.Bătineţu-Giurgiu, Daniel Sitaru-Romania***

***Solution 1 by Adrian Popa-Romania***

$$\frac{\left(\frac{x}{z}+\frac{y}{z}\right)^2}{1} + \frac{\left(\frac{y}{x}+\frac{z}{x}\right)^2}{1} + \frac{\left(\frac{x}{y}+\frac{z}{y}\right)^2}{1} \geq \frac{\left(\frac{x}{z}+\frac{z}{x}+\frac{y}{z}+\frac{z}{y}+\frac{y}{x}+\frac{x}{y}\right)^2}{3} = 12$$

$$\frac{a^4}{1} + \frac{b^4}{1} + \frac{c^4}{1} \geq \frac{(a^2+b^2+c^2)^2}{3} \geq \frac{\left(4\sqrt{3}F\right)^2}{3} = 16F^2$$

*We must show that: $12 + 16F^2 \geq 16\sqrt{3}F$ true from Am-Gm.*

***Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand***

*For $x, y, z > 0$ and in any $\Delta ABC$ we have*

$$\left(\frac{x+y}{z}\right)^2+\left(\frac{y+z}{x}\right)^2+\left(\frac{z+x}{y}\right)^2+a^4+b^4+c^4\ge\frac{\left(\frac{x+y}{z}+\frac{y+z}{x}+\frac{z+x}{y}+a^2+b^2+c^2\right)^2}{6}$$

$$\ge\frac{(6+a^2+b^2+c^2)^2}{6}\ge\frac{(6+4\sqrt{3}S)^2}{6}=16\sqrt{3}S\quad (1)$$

$$\left(\frac{x+y}{z}\right)^2+\left(\frac{y+z}{x}\right)^2+\left(\frac{z+x}{y}\right)^2+a^4+b^4+c^4\ge 3\sqrt[3]{\left[\left(\frac{x+y}{z}\right)\left(\frac{y+z}{x}\right)\left(\frac{z+x}{y}\right)\right]^2}+a^4+b^4+c^4$$

$$=12+a^4+b^4+c^4\ge 12+\frac{(a^2+b^2+c^2)^2}{3}$$

$$\ge 12+\frac{(4\sqrt{3}S)^2}{3}\ge 16\sqrt{3}S\quad (2)$$

***From (1)+(2) the inequality is proved.***

**SP.262. If $x,y,z>0;u\ge 0$ then in $\Delta ABC$ the following relationship holds:**

$$\frac{y+z+u}{x}a^2+\frac{z+x+u}{y}b^2+\frac{x+y+u}{z}c^2\ge 8\sqrt{3}F+\frac{12u\sqrt{3}F}{x+y+z}$$

***Proposed by D.M.Bătineţu-Giurgiu, Daniel Sitaru-Romania***

***Solution 1 by Tran Hong-Dong Thap-Vietnam***

$$\Omega=\frac{y+z+u}{x}a^2+\frac{z+x+u}{y}b^2+\frac{x+y+u}{z}c^2$$

$$=\left(\frac{y}{x}a^2+\frac{z}{y}b^2+\frac{x}{z}c^2\right)+\left(\frac{z}{x}a^2+\frac{x}{y}b^2+\frac{y}{z}c^2\right)+u\left(\frac{1}{x}a^2+\frac{1}{y}b^2+\frac{1}{z}c^2\right)$$

$$\frac{y}{x}a^2+\frac{z}{y}b^2+\frac{x}{z}c^2\ge 4\sqrt{\frac{z}{x}+\frac{x}{y}+\frac{y}{z}}\cdot F\overset{Am-Gm}{\geq}4\sqrt{3}F\quad (1)$$

$$\frac{z}{x}a^2+\frac{x}{y}b^2+\frac{y}{z}c^2\ge 4\sqrt{\frac{z}{x}+\frac{x}{y}+\frac{y}{z}}\cdot F\overset{Am-Gm}{\geq}4\sqrt{3}F\quad (2)$$

$$u\left(\frac{1}{x}a^2+\frac{1}{y}b^2+\frac{1}{z}c^2\right)\ge 4\cdot\sqrt{\frac{z}{x}+\frac{x}{y}+\frac{y}{z}}\cdot F=4\cdot\sqrt{\frac{x+y+z}{xyz}}\cdot F$$

$$\overset{Am-Gm}{\geq}4\cdot\sqrt{\frac{27(x+y+z)}{(x+y+z)^3}}\cdot F=\frac{12u\sqrt{3}\cdot F}{x+y+z}\quad (3)\xrightarrow{(1)+(2)+(3)}$$

$$\Omega\ge 8\sqrt{3}\cdot F+\frac{12u\sqrt{3}\cdot F}{x+y+z}$$ ***.Proved.***

**Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand**

For $x, y, z > 0, u \geq 0$ and triangle $\Delta ABC$ we have

$$\frac{y+z+u}{x}a^2+\frac{z+x+u}{y}b^2+\frac{x+y+u}{z}c^2=$$

$$=\frac{y}{x}a^2+\frac{z}{x}a^2+\frac{u}{x}a^2+\frac{z}{y}b^2+\frac{x}{y}b^2+\frac{u}{y}b^2+\frac{x}{z}c^2+\frac{y}{z}c^2+\frac{u}{z}c^2$$

$$=\left(\frac{y}{x}a^2+\frac{x}{y}b^2\right)+\left(\frac{z}{y}b^2+\frac{y}{z}c^2\right)+\left(\frac{z}{x}a^2+\frac{x}{z}c^2\right)+u\left(\frac{a^2}{x}+\frac{b^2}{y}+\frac{c^2}{z}\right)$$

$$\geq 2ab+2bc+2ca+u\frac{(a+b+c)^2}{x+y+z}=\geq 2(ab+bc+ca)+\frac{3u(ab+bc+ca)}{x+y+z}$$

$$\geq 8\sqrt{3}F+\frac{12\sqrt{3}uF}{x+y+z} \text{ true, because } ab+bc+ca\geq 4\sqrt{3}F$$

**SP263. In $\Delta ABC$ the following relationship holds:**

$$12\leq\sum_{cyc}\left(\frac{a+b}{c}\right)^2\leq 18\left(\frac{R}{2r}\right)^2-6$$

*Proposed by George Apostolopoulos-Messolonghi-Greece*

**Solution by Soumava Chakraborty-Kolkata-India**

$$\sum\left(\frac{a+b}{c}\right)^2-3=\sum\left[\left(\frac{b+c}{a}\right)^2-1\right]=\sum\left[\frac{(b+c+a)(b+c-a)}{a^2}\right]=\sum\left[\frac{4s(s-a)}{a^2}\right]\leq$$

$$\leq\sum\left[\frac{4m_a^2}{a^2}\right]=\sum\left[\frac{2b^2+2c^2-a^2}{a^2}\right]$$

$$=\sum\frac{(2b^2+2c^2+2a^2)-3a^2}{a^2}=2\left(\sum a^2\right)\left(\sum\frac{1}{a^2}\right)-9\overset{Leibniz}{\leq}18R^2\left(\frac{\sum a^2b^2}{16R^2r^2s^2}\right)-9\leq$$

$$\overset{Goldstone}{\leq}\frac{18(4R^2s^2)}{16r^2s^2}-9=18\left(\frac{R}{2r}\right)^2-9\Rightarrow\sum\left(\frac{a+b}{c}\right)^2\overset{(1)}{\leq}18\left(\frac{R}{2r}\right)^2-6$$

$$Again, \sum\left(\frac{a+b}{c}\right)^2-3=\sum\left[\frac{4s(s-a)}{a^2}\right]\geq 4\sum\frac{w_a^2}{a^2}=4\sum\left[\frac{\left(\frac{4b^2c^2s(s-a)}{bc(b+c)^2}\right)}{a^2}\right]=$$

$$= 4\sum\left[\frac{\left(\frac{bc(b+c+a)(b+c-a)}{(b+c)^2}\right)}{a^2}\right] = 4\sum\left[\frac{\frac{bc\{(b+c)^2-a^2\}}{(b+c)^2}}{a^2}\right] = 4\sum\left[\frac{bc-\frac{a^2bc}{(b+c)^2}}{a^2}\right] =$$

$$= 4\sum\frac{bc}{a^2} - 4\sum\frac{bc}{(b+c)^2} \overset{A-G}{\gtrapprox} 12\sqrt[3]{\prod\left(\frac{bc}{a^2}\right)} - \sum\frac{4bc}{(b+c)^2} \overset{A-G}{\gtrapprox} 12 - \sum(1) = 9 \Rightarrow \sum\left(\frac{a+b}{c}\right)^2 \overset{(2)}{\gtrapprox} 12$$

$$\therefore combining\ (1)\ and\ (2), 12 \le \sum\left(\frac{a+b}{c}\right)^2 \le 18\left(\frac{R}{2r}\right)^2 - 6\ (Proved)$$

**SP.264 Let $a, b, c$ be the lengths of the sides of triangle $ABC$ with iradius $r$ circumradius $R$ and radii of excircles $r_a, r_b, r_c$ at angles $A, B, C$, respectively.**

**Prove that:**

$$\frac{1}{4R^4} \le \frac{r_a^2 + r_b^2}{c^2(a^4+b^4)} + \frac{r_b^2 + r_c^2}{a^2(b^4+c^4)} + \frac{r_c^2 + r_a^2}{b^2(c^4+a^4)} \le \frac{1}{64r^4}$$

***Proposed by George Apostolopoulos-Messolonghi-Greece***

***Solution by proposer***

*First, will prove that:* $r_a \le \frac{a^2}{4r}, r_b \le \frac{b^2}{4r}, r_c \le \frac{c^2}{4r}$ *and* $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \le \frac{1}{4r^2}$

*We have:* $\frac{a^2}{4rs} = \frac{a^2}{4F}$ *where* $F$ *denotes the area of* $\Delta ABC$*. So:*

$$\frac{a^2}{4rs} = \frac{a^2}{4\left(\frac{1}{2}bc\sin A\right)} = \frac{(2R\sin A)^2}{2bc\sin A} = \frac{2R^2\sin A}{bc} = \frac{2R^2\sin A}{(2R\sin B)(2R\sin C)}$$

$$= \frac{\sin A}{2\sin B\cdot\sin C} = \frac{\sin A}{\cos(B-C)-\cos(B+C)} \ge \frac{\sin A}{1+\cos A}$$

$$= \frac{2\sin\frac{A}{2}\cos\frac{A}{2}}{1+2\cos^2\frac{A}{2}-1} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = \tan\frac{A}{2}$$

*So,* $\tan\frac{A}{2} \le \frac{a^2}{4rs} \Leftrightarrow s\cdot\tan\frac{A}{2} \le \frac{a^2}{4r}$ *or* $r_a \le \frac{a^2}{4r}$*. Similarly* $r_b \le \frac{b^2}{4r}, r_c \le \frac{c^2}{4r}$

*Also we have:* $(b-c)^2 \ge 0 \Leftrightarrow a^2-(b-c)^2 \le a^2 \Leftrightarrow \frac{1}{a^2} \le \frac{1}{a^2-(b-c)^2} = \frac{1}{(a+b-c)(a-b+c)}$

*Let:* $2s = a+b+c$ *is the perimeter of triangle* $ABC$*. Then:* $\frac{1}{a^2} \le \frac{1}{a^2-(b-c)^2} = \frac{1}{4(s-c)(s-b)}$

***Similarly*** $\frac{1}{b^2} \le \frac{1}{4(s-c)(s-a)}$; $\frac{1}{c^2} \le \frac{1}{4(s-a)(s-b)}$

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} \le \frac{1}{4}\left(\frac{1}{(s-b)(s-c)}+\frac{1}{(s-c)(s-a)}+\frac{1}{(s-a)(s-b)}\right)$$

$$=\frac{1}{4}\cdot\frac{s-a+s-b+s-c}{(s-a)(s-b)(s-c)}=\frac{1}{4}\cdot\frac{s(3s-2s)}{s(s-a)(s-b)(s-c)}=\frac{1}{4}\cdot\frac{s^2}{F^2}=\frac{1}{4r^2}$$

$\Rightarrow \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} \le \frac{1}{4r^2}$. ***Now, we have:*** $r_a \le \frac{a^2}{4r} \Leftrightarrow r_a^2 \le \frac{a^4}{16r^2}$

***Also we have:*** $r_b^2 \le \frac{b^4}{16r^2}$ ***and*** $r_c^2 \le \frac{c^4}{16r^2}$.

***So*** $r_a^2+r_b^2 \le \frac{a^4+b^4}{16r^2} \Leftrightarrow \frac{r_a^2+r_b^2}{a^4+b^4} \le \frac{1}{16r^2} \Leftrightarrow \frac{r_a^2+r_b^2}{c^2(a^4+b^4)} \le \frac{1}{16r^2\cdot c^2}$

***Similarly*** $\frac{r_b^2+r_c^2}{a^2(b^4+c^4)} \le \frac{1}{16r^2\cdot a^2}$ ***and*** $\frac{r_c^2+r_a^2}{b^2(a^4+c^4)} \le \frac{1}{16r^2\cdot b^2}$

$$\frac{r_a^2+r_b^2}{c^2(a^4+b^4)}+\frac{r_b^2+r_c^2}{a^2(b^4+c^4)}+\frac{r_c^2+r_a^2}{b^2(a^4+c^4)} \le \frac{1}{16r^2}\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right) \le \frac{1}{16r^2}\cdot\frac{1}{4r^2}=\frac{1}{64r^4}$$

***For the left inequality, we have:***

$$\frac{r_a^2+r_b^2}{c^2(a^4+b^4)}+\frac{r_b^2+r_c^2}{a^2(b^4+c^4)}+\frac{r_c^2+r_a^2}{b^2(a^4+c^4)} \ge \frac{2r_ar_b}{c^2(2a^2b^2)}+\frac{2r_br_c}{a^2(2b^2c^2)}+\frac{2r_cr_a}{b^2(2c^2a^2)}$$

$$=\frac{r_ar_b+r_br_c+r_cr_a}{(abc)^2}=\frac{s^2}{(4Rrs)^2}=\frac{1}{16R^2r^2} \ge \frac{1}{16R^2\left(\frac{R}{2}\right)^2}=\frac{1}{4R^4}$$

***We know that:*** $r_ar_b+r_br_c+r_cr_a=s^2, abc=4Rrs, R \ge 2r$ ***(Euler)***

***So:*** $\frac{1}{4R^4} \le \frac{r_a^2+r_b^2}{c^2(a^4+b^4)}+\frac{r_b^2+r_c^2}{a^2(b^4+c^4)}+\frac{r_c^2+r_a^2}{b^2(c^4+a^4)} \le \frac{1}{64r^4}$

**SP.265. In acute $\Delta ABC$ the following relationship holds:**

$$(cosA)^A\cdot(cosB)^B\cdot(cosC)^C \le 2^{-\pi}$$

***Proposed by Florentin Vişescu-Romania***

***Solution 1 by Radu Butelcă-Romania***

***As the members are positive, we can apply the function*** $x \to log(x)$

$$log((cosA)^A\cdot(cosB)^B\cdot(cosC)^C) \le log(2^{-\pi}) \Leftrightarrow$$

$$Alog(A)+Blog(B)+Clog(C) \le -\pi log2 \quad \textbf{(1)}$$

***Let*** $f:[0,\pi] \to \mathbb{R}, f(x)=xlog(x)$

$$f'(x) = log(cosx) - xtanx$$

$f''(x) = -2tanx - \frac{x}{cos^2x} < 0, \forall x \in [0,\pi]$ **then f is concave**

**By Jensen, we have** $f\left(\frac{A+B+C}{3}\right) \geq \frac{f(A)+f(B)+f(C)}{3} \Leftrightarrow$

$$\frac{A+B+C}{3}log\left(cos\left(\frac{A+B+C}{3}\right)\right) \geq \frac{A\cdot log(cosA) + B\cdot log(cosB) + C\cdot log(cosC)}{3}$$

$$\pi log\left(cos\frac{\pi}{3}\right) \geq A\cdot log(cosA) + B\cdot log(cosB) + C\cdot log(cosC)$$

$$-\pi log2 \geq A\cdot log(cosA) + B\cdot log(cosB) + C\cdot log(cosC) \quad (2)$$

**From (1)+(2) proved.**

## Solution 2 by Adrian Popa-Romania

$$P = (cosA)^A\cdot(cosB)^B\cdot(cosC)^C \leq 2^{-\pi}$$

$$P = (cosA)^A\cdot(cosB)^B\cdot(cosC)^C \overset{Am-Gm}{\leq} \left(\frac{A\cdot cosA + B\cdot cosB + C\cdot cosC}{A+B+C}\right)^{A+B+C}$$

$$= \left(\frac{A\cdot cosA + B\cdot cosB + C\cdot cosC}{\pi}\right)^{\pi}$$

**Let:** $f(x) = xcosx, x \in \left(0,\frac{\pi}{2}\right)$

$f'(x) = -2sinx - xcosx < 0$, **because** $sinx > 0; cosx > 0, \forall x \in \left(0,\frac{\pi}{2}\right)$

**So,** $f$ **is concave** $\overset{Jensen}{\Longrightarrow} \frac{f(A)+f(B)+f(C)}{3} \leq f\left(\frac{A+B+C}{3}\right)$ **then**

$$A\cdot cosA + B\cdot cosB + C\cdot cosC \leq 3\left(\frac{A+B+C}{3}\right)cos\left(\frac{A+B+C}{3}\right) = \frac{\pi}{2} \Rightarrow$$

$$P < 2^{-\pi}$$

**SP.266. If** $A, B \in M_4(\Omega)$;

$$AB = \begin{pmatrix} p & p & p & p \\ 0 & -p & -p & -p \\ 0 & 0 & p & p \\ 0 & 0 & 0 & -p \end{pmatrix}; p \in \mathbb{C}, p \neq 0;\ \Omega_1 = BA;\ \Omega_2 = (BA)^{-1}$$ **then**

**find:** $\Omega = \Omega_1^2 + \left(p^2\Omega_2^{-1}\right)^2$

***Proposed by Marian Ursărescu-Romania***

***Solution by Florentin Vişescu-Romania***

$$P_{AB}(x) = det(xI_4 - AB) = det\begin{pmatrix} x-p & -p & -p & -p \\ 0 & x+p & p & p \\ 0 & 0 & x-p & -p \\ 0 & 0 & 0 & x+p \end{pmatrix} =$$

$$= (x-p)^2(x+p)^2 = x^4 - 2p^2x^2 + p^4 \Rightarrow$$

$$P_{AB}(x) = x^4 - 2p^2x^2 + p^4 \Leftrightarrow (BA)^4 - 2p^2(BA)^2 + p^4I_4 = O_4 \Leftrightarrow$$

$$(BA)^4 + p^4I_4 = 2p^2(BA)^2/_sB^{-1}/_dA^{-1} \Rightarrow (AB)^3 + p^4B^{-1}A^{-1} = 2p^2AB/_sA^{-1}/_dB^{-1} \Rightarrow$$

$$(BA)^2 + p^4(BA)^{-1}(BA)^{-1} = 2p^2, (BA)^2 + (p^2(BA)^{-1})^2 = 2p^2I_4$$

$$\Omega = \Omega_1^2 + \left(p^2\Omega_2^{-1}\right)^2 = 2p^2I_4$$

**SP.267. In $\Delta ABC$; $AA_1, BB_1, CC_1$ –internal bisectors and $A_2B_2C_2$ the circumcevian triangle of incenter.Prove that:**

$$\left(\frac{r}{R}\right)^2 \le \frac{[A_1B_1C_1]}{[A_2B_2C_2]} \le \frac{1}{4}$$

***Proposed by Marian Ursărescu-Romania***

***Solution by Marian Dinca-Romania***

$$[A_1B_1C_1] = \frac{2abc}{(a+b)(b+c)(c+a)} \cdot [ABC] =$$

$$= \frac{2abc}{(a+b)(b+c)(c+a)} \cdot 2R^2\sin A\sin B\sin C$$

$$\measuredangle A_2 = \frac{B+C}{2}; \measuredangle B_2 = \frac{A+C}{2}; \measuredangle C_2 = \frac{A+B}{2}$$

$$[A_2B_2C_2] = 2R^2\sin\left(\frac{B+C}{2}\right)\sin\left(\frac{A+C}{2}\right)\sin\left(\frac{A+B}{2}\right)$$

$$\frac{[A_1B_1C_1]}{[A_2B_2C_2]} = \frac{\frac{2abc}{(a+b)(b+c)(c+a)} \cdot 2R^2\sin A\sin B\sin C}{2R^2\sin\left(\frac{B+C}{2}\right)\sin\left(\frac{A+C}{2}\right)\sin\left(\frac{A+B}{2}\right)}$$

$$\sin A\sin B \le \sin^2\left(\frac{A+B}{2}\right) \Leftrightarrow \cos(B-A) - \cos(A+B) \le 1 - \cos(A+B) \Leftrightarrow$$

$$\cos(A-B) \le 1$$

$$Similarly: \sin C \sin B \le \sin^2\left(\frac{C+B}{2}\right) \text{ and } \sin A \sin C \le \sin^2\left(\frac{A+C}{2}\right)$$

***Multiplying the three inequalities and then extracting the square root we obtain the***

***inequality:*** $\sin A \sin B \sin C \le \sin\left(\frac{B+C}{2}\right)\sin\left(\frac{A+C}{2}\right)\sin\left(\frac{A+B}{2}\right)$

***And use Am-Gm result:*** $(a+b)(b+c)(c+a) \ge 8abc(Cesaro\ inequality)$

***We obtain:*** $\frac{[A_1B_1C_1]}{[A_2B_2C_2]} \le \frac{1}{4}$

$$\frac{[A_1B_1C_1]}{[A_2B_2C_2]} = \frac{\frac{2abc}{(a+b)(b+c)(c+a)} \cdot 2R^2 \sin A \sin B \sin C}{2R^2 \sin\left(\frac{B+C}{2}\right)\sin\left(\frac{A+C}{2}\right)\sin\left(\frac{A+B}{2}\right)}$$

$$= \frac{2abc}{(a+b)(b+c)(c+a)} \cdot 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{2abc}{(a+b)(b+c)(c+a)} \cdot \frac{2r}{R}$$

$$= \frac{4abc}{(a+b)(b+c)(c+a)} \cdot \frac{r}{R} \ge \left(\frac{r}{R}\right)^2 \Leftrightarrow \frac{4abc}{(a+b)(b+c)(c+a)} \ge \frac{r}{R} \Leftrightarrow$$

$$\frac{4abc}{(ab+bc+ca)(a+b+c)-abc} = \frac{16Rrs}{(s^2+r^2+4Rr)\cdot 2s - 4Rrs}$$

$$= \frac{8Rr}{(s^2+r^2+4Rr)-2Rr} = \frac{8Rr}{s^2+r^2+2Rr} \ge \frac{r}{R} \Leftrightarrow$$

$$8R^2 \ge s^2+r^2+2Rr \Leftrightarrow 8R^2 - r^2 - 2Rr \ge s^2$$

$$s^2 \le 4R^2+4Rr+3r^2 \ldots Gerretsen\ inequality$$

$$and: 4R^2+4Rr+3r^2 \le 8R^2 - r^2 - 2Rr \Leftrightarrow$$

$$4R^2 - 6Rr - 4r^2 \ge 0 \Leftrightarrow 2R^2 - 3Rr - 2r^2 \ge 0$$

$$\Leftrightarrow (R-2r)(2R+r) \ge 0 \Leftrightarrow R - 2r \ge 0 \ldots (Euler). Done!$$

**SP.268. If $A \in M_2(\mathbb{R}); Tr\ A = \det A = 1$ then:**

$$\det(A^2+3A+3I_2) \ge 5Tr(A^{-1}) + 3$$

***Proposed by Marian Ursărescu – Romania***

***Solution by Florentin Vișescu – Romania***

$$Tr\ A = \det A = 1 \Rightarrow A^2 - A + I_2 = O_2$$

$$A^2 = A - I_2; A^2 - A = -I_2 \Rightarrow A - A^2 = I_2 \Rightarrow A(I_2 - A) = I_2 \Rightarrow A^{-1} = I_2 - A$$

**(*)** $5Tr(A^{-1})+3=5\left(Tr(I_2-A)\right)+3=5(2-1)+3=5+3=8$

**(**)** $\det(A^2+3A+3I_2)=\det(A-I_2+3A+3I_2)$

$$=\det(4A+2I_2)=\det\left(2(2A+I_2)\right)=4\det(2A+I_2)$$

*So, we have to prove that* $\det(2A+I_2)\ge 2$

*Let be* $A=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in M_2(\mathbb{R}); Tr\ A=1\Rightarrow a+d=1$

$$\Rightarrow d=1-a\Rightarrow A=\begin{pmatrix} a & b \\ c & 1-a\end{pmatrix}\det A=1\Rightarrow a-a^2-bc=1\Rightarrow a-a^2-1=bc$$

*If* $b=0\Rightarrow a^2-a+1=0\Rightarrow a\in\mathbb{C}$ *(False). So,* $b\ne 0\Rightarrow c=\frac{a-a^2-1}{b}\Rightarrow$

$$A=\begin{pmatrix} a & b \\ \frac{a-a^2-1}{b} & 1-a\end{pmatrix}\Rightarrow 2A+I_2=\begin{pmatrix} 2a+1 & 2b \\ \frac{2a-2a^2-2}{b} & 3-2a\end{pmatrix}$$

$$\det(2A+I_2)=(2a+1)(3-2a)-4a+4a^2+4=$$

$$=6a-4a^2+3-2a-4a+4a^2+4=7$$

**SP.269. If in $\Delta ABC; s=\frac{1}{2}$ then:**

$$a\cdot e^{\frac{m_a}{a}}+b\cdot e^{\frac{m_b}{b}}+c\cdot e^{\frac{m_c}{c}}\ge e^{m_a+m_b+m_c}$$

***Proposed by Daniel Sitaru – Romania***

***Solution 1 by Marian Ursărescu-Romania***

$$s=\frac{1}{2}\Rightarrow\frac{a+b+c}{2}=\frac{1}{2}\Rightarrow a+b+c=1\Rightarrow$$

$$ae^{\frac{m_a}{a}}+be^{\frac{m_b}{b}}+ce^{\frac{m_c}{c}}=\frac{a}{a+b+c}e^{\frac{m_a}{a}}+\frac{b}{a+b+c}\cdot e^{\frac{m_b}{b}}+\frac{c}{a+b+c}e^{\frac{m_c}{c}}\ge$$

$$\ge e^{\frac{m_a}{a}\cdot\frac{a}{a+b+c}}\cdot e^{\frac{m_b}{b}\cdot\frac{b}{a+b+c}}\cdot e^{\frac{m_c}{c}\cdot\frac{c}{a+b+c}}=e^{m_a}\cdot e^{m_b}\cdot e^{m_c}=e^{m_a+m_b+m_c}$$

***Solution 2 by Tran Hong-Dong Thap-Vietnam***

$$s=\frac{1}{2}\Rightarrow\frac{a+b+c}{2}=\frac{1}{2}\Rightarrow a+b+c=1$$

*Using Jensen's inequality with* $f(x)=e^x$

$$LHS=af\left(\frac{m_a}{a}\right)+bf\left(\frac{m_b}{b}\right)+cf\left(\frac{m_c}{c}\right)\ge(a+b+c)f\left(\frac{\frac{m_a}{a}+\frac{m_b}{b}+\frac{m_c}{c}}{a+b+c}\right)$$

$$= f\left(\frac{m_a}{a} + \frac{mb}{b} + \frac{m_c}{c}\right) = e^{\frac{m_a}{a}+\frac{m_b}{b}+\frac{m_c}{c}} = e^{\Omega}$$

$$\text{Suppose } a \ge b \ge c \Rightarrow \begin{cases} m_a \le m_b \le m_c \\ \frac{1}{a} \le \frac{1}{b} \le \frac{1}{c} \end{cases}$$

$$\Rightarrow \Omega \overset{Chebyshev}{\ge} \frac{1}{3}(m_a + m_b + m_c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$$

$$\ge \frac{1}{3}(m_a + m_b + m_c) \cdot \frac{9}{a + b + c} = \frac{1}{3}(m_a + m_b + m_c) \cdot \frac{9}{3} = m_a + m_b + m_c$$

$$\Rightarrow e^{\Omega} \ge e^{m_a+m_b+m_c}. \text{ Proved.}$$

## Solution 3 by proposer

$$\text{Let be } f_1, f_2, f_3: (0, \infty) \to \mathbb{R}$$

$$f_1(x) = ax\ln x - (a + m_a)x; f_2(x) = bx\ln x - (b + m_b)x$$

$$f_3(x) = cx\ln x - (c + m_c)x$$

$$f'_1(x) = a(\ln x + 1) - (a + m_a) = a\ln x - m_a$$

$$f'_1(x) = 0 \Rightarrow a\ln x = m_a \Rightarrow \ln x = \frac{m_a}{a}$$

$$\ln x = \ln e^{\frac{m_a}{a}} \Rightarrow x = e^{\frac{m_a}{a}}$$

$$\min f_1(x) = f_1\left(e^{\frac{m_a}{a}}\right) == a \cdot e^{\frac{m_a}{a}} \cdot \ln e^{\frac{m_a}{a}} - (a + m_a) \cdot e^{\frac{m_a}{a}} =$$

$$= m_a \cdot e^{\frac{m_a}{a}} - ae^{\frac{m_a}{a}} - m_a \cdot e^{\frac{m_a}{a}} = -ae^{\frac{m_a}{a}}$$

$$\text{Analogous: } \min f_2(x) = -be^{\frac{m_b}{b}}; \min f_3(x) = -ce^{\frac{m_c}{c}}$$

$$f_1 + f_2 + f_3: (0, \infty) \to \mathbb{R}$$

$$(f_1 + f_2 + f_3)(x) = f_1(x) + f_2(x) + f_3(x)$$

$$\min(f_1 + f_2 + f_3)(x) = -(a + b + c)e^{\frac{m_a+m_b+m_c}{a+b+c}}$$

$$\min f_1(x) + \min f_2(x) + \min f_3(x) \le \min(f_1 + f_2 + f_3)(x)$$

$$-ae^{\frac{m_a}{a}} - be^{\frac{m_b}{b}} - ce^{\frac{m_c}{c}} \le -(a + b + c)e^{\frac{m_a+m_b+m_c}{a+b+c}}$$

$$ae^{\frac{m_a}{a}} + be^{\frac{m_b}{b}} + ce^{\frac{m_c}{c}} \ge (a + b + c)e^{\frac{m_a+m_b+m_c}{a+b+c}} =$$

$$= (2s) \cdot e^{\frac{m_a+m_b+m_c}{2s}} = \left(2 \cdot \frac{1}{2}\right) \cdot e^{\frac{m_a+m_b+m_c}{2 \cdot \frac{1}{2}}} = e^{m_a+m_b+m_c}$$

*Equality holds for $a = b = c = \frac{1}{6}$.*

## Solution 4 by Ravi Prakash-New Delhi-India

$$\frac{a^{e^{m_a/a}} + be^{m_b/b} + ce^{m_c/c}}{a+b+c} \ge \left[\left(e^{m_a/a}\right)^a\left(e^{m_b/b}\right)^b\left(e^{m_c/c}\right)^c\right]^{\frac{1}{a+b+c}}$$

$$\Rightarrow a^{e^{m_a/a}} + be^{m_b/b} + ce^{m_c/c} \ge \left(e^{m_a+m_b+m_c}\right)\left[\because s = \frac{1}{2}\right]$$

**SP.270. If $x_n > 0; n \in \mathbb{N}$ is a sequence such that exists**

$$\lim_{n\to\infty}\frac{x_{n+3}\cdot x_{n+1}^3}{x_{n+2}^2\cdot x_n} = \pi \text{ then find: } \Omega = \lim_{n\to\infty}\sqrt[n^3]{x_n}$$

***Proposed by D.M.Batineţu Giurgiu, Neculai Stanciu-Romania***

## Solution by Marian Ursărescu-Romania

$$\log(\Omega) = \lim_{n\to\infty}\log\left(\sqrt[n^3]{x_n}\right) = \lim_{n\to\infty}\frac{\log(x_n)}{n^3} \overset{L.C-S}{=} \lim_{n\to\infty}\frac{\log(x_{n+1}) - \log(x_n)}{(n+1)^3 - n^3}$$

$$= \lim_{n\to\infty}\frac{\log\left(\frac{x_{n+1}}{x_n}\right)}{3n^2+3n+1} = \lim_{n\to\infty}\frac{n^2}{3n^2+3n+1}\cdot\frac{\log\left(\frac{x_{n+1}}{x_n}\right)}{n^2}$$

$$\overset{L.C-S}{=} \frac{1}{3}\lim_{n\to\infty}\frac{\log\left(\frac{x_{n+2}}{x_{n+1}}\right) - \log\left(\frac{x_{n+1}}{x_n}\right)}{(n+1)^2 - n^2} = \frac{1}{3}\lim_{n\to\infty}\frac{\log\left(\frac{x_{n+2}\cdot x_n}{x_{n+1}^2}\right)}{2n+1}$$

$$= \frac{1}{3}\lim_{n\to\infty}\frac{n}{2n+1}\cdot\frac{\log\left(\frac{x_{n+2}\cdot x_n}{x_{n+1}^2}\right)}{n} \overset{L.C-S}{=} \frac{1}{6}\lim_{n\to\infty}\frac{\log\left(\frac{x_{n+3}\cdot x_{n+1}}{x_{n+2}^2}\right) - \log\left(\frac{x_{n+2}\cdot x_n}{x_{n+1}^2}\right)}{n+1-n}$$

$$= \frac{1}{6}\lim_{n\to\infty}\log\left(\frac{x_{n+3}\cdot x_{n+1}^3}{x_{n+2}^3\cdot x_n}\right) = \frac{1}{6}\log(\pi) = \log\left(\sqrt[6]{\pi}\right) \Rightarrow$$

$$\log(\Omega) = \log\left(\sqrt[6]{\pi}\right) \Rightarrow \Omega = \sqrt[6]{\pi}$$

**UP.256. If $m, n \in \mathbb{N}; a, b, c > 0$ then:**

$$\left(m + (a+b)^{m+1}\right)\left(n + \frac{1}{c^{n+1}}\right) + \left(m + (b+c)^{m+1}\right)\left(n + \frac{1}{a^{n+1}}\right)$$
$$+ \left(m + (c+a)^{m+1}\right)\left(n + \frac{1}{b^{n+1}}\right) \ge 6(m+1)(n+1)$$

***Proposed by D.M.Bătineţu-Giurgiu, Daniel Sitaru-Romania***

*Solution by George Florin Şerban-Romania*

$$m+(a+b)^{m+1}\ge\underbrace{1+1+\dots+1}_{m\ timmes}+(a+b)^{m+1}\overset{Am-Gm}{\ge}$$

$$\ge(m+1)\cdot\sqrt[m+1]{\underbrace{1\cdot1\cdot\ldots\cdot1}_{m\ timmes}\cdot(a+b)^{m+1}}=(m+1)\cdot\sqrt[m+1]{(a+b)^{m+1}}=(m+1)(a+b)$$

$$n+\frac{1}{c^{n+1}}=\underbrace{1+1+\dots+1}_{n\ timmes}+\frac{1}{c^{n+1}}\overset{Am-Gm}{\ge}(n+1)\cdot\sqrt[n+1]{\underbrace{1\cdot1\cdot\ldots\cdot1}_{n\ timmes}\cdot\frac{1}{c^{n+1}}}$$

$$=(n+1)\cdot\sqrt[n+1]{\frac{1}{c^{n+1}}}=\frac{n+1}{c}$$

$$\sum_{cyc}(m+(a+b)^{m+1})\left(n+\frac{1}{c^{n+1}}\right)\ge\sum_{cyc}(m+1)(a+b)\cdot\frac{(n+1)}{c}$$

$$=(m+1)(n+1)\sum_{cyc}\frac{a+b}{c}=(m+1)(n+1)\left(\frac{a}{b}+\frac{b}{a}+\frac{c}{b}+\frac{b}{c}+\frac{c}{a}+\frac{a}{c}\right)$$

$$\overset{Am-Gm}{\ge}(m+1)(n+1)\cdot6\cdot\sqrt[6]{\frac{a}{b}\cdot\frac{b}{a}\cdot\frac{c}{b}\cdot\frac{b}{c}\cdot\frac{c}{a}\cdot\frac{a}{c}}=6(m+1)(n+1)$$

**UP.257. If $A\in M_6(\mathbb{R})$ such that**

$$det(A^4+pA^2+p^2I_6)=det(A^2+qI_6)=0,p,q\in\mathbb{R}.$$

**Find:** $\Omega=det(A)$

*Proposed by Marian Ursărescu-Romania*

*Solution by proposer*

*Let:* $f(x)=x^4+px^2+p^2;g(x)=x^2+q;f,g\in\mathbb{R}[x]$

$P_A$ *–the characteristic polynomial of matrix* $A$.

*We must show that:* $(f,P_A)\neq1$.

*Suppose that:* $(f,P_A)=1\Rightarrow\exists u,v\in\mathbb{R}[x]$ *such that:* $f(x)u(x)+P_A(x)v(x)=1\Rightarrow$

$f(A)u(A)+P_A(A)v(A)=I_6\Rightarrow I_6=O_6$ *absurd.*

*Similary* $(f,g)\neq1\Rightarrow f/P_A$ *and* $g/P_A$*, for degree* $P_A=6$

*But* $(f,g)=1\Rightarrow P_A(x)=f(x)\cdot g(x)\Rightarrow det(A)=P_A(0)=f(0)\cdot g(0)=p^2q$

**UP.258. If $m \geq 0; x, y > 0$ then in $\Delta ABC$ the following relationship holds:**

$$\frac{a^{3m+4}}{(ax+by)^m}+\frac{b^{3m+4}}{(bx+cy)^m}+\frac{c^{3m+4}}{(cx+ay)^m}\geq\frac{4^{m+2}\cdot F^{m+2}}{(\sqrt{3})^m\cdot(x+y)^m}$$

***Proposed by D.M.Bătineţu-Giurgiu, Daniel Sitaru-Romania***

***Solution by Tran Hong-Dong Thap-Vietnam***

***For*** $x, y > 0, m \geq 0$

$$(ax+by)(bx+cy)(cx+ay)\overset{Am-Gm}{\leq}\frac{(ax+by+bx+cy+cx+ay)^3}{3^3}$$

$$=\frac{(a+b+c)^3(x+y)^3}{3^3}\Rightarrow$$

$$[(ax+by)(bx+cy)(cx+ay)]^m\leq\frac{)a+b+c)^{3m}(x+y)^{3m}}{3^{3m}}\ldots(1)$$

***Now,*** $\frac{a^{3m+4}}{(ax+by)^m}+\frac{b^{3m+4}}{(bx+cy)^m}+\frac{c^{3m+4}}{(cx+ay)^m}\overset{Am-Gm}{\geq}3\cdot\sqrt[3]{\frac{(abc)^{3m+4}}{[(ax+by)(bx+cy)(cx+ay)]^m}}$

$$\overset{(1)}{\geq}3\cdot\sqrt[3]{\frac{3^{3m}\cdot(abc)^{3m+4}}{(a+b+c)^{3m}(x+y)^{3m}}}$$

***We must show that:*** $3\cdot\sqrt[3]{\frac{3^{3m}\cdot(abc)^{3m+4}}{(a+b+c)^{3m}(x+y)^{3m}}}\geq\frac{4^{m+2}\cdot F^{m+2}}{(\sqrt{3})^m\cdot(x+y)^m}$

$$\Leftrightarrow\frac{3^{3m+3}\cdot(abc)^{3m+4}}{(a+b+c)^{3m}(x+y)^{3m}}\geq\frac{4^{3(m+2)}\cdot F^{3(m+2)}}{(\sqrt{3})^{3m}\cdot(x+y)^{3m}}$$

$$\Leftrightarrow\frac{3^{3m+3}\cdot(4RF)^{3m+4}}{(2s)^{3m}}\geq\frac{4^{3(m+2)}\cdot F^{3(m+2)}}{(3\sqrt{3})^m}$$

$$\Leftrightarrow 3^{3m+3}\cdot(3\sqrt{3})^m\cdot(4R)^{3m+4}\geq(2s)^{3m}\cdot 4^{3(m+2)}\cdot s^2r^2$$

$$\Leftrightarrow 3^3\cdot(3^4\cdot\sqrt{3})^m\cdot R^{3m+4}\geq 4^2\cdot(2s)^{3m}\cdot s^2r^2\ldots(*)$$

$$\therefore s\leq\frac{3\sqrt{3}}{2}R\Rightarrow 2s\leq 3\sqrt{3}R\Rightarrow(2s)^{3m}\leq(3\sqrt{3}R)^{3m}=(3\sqrt{3})^{3m}R^{3m}\ldots(1)$$

$$\therefore s\leq\frac{3\sqrt{3}}{2}R\Rightarrow s^2\leq\frac{3^3}{4}R^2\ldots(2)$$

$$\therefore r \le \frac{R}{2} \Rightarrow r^2 \le \frac{R^2}{4} \dots (3)$$

$$\overset{(1),(2),(3)}{\Longrightarrow} (2s)^{3m} \cdot s^2 \cdot r^2 \le \left(3\sqrt{3}\right)^{3m} \cdot \frac{3^3}{4} \cdot \frac{1}{4} \cdot R^{3m} \cdot R^2 \cdot R^2$$

$$\Leftrightarrow 4^2 \cdot (2s)^{3m} \cdot s^2 \cdot r^2 \le 3^3 \cdot \left(3^4\sqrt{3}\right)^m \cdot R^{3m+4} \Rightarrow (*) true. Proved.$$

**UP.259. If $0; a \le b; n \in \mathbb{N}; n \ge 1$ then:**

$$\left(\int_0^{\sqrt{ab}} x^n e^{x^2} dx\right)\left(\int_0^{\frac{a+b}{2}} x^{n-1} e^{x^2} dx\right) \le \left(\int_0^{\sqrt{ab}} x^{n-1} e^{x^2} dx\right)\left(\int_0^{\frac{a+b}{2}} x^n e^{x^2} dx\right)$$

***Proposed by Daniel Sitaru-Romania***

***Solution 1 by Tran Hong-Dong Thap-Vietnam***

$$0 < a \le b \Rightarrow \sqrt{ab} \le \frac{a+b}{2}. \textit{ Put:}$$

$$Let: \varphi(x) = \frac{\int_0^x t^n e^{t^2} dt}{\int_0^x t^{n-1} e^{t^2} dt}; x > 0$$

$$\varphi'(x) = \frac{\left(x^n e^{x^2} \int_0^x t^{n-1} e^{t^2} dt\right) - \left(x^{n-1} e^{x^2} \int_0^x t^n e^{t^2} dt\right)}{\left(\int_0^x t^{n-1} e^{t^2} dt\right)^2}$$

$$= \frac{x^{n-1} e^{x^2}\left(x \int_0^x t^{n-1} e^{t^2} dt - \int_0^x t^n e^{t^2} dt\right)}{\left(\int_0^x t^{n-1} e^{t^2} dt\right)^2} = f(x)$$

$$g(x) = x \int_0^x t^{n-1} e^{t^2} dt - \int_0^x t^n e^{t^2} dt, x > 0$$

$$g'(x) = \int_0^x t^{n-1} e^{t^2} dt + x \cdot x^{n-1} \cdot e^{x^2} - x^n e^{x^2} = \int_0^x t^{n-1} e^{t^2} dt > 0, \forall t \in (0; x)$$

$$\Rightarrow g(x) > 0 \Rightarrow f(x) > 0, \forall x > 0 \Rightarrow \varphi(x) \uparrow (0, \infty) \Rightarrow \varphi(\sqrt{ab}) \le \varphi\left(\frac{a+b}{2}\right)$$

$$\Rightarrow \frac{\int_0^{\sqrt{ab}} t^n e^{t^2} dt}{\int_0^{\sqrt{ab}} t^{n-1} e^{t^2} dt} \le \frac{\int_0^{\frac{a+b}{2}} t^n e^{t^2} dt}{\int_0^{\frac{a+b}{2}} t^{n-1} e^{t^2} dt}$$

**Solution 2 by Sergio Esteban-Argentina**

*Consider the following*

$$\varphi = \int_0^{\sqrt{ab}} \left[ \int_{\sqrt{ab}}^{\frac{a+b}{2}} x^n \cdot e^{x^2} \cdot y^n \cdot e^{y^2} \cdot \left(\frac{1}{y} - \frac{1}{x}\right) dx \right] dy \ge 0$$

*With* $x \in \left[\sqrt{ab}; \frac{a+b}{2}\right]$ *and* $y \in [0; \sqrt{ab}]$

*We can notice that* $f(\tau) = \tau^n \cdot e^{\tau^2}$ *is positive, because* $f'(\tau) > 0$ *then f is increasing*

$f(0) = 0$*. Then:* $x^n \cdot e^{x^2} \cdot y^n \cdot e^{y^2} > 0$*.*

*Since* $0 < a \le \sqrt{ab} \le \frac{a+b}{2}$*, then* $x \ge y$ *and* $\frac{1}{x} \le \frac{1}{y}$

$\therefore \varphi \ge 0$ *finally, expanding* $\varphi$ *we obtain*

$$\varphi = \int_0^{\sqrt{ab}} \int_{\sqrt{ab}}^{\frac{a+b}{2}} \left(x^n \cdot e^{x^2} \cdot y^{n-1} \cdot e^{y^2} - y^n \cdot e^{y^2} \cdot x^{n-1} \cdot e^{x^2}\right) dxdy$$

$$= \int_{\sqrt{ab}}^{\frac{a+b}{2}} x^n \cdot e^{x^2} dx \cdot \int_0^{\sqrt{ab}} y^{n-1} \cdot e^{y^2} dy - \int_0^{\sqrt{ab}} y^n \cdot e^{y^2} dy \cdot \int_{\sqrt{ab}}^{\frac{a+b}{2}} x^{n-1} \cdot e^{x^2} dx$$

$$= \int_{\sqrt{ab}}^{\frac{a+b}{2}} x^n \cdot e^{x^2} dx \cdot \int_0^{\sqrt{ab}} x^{n-1} \cdot e^{x^2} dx - \int_0^{\sqrt{ab}} x^n \cdot e^{x^2} dx \cdot \int_{\sqrt{ab}}^{\frac{a+b}{2}} x^{n-1} \cdot e^{x^2} dx \ge 0$$

**UP.260. If** $0 < a \le b < \frac{\pi}{2}$ **then:**

$$\left(\int_0^{\sqrt{ab}} \left(\frac{\sin t}{1+e^t}\right) dt\right)\left(\int_0^{\frac{a+b}{2}} \left(\frac{\cos t}{1+e^t}\right) dt\right) \le \left(\int_0^{\sqrt{ab}} \left(\frac{\cos t}{1+e}\right) dt\right)\left(\int_0^{\frac{a+b}{2}} \left(\frac{\sin t}{1+e^t}\right) dt\right)$$

***Proposed by Daniel Sitaru – Romania***

***Solution by Sergio Esteban-Argentina***

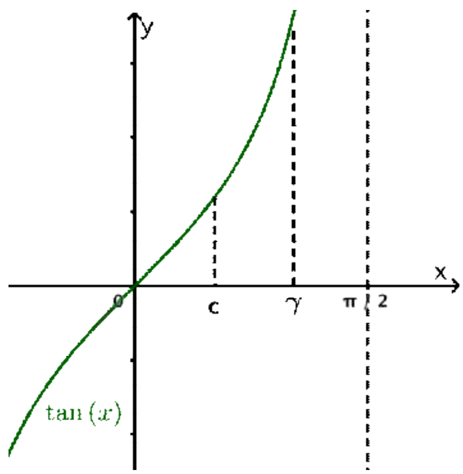
*Sea* $f, g: \left(0, \frac{\pi}{2}\right) \to \mathbb{R}, f(T) = \frac{\sin\tau}{(1+e^T)}$, *y* $g(\tau) = \frac{\cos\tau}{(1+e^t)}$. *Sabemos que* $\frac{f(T)}{g(T)} = \tan\tau$

*la desigualdad puede set escrita como:* $\frac{\int_0^{\sqrt{ab}} f(T)d\tau}{\int_0^{\sqrt{ab}} g(\tau)d\tau} \le \frac{\int_{\sqrt{ab}}^{\frac{a+b}{2}} f(T)d\tau}{\int_{\sqrt{ab}}^{\frac{a+b}{2}} g(\tau)d\tau}$

*Por el TVM de Cauchy:* $\exists c \in (0, \sqrt{ab})$ *tal que* $\frac{\int_0^{\sqrt{ab}} f(\tau)d\tau}{\int_0^{\sqrt{ab}} g(\tau)d\tau} = \frac{f(c)}{g(c)} = \tan c$

*y tambien* $\exists\, \gamma \in \left(\sqrt{ab}, \frac{a+b}{2}\right)$ *tal que* $\frac{f(T)\, d\tau}{\int_{\sqrt{ab}}^{\frac{a+b}{2}} g(T)d\tau} = \frac{f(\gamma)}{g(\gamma)} = \tan\gamma$

***Sabemos que:***




*Es creciente de* $\left(0, \frac{\pi}{2}\right) \therefore \tan c < \tan\gamma$

**UP.261. If $0 < a \le b < \frac{\pi}{2}$ then:**

$$\left(\int_a^{\sqrt{ab}} e^{-x^2} sinxdx\right)\left(\int_{\frac{a+b}{2}}^{b} e^{-x^2} cosxdx\right) \le \left(\int_a^{\sqrt{ab}} e^{-x^2} cosxdx\right)\left(\int_{\frac{a+b}{2}}^{b} e^{-x^2} sinxdx\right)$$

***Proposed by Daniel Sitaru-Romania***

***Solution by proposer***

*By Cauchy's theorem* $\exists c \in [a, \sqrt{ab}]; d \in \left[\frac{a+b}{2}, b\right]$ *such that:*

$$\frac{\int_a^{\sqrt{ab}} e^{-x^2} sinxdx}{\int_a^{\sqrt{ab}} e^{-x^2} cosxdx} = \frac{sinc}{cosc} = tanc$$

$$\frac{\int_{\frac{a+b}{2}}^{b} e^{-x^2} sinxdx}{\int_{\frac{a+b}{2}}^{b} e^{-x^2} cosxdx} = \frac{sind}{cosd} = tand$$

$$0 < a \leq c \leq \sqrt{ab} \overset{Am-Gm}{\leq} \frac{a+b}{2} \leq d \leq b < \frac{\pi}{2}, tanx - \text{increasing on } \left(0, \frac{\pi}{2}\right)$$

$$\Rightarrow tan(c) \leq tan(d)$$

$$\frac{\int_a^{\sqrt{ab}} e^{-x^2} sinxdx}{\int_a^{\sqrt{ab}} e^{-x^2} cosxdx} \leq \frac{\int_{\frac{a+b}{2}}^{b} e^{-x^2} sinxdx}{\int_{\frac{a+b}{2}}^{b} e^{-x^2} cosxdx} \Leftrightarrow$$

$$\left(\int_a^{\sqrt{ab}} e^{-x^2} sinxdx\right)\left(\int_{\frac{a+b}{2}}^{b} e^{-x^2} cosxdx\right) \leq \left(\int_a^{\sqrt{ab}} e^{-x^2} cosxdx\right)\left(\int_{\frac{a+b}{2}}^{b} e^{-x^2} sinxdx\right)$$

**UP.262. If $u, v, w: 0, x_n, y_n, z_n > 0, n \in \mathbb{N}$ sequence such that:**

$$\lim_{n\to\infty} \frac{x_{n+1}}{nx_n} = a, \lim_{n\to\infty} \frac{y_{n+1}}{ny_n} = b, \lim_{n\to\infty} \frac{z_{n+1}}{nz_n} = c,$$

**Where *a,b,c* are sides in $\Delta ABC$ with circumradii *R* then in $\Delta ABC$ the following relationship holds:**

$$\lim_{n\to\infty} \left(\frac{1}{n^2}\left(vw\sqrt[n]{x_n^2} + wu\sqrt[n]{y_n^2} + uv\sqrt[n]{z_n^2}\right)\right) \leq \frac{(u+v+w)^2 \cdot R^2}{e^2}$$

***Proposed by D.M.Bătineţu-Giurgiu, Neculai Stanciu-Romania***

***Solution by Marian Ursărescu-Romania***

$$\lim_{n\to\infty} \frac{1}{n^2}\sqrt[n]{x_n^2} = \lim_{n\to\infty} \sqrt[n]{\frac{x_n^2}{n^{2n}}} \overset{C-D'A}{=} \lim_{n\to\infty} \frac{x_{n+1}^2}{(n+1)^{2n+2}} \cdot \frac{n^{2n}}{x_n^2}$$

$$= \lim_{n\to\infty} \frac{x_{n+1}^2}{n^2 x_n^2} \cdot \frac{n^2}{(n+1)^2} \cdot \left(\frac{n}{n+1}\right)^{2n} = a^2 \cdot 1 \cdot \frac{1}{e^2} \cdot e = \frac{a}{e^2}$$

***And similarly,*** $\lim_{n\to\infty} \frac{1}{n^2}\sqrt[n]{y_n^2} = \frac{b}{e^2}, \lim_{n\to\infty} \frac{1}{n^2}\sqrt[n]{z_n^2} = \frac{c}{e^2}$

*We must show:* $(vwa^2+wub^2+uvc^2)\le(u+v+w)^2\cdot R^2\ \ldots (1)$

*Now, use Klamkin theorem:* $\forall M\in P,\forall x,y,z\in\mathbb{R}$

$$(x+y+z)(xMA^2+yMB^2+zMC^2)\ge a^2yz+b^2zx+c^2xy\ \ldots (2)$$

*Let:* $x=u, y=v, z=w$ *and* $M=O\Rightarrow MA^2=MB^2=MC^2=R^2\overset{(2)}{\Rightarrow}$

$$vwa^2+wub^2+uvc^2\le(u+v+w)(uOA^2+vOB^2+wOC^2)\Leftrightarrow$$

$$vwa^2+wub^2+uvc^2\le(u+v+w)^2\cdot R^2\Rightarrow (1)\text{ it's true.}$$

**UP.263. If** $m>0; f,g:(0,\infty)\to(0,\infty), \lim_{x\to\infty}\frac{f(x)}{x}=a>0$,

$\lim_{x\to\infty}g(x)\cdot x^{\frac{1}{m}}=b>0$ **then find:**

$$\Omega=\lim_{x\to\infty}\left(\left(\Gamma(x+2)\right)^{\frac{1}{m(x+1)}}-\left(\Gamma(x+1)\right)^{\frac{1}{mx}}\right)f(x)g(x)$$

***Proposed by D.M.Bătineţu-Giurgiu and Neculai Stanciu-Romania***

***Solution by Marian Ursărescu-Romania***

$$\Gamma(n+1)=(n+1)!, \Gamma(n)=n!$$

*We must find:* $\Omega=\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!^{\frac{1}{m}}}-\sqrt[n]{n!^{\frac{1}{m}}}\right)f(n)g(n)$

$$=\lim_{n\to\infty}\frac{f(n)}{n}\cdot g(n)\cdot n^{\frac{1}{m}}\cdot\frac{\sqrt[n]{n!^{\frac{1}{m}}}}{n^{\frac{1}{m}}}\cdot n\cdot\left(\frac{\sqrt[n+1]{(n+1)!^{\frac{1}{m}}}}{\sqrt[n]{n!^{\frac{1}{m}}}}-1\right)\ldots (1)$$

$$\lim_{n\to\infty}\frac{f(n)}{n}=a\ and\ \lim_{n\to\infty}g(n)\cdot n^{\frac{1}{m}}=b\ \ldots (2)$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{n!}{n^n}}\overset{C-D'Alembert}{=}\lim_{n\to\infty}\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!}$$

$$=\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n=\frac{1}{e}\Rightarrow\lim_{n\to\infty}\left(\frac{\sqrt[n]{n!}}{n}\right)^{\frac{1}{m}}=\left(\frac{1}{e}\right)^{\frac{1}{m}}\ldots (3)$$

*Let:* $x_n=\frac{\sqrt[n+1]{(n+1)!^{\frac{1}{m}}}}{\sqrt[n]{n!^{\frac{1}{m}}}}\Rightarrow\lim_{n\to\infty}n(x_n-1)=\lim_{n\to\infty}n\left(e^{\log x_n}-1\right)$

$$= \lim_{n\to\infty}\frac{n\left(e^{logx_n}-1\right)}{logx_n}\cdot logx_n = \lim_{n\to\infty} nlogx_n = \lim_{n\to\infty} logx_n^n = log\left(\lim_{n\to\infty} x_n^n\right)$$

$$= log\left(\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{(n+1)!^{\frac{1}{m}}}}{\sqrt[n]{n!^{\frac{1}{m}}}}\right)^n\right) = log\left(\lim_{n\to\infty}\frac{\left(\sqrt[n+1]{(n+1)!^{\frac{1}{m}}}\right)^n}{n!^{\frac{1}{m}}}\right)$$

$$= log\left(\lim_{n\to\infty}\frac{(n+1)!^{\frac{1}{m}}}{n!^{\frac{1}{m}}\cdot\sqrt[n+1]{(n+1)!^{\frac{1}{m}}}}\right) = log\left(\lim_{n\to\infty}\frac{(n+1)^{\frac{1}{m}}}{\sqrt[n+1]{(n+1)!^{\frac{1}{m}}}}\right)$$

$$= \lim_{n\to\infty}\left(\frac{n+1}{\sqrt[n]{(n+1)!}}\right)^{\frac{1}{m}} \overset{(3)}{=} log\left(e^{\frac{1}{m}}\right) = \frac{1}{m}\ldots(4)$$

*From (1)+(2)+(3)+(4)⇒* $\Omega = a\cdot b\cdot\left(\frac{1}{e}\right)^{\frac{1}{m}}\cdot\frac{1}{m} = \frac{ab}{me^{\frac{1}{m}}}$

**UP.264. If** $r,s\geq 0;\ a_n,b_n>0; n\geq 0; n\in\mathbb{N}$

$$\lim_{n\to\infty}\frac{a_{n+1}}{n^r\cdot a_n} = a > 0;\ \lim_{n\to\infty}\frac{b_{n+1}}{n^{s+1}\cdot b_n} = b > 0$$

**then find:**

$$\Omega = \lim_{n\to\infty}\left(\left(\frac{\sqrt[n]{a_n}}{n^{r+s}} - \frac{\sqrt[n+1]{b_{n+1}}}{(n+1)^{r+s}}\right)\cdot\sqrt[n]{b_n}\right)$$

***Proposed by D.M.Bătineţu-Giurgiu, Neculai Stanciu-Romania***

***Solution 1 by Marian Ursărescu-Romania***

$$\Omega = \lim_{n\to\infty}\left(-\sqrt[n]{b_n}\cdot\frac{\sqrt[n]{a_n}}{n^{r+s}}\cdot\left(\frac{\sqrt[n+1]{a_{n+1}}}{(n+1)^{r+s}}\cdot\frac{n^{r+s}}{\sqrt[n]{a_n}} - 1\right)\right)$$

$$= -\lim_{n\to\infty}\left[\frac{\sqrt[n]{a_n}}{n^r}\cdot\frac{\sqrt[n]{b_n}}{n^{s+1}}\cdot n\left(\left(\frac{n}{n+1}\right)^{r+s}\cdot\frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}} - 1\right)\right]\ldots(1)$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{n^r} = \lim_{n\to\infty}\sqrt[n]{\frac{a_n}{n^{nr}}} \overset{C-D'Alembert}{=} \lim_{n\to\infty}\frac{a_{n+1}}{(n+1)^{(n+1)r}}\cdot\frac{n^{nr}}{a_n}$$

$$= \lim_{n\to\infty} \frac{a_{n+1}}{(n+1)^r \cdot a_n} \cdot \frac{n^{nr}}{(n+1)^{nr}}$$

$$= \lim_{n\to\infty} \frac{n^r}{(n+1)^r} \cdot \frac{a_{n+1}}{n^r \cdot a_n} \cdot \left(\left(\frac{n}{n+1}\right)^n\right)^r = \frac{a}{e^r} \dots (2)$$

$$\lim_{n\to\infty} \frac{\sqrt[n]{b_n}}{n^{s+1}} = \lim_{n\to\infty} \sqrt[n]{\frac{b_n}{n^{n(s+1)}}} \overset{C-D'Alembert}{=} \lim_{n\to\infty} \frac{b_{n+1}}{(n+1)^{(n+1)(s+1)}} \cdot \frac{n^{n(s+1)}}{b_n}$$

$$= \lim_{n\to\infty} \frac{b_{n+1}}{(n+1)^{s+1} \cdot b_n} \cdot \frac{n^{n(s+1)}}{(n+1)^{n(s+1)}}$$

$$= \lim_{n\to\infty} \frac{n^{s+1}}{(n+1)^{s+1}} \cdot \frac{b_{n+1}}{n^{s+1} \cdot b_n} \cdot \left(\left(\frac{n}{n+1}\right)^n\right)^{s+1} = \frac{b}{e^{s+1}} \dots (3)$$

***Let:*** $x_n = \left(\frac{n}{n+1}\right)^{r+s} \cdot \frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}}$

$$\Rightarrow \lim_{n\to\infty} n(x_n - 1) = \lim_{n\to\infty} \frac{n(e^{logx_n} - 1)}{logx_n} \cdot logx_n = \lim_{n\to\infty} nlogx_n = \lim_{n\to\infty} logx_n^n$$

$$= log\left(\lim_{n\to\infty} x_n^n\right) = log\left(\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{n(r+s)} \cdot \frac{\sqrt[n+1]{a_{n+1}^n}}{a_n}\right)$$

$$= log\left(\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{n(r+s)} \cdot \frac{a_{n+1}}{n^r \cdot a_n} \cdot \frac{n^r}{\sqrt[n+1]{a_{n+1}}}\right)$$

$$= log\left(\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{n(r+s)} \cdot \frac{a_{n+1}}{n^r \cdot a_n} \cdot \frac{n^r}{(n+1)^r} \cdot \frac{(n+1)^r}{\sqrt[n+1]{a_{n+1}}}\right)$$

$$= log\left(e^{-r-s} \cdot a \cdot 1 \cdot \frac{e^r}{a}\right) = log(e^{-s}) = -s \dots (4)$$

***From (1)+(2)+(3)+(4) we have*** $\Omega = -\frac{a}{e^r} \cdot \frac{b}{e^{s+1}} \cdot (-s) = \frac{abs}{e^{r+s+1}}$

***Solution 2 by Soumitra Mandal-Chandar Nagore-India***

$$\lim_{n\to\infty} \frac{\sqrt[n]{b_n}}{n^{s+1}} = \lim_{n\to\infty} \sqrt[n]{\frac{b_n}{n^{n(s+1)}}} \overset{C-D'Alembert}{=}$$

$$= \lim_{n\to\infty} \frac{b_{n+1}}{n^{s+1} \cdot b_n} \cdot \left(\frac{n}{1+n}\right)^{s+1} \cdot \frac{1}{\left(1 + \frac{1}{n}\right)^{n(s+1)}} = \frac{b}{e^{s+1}}$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{n^r}=\lim_{n\to\infty}\sqrt[n]{\frac{a_n}{n^{nr}}}\overset{C-D'Alembert}{=}\lim_{n\to\infty}\frac{a_{n+1}}{n^r\cdot a_n}\cdot\left(\frac{n}{1+n}\right)^r\cdot\frac{1}{\left(1+\frac{1}{n}\right)^{nr}}=\frac{a}{e^r}$$

$$Let: u_n=\frac{\sqrt[n+1]{a_{n+1}}}{(n+1)^{r+s}}\cdot\frac{n^{r+s}}{\sqrt[n]{a_n}}\ for\ all\ n\in\mathbb{N}$$

$$\therefore\lim_{n\to\infty}u_n=\lim_{n\to\infty}\frac{\sqrt[n+1]{a_{n+1}}}{(n+1)^{r+1}}\cdot\frac{n^r}{\sqrt[n]{a_n}}\cdot\left(\frac{1+n}{n}\right)^r\cdot\left(\frac{n}{1+n}\right)^{r+s}=1$$

$$Hence: \frac{u_n-1}{log(u_n)}\xrightarrow{n\to\infty}1$$

$$\lim_{n\to\infty}u_n^n=\lim_{n\to\infty}\frac{a_{n+1}}{n^r\cdot b_n}\cdot\frac{(1+n)^r}{\sqrt[n]{a_{n+1}}}\cdot\left(\frac{n}{1+n}\right)^r\cdot\frac{1}{\left(1+\frac{1}{n}\right)^{n(r+s)}}=a\cdot\frac{e^r}{a}\cdot\frac{1}{e^{r+s}}=e^{-s}$$

$$\Omega=-\lim_{n\to\infty}\frac{\sqrt[n]{b_n}}{n^{s+1}}\cdot\frac{\sqrt[n]{a_n}}{n^r}\cdot\frac{u_n-1}{log(u_n)}\cdot log(u_n^n)=s\cdot\frac{ab}{e^{r+s+1}}$$

**UP.265. If $r,s\geq 0; a_n, b_n>0; n\geq 0; n\in\mathbb{N}$;**

$$\lim_{n\to\infty}\frac{a_{n+1}}{n^r\cdot a_n}=a>0;\ \lim_{n\to\infty}\frac{b_{n+1}}{n^s\cdot b_n}=b>0\ \text{ then find:}$$

$$\Omega=\lim_{n\to\infty}\left(\left(a_{n+1}^{\frac{n^s}{(n+1)\sqrt[n]{b_n}}}-a_n^{\frac{n^{s-1}}{\sqrt[n]{b_n}}}\right)\cdot n^{1-\frac{r\cdot n^s}{\sqrt[n]{b_n}}}\right)$$

***Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania***

***Solution 1 by Remus Florin Stanca-Romania***

$$\Omega=\lim_{n\to\infty}a_n^{\frac{n^s}{n\sqrt[n]{b_n}}}\cdot n^{1-\frac{rn^s}{\sqrt[n]{b_n}}}\left(\left(\frac{a_{n+1}^{\frac{1}{n+1}}}{a_n^{\frac{1}{n}}}\right)^{\frac{n^s}{\sqrt[n]{b_n}}}-1\right)=\lim_{n\to\infty}n\cdot\left(\frac{a_n^{\frac{1}{n}}}{n^r}\right)^{\frac{n^s}{\sqrt[n]{b_n}}}\cdot\frac{n^s}{\sqrt[n]{b_n}}\cdot\ln\left(\frac{a_{n+1}^{\frac{1}{n+1}}}{a_n^{\frac{1}{n}}}\right)$$

***(1)***

$$\lim_{n\to\infty}\frac{n^s}{\sqrt[n]{b_n}}=\lim_{n\to\infty}\sqrt[n]{\frac{n^{ns}}{b_n}}=\lim_{n\to\infty}\frac{(n+1)^{(n+1)s}}{b_{n+1}}\cdot\frac{b_n}{n^{ns}}=e^s\cdot\frac{1}{b}=\frac{e^s}{b}\qquad (2)$$

$$\lim_{n\to\infty}\frac{a_n^{\frac{1}{n}}}{n^r}=\lim_{n\to\infty}\sqrt[n]{\frac{a_n}{n^{nr}}}=\lim_{n\to\infty}\frac{a_{n+1}}{(n+1)^{(n+1)r}}\cdot\frac{a^{nr}}{a_n}=\frac{a}{e^r}\quad (3)$$

$$\overset{(2);(3)}{\Rightarrow}\lim_{n\to\infty}\left(\frac{a_n^{\frac{1}{n}}}{n^r}\right)^{\frac{n^s}{\sqrt[n]{b_n}}}=\left(\frac{a}{e^r}\right)^{\frac{e^s}{b}}\quad (4)$$

$$\overset{(1);(2);(4)}{\Rightarrow}\Omega=\frac{e^s}{b}\cdot\left(\frac{a}{e^r}\right)^{\frac{e^s}{b}}\cdot\lim_{n\to\infty}\ln\left(\frac{a_{n+1}^{\frac{n}{n+1}}}{a_n}\right)=\frac{e^s}{b}\cdot\left(\frac{a}{e^r}\right)^{\frac{e^s}{b}}\cdot\lim_{n\to\infty}\ln\left(\frac{a_n+1}{a_n}\cdot\frac{1}{\sqrt[n+1]{a_{n+1}}}\right)=$$

$$=\frac{e^s}{b}\cdot\left(\frac{a}{e^r}\right)^{\frac{e^s}{b}}\cdot\lim_{n\to\infty}\ln\left(\frac{a_{n+1}}{n^r a_n}\cdot\frac{(n+1)^r}{\sqrt[n+1]{a_{n+1}}}\right)=\frac{e^s}{b}\cdot\left(\frac{a}{e^r}\right)^{\frac{e^s}{b}}\cdot\ln\left(a\cdot\lim_{n\to\infty}\left(\frac{(n+1)^{(n+1)r}}{a_{n+1}}\cdot\frac{a_n}{n^{nr}}\right)\right)=$$

$$=\frac{e^s}{b}\cdot\left(\frac{a}{e^r}\right)^{\frac{e^s}{b}}\cdot\ln\left(a\cdot\frac{e^r}{a}\right)=r\cdot\frac{e^s}{b}\cdot\left(\frac{a}{e^r}\right)^{\frac{e^s}{b}}\Rightarrow\Omega=r\cdot\frac{e^s}{b}\cdot\left(\frac{a}{e^r}\right)^{\frac{e^s}{b}}$$

***Solution 2 by Soumitra Mandal-Chandar Nagore-India***

$$\lim_{n\to\infty}\frac{a_{n+1}}{n^r a_n}=a \textit{ and } \lim_{n\to\infty}\frac{b_{n+1}}{n^s b_n}=b$$

$$\lim_{x\to\infty}\frac{x^s}{\sqrt[x]{b_x}}=\lim_{\substack{x\to\infty\\ n\in\mathbb{N}}}\frac{n^s}{\sqrt[n]{b_n}}=\lim_{x\to\infty}\frac{n^s}{\sqrt[n]{b_n}}=\lim_{x\to\infty}\sqrt[n]{\frac{n^{ns}}{b_n}}\overset{\substack{CAUCHY\\ D'ALEMBERT}}{=}$$

$$=\lim_{x\to\infty}\left(\frac{n^s\cdot b_n}{b_n+1}\cdot\left(1+\frac{1}{n}\right)^{ns}\cdot\left(1+\frac{1}{n}\right)^s\right)=\frac{e^s}{b}.\ \textit{Again}$$

$$\lim_{x\to\infty}\frac{\sqrt[n]{a_n}}{n^r}=\lim_{x\to\infty}\sqrt[n]{\frac{a_n}{n^{nr}}}\overset{\substack{CAUCHY\\ D'ALEMBERT}}{=}\lim_{n\to\infty}\frac{a_{n+1}}{n^r a_n}\cdot\frac{1}{\left(1+\frac{1}{n}\right)^{nr}}\cdot\left(\frac{n}{1+n}\right)^r=\frac{a}{e^r}$$

$$\textit{Let } u_n=\left(\frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}}\right)^{\frac{x^s}{\sqrt[x]{b_x}}} \textit{ for all } n\in\mathbb{N}^*$$

$$\textit{then } \lim_{n\to\infty}u_n=\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{a_{n+1}}}{(n+1)^r}\cdot\frac{n^r}{\sqrt[n]{a_n}}\left(1+\frac{1}{n}\right)^r\right)^{\frac{x^s}{\sqrt[x]{b_x}}}=1 \textit{ hence } \frac{u_n-1}{\ln u_n}\to 1 \textit{ when}$$

$$u_n\to 1$$

$$\therefore\lim_{n\to\infty}u_n^n=\lim_{n\to\infty}\left(\frac{a_{n+1}}{n^r a_n}\cdot\frac{(n+1)^r}{\sqrt[n+1]{a_{n+1}}}\cdot\left(\frac{n}{n+1}\right)^r\right)^{\frac{x^s}{\sqrt[x]{b_x}}}=e^{\frac{rx^s}{\sqrt[x]{b_x}}}$$

$$\lim_{n\to\infty}\left(\left(a_{n+1}^{\frac{n^s}{(n+1)\sqrt[n]{b_n}}}-a_n^{\frac{n^{s-1}}{\sqrt[n]{b_n}}}\right)n^{1-\frac{rn^s}{n\sqrt{b_n}}}\right)=\lim_{n\to\infty}\left(\lim_{n\to\infty}\left(a_{n+1}^{\frac{x^s}{(n+1)\sqrt[x]{bx}}}-a_n^{\frac{x^s}{n\sqrt[x]{b_x}}}\right)\cdot n^{1-\frac{x^s}{\sqrt[x]{b_x}}}\right)$$

$$=\lim_{n\to\infty}\left(\lim_{n\to\infty}\left(\frac{\sqrt[n]{a_n}}{n^r}\right)^{\frac{x^s}{\sqrt[x]{b_x}}}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n\right)$$

$$=\lim_{x\to\infty}\left(r^{\frac{x^s}{\sqrt[x]{b_x}}}\left(\frac{a}{e^r}\right)^{\frac{x^s}{\sqrt[x]{b_x}}}\right)=r^{\frac{e^s}{b}}\left(\frac{a}{e^r}\right)^{\frac{e^s}{b}}\ \textit{(Answer)}$$

**UP.266. If $m\ge 0; x,y,z>0$ then in $\Delta ABC$ the following relationship holds:**

$$\left(\frac{a^2x}{y+z}\right)^{m+1}+\left(\frac{b^2y}{z+x}\right)^{m+1}+\left(\frac{c^2z}{x+y}\right)^{m+1}\ge 2^{m-1}\cdot\left(\sqrt{3}\right)^{1-m}\cdot F^{m+1}$$

***Proposed by D.M.Bătineţu-Giurgiu, Neculai Stanciu-Romania***

***Solution by Tran Hong-Dong Thap-Vietnam***

$$LHS=\left(\frac{a^2x}{y+z}\right)^{m+1}+\left(\frac{b^2y}{z+x}\right)^{m+1}+\left(\frac{c^2z}{x+y}\right)^{m+1}\ge\frac{\left(\frac{a^2x}{y+z}+\frac{b^2y}{z+x}+\frac{c^2z}{x+y}\right)^{m+1}}{3^m}=\Phi$$

$$Let:\Omega=\frac{a^2x}{y+z}+\frac{b^2y}{z+x}+\frac{c^2z}{x+y}$$

$$\overset{Weizenbock}{\ge}4\cdot\sqrt{\frac{xy}{(z+y)(z+x)}+\frac{yz}{(x+z)(x+y)}+\frac{xz}{(y+z)(y+x)}}\cdot F$$

$$\sum_{cyc}\frac{xy}{(z+y)(z+x)}\ge\frac{3}{4}\dots(1)$$

$$\Leftrightarrow 4[xy(x+y)+yz(y+z)+zx(x+z)]\ge 3(x+y)(y+z)(z+x)$$

$$\Leftrightarrow yx^2+yz^2+zx^2+xy^2+zy^2+xz^2\ge 6xyz\ \ \textit{(true from Am-Gm)}$$

$$\Rightarrow(1)true\Rightarrow\Omega\ge 4\cdot\sqrt{\frac{3}{4}}\cdot F=2\sqrt{3}\cdot F$$

$$\Rightarrow \Phi \ge \frac{\left(2\sqrt{3}\cdot F\right)^{m+1}}{3^m} = 2^{m+1}\cdot\frac{3^{\frac{m+1}{2}}}{3^m}\cdot F^{m+1} = 2^{m+1}\cdot\left(\sqrt{3}\right)^{1-m}\cdot F^{m+1}$$

$$\Rightarrow LHS \ge 2^{m+1}\cdot\left(\sqrt{3}\right)^{1-m}\cdot F^{m+1}. Proved.$$

**UP.267. If $m \in \mathbb{N}; m \ge 1; x, y, z > 0$ then in $\Delta ABC$ the following relationship holds:**

$$m\cdot\sum_{cyc}\frac{x}{y+z} + \sum_{cyc}\frac{x\cdot a^{4(m+1)}}{y+z} \ge 8(m+1)F^2$$

***Proposed by D.M.Bătineţu-Giurgiu-Romania***

***Solution by Marian Ursărescu-Romania***

*We must show:* $\sum_{cyc}\frac{x\cdot a^{4(m+1)}}{y+z} \ge 8(m+1)F^2 \dots (1)$

$$m + a^{4(m+1)} = \underbrace{1+1+\dots+1}_{m} + a^{4(m+1)}$$

$$\ge (m+1)\sqrt[m+1]{a^{4(m+1)}} = (m+1)a^4 \dots (2)$$

*From (1)+(2) we must show:* $\sum_{cyc}\frac{x\cdot a^4}{y+z} \ge 8F^2 \dots (3)$

$$\sum_{cyc}\frac{x\cdot a^4}{y+z} = \sum_{cyc}\frac{(xa^2)^2}{xy+xz} \overset{Bergstrom}{\ge} \frac{(xa^2+yb^2+zc^2)^2}{2(xy+yz+zx)} \dots (4)$$

*For Oppenheimer inequality, we have:* $xa^2+yb^2+zc^2 \ge 4F\sqrt{xy+yz+zx}$

$$(xa^2+yb^2+zc^2)^2 \ge 16F^2(xy+yz+zx) \dots (5)$$

*From (4)+(5) we have* $\sum_{cyc}\frac{x\cdot a^4}{y+z} \ge 8F^2 \Rightarrow$ *(3) it's true*

*Observation: Oppenheimer inequality is also known as Klamkin inequality.*

**UP.268. In $\Delta ABC$ the following relationship holds:**

$$20a^4 + 5b^4 + 2c^4 \ge 80F^2$$

***Proposed by D.M.Bătineţu-Giurgiu-Romania***

***Solution 1 by Tran Hong-Dong Thap-Vietnam***

$$LHS = 20a^4 + 5b^4 + 2c^4 \overset{Am-Gm}{\geq} 3\sqrt[3]{20a^4 \cdot 5b^4 \cdot 2c^4} = 3\sqrt[3]{200(4Rrs)^4}$$

***We need to prove:***

$$3\sqrt[3]{200(4Rrs)^4} \geq 80F^2 = 80(sr)^2 \Leftrightarrow 3^3 \cdot 200 \cdot 4^4 \cdot (Rrs)^4 \geq [80(sr)^2]^3 \Leftrightarrow$$

$$R^4 \geq \frac{10}{27} r^2 s^2 \ldots (*)$$

**But** $R \geq 2r \Rightarrow R^2 \geq 4r^2$

$$R \geq \frac{2}{3\sqrt{3}} s \Rightarrow R^2 \geq \frac{4}{27} s^2 \Rightarrow R^4 \geq \frac{16}{27} s^2 r^2 \geq \frac{10}{27} s^2 r^2$$

***Solution 2 by Marian Dincă-Romania***

$$Let: b^2 + c^2 - a^2 = 2x;\ c^2 + a^2 - b^2 = 2y;\ b^2 + a^2 - c^2 = 2z \Rightarrow$$

$$a^2 = y + z;\ b^2 = x + z;\ c^2 = x + y$$

$$20a^4 + 5b^4 + 2c^4 = 20(y+z)^2 + 5(x+z)^2 + 2(x+y)^2$$

$$= 7x^2 + 22y^2 + 25z^2 + 4xy + 10xz + 40yz$$

$$16F^2 = \sum_{cyc} a^2(b^2 + c^2 - a^2) = \sum_{cyc} (y+z)x = 2(xy + yz + zx) \Leftrightarrow$$

$$80F^2 = 10(xy + yz + zx)$$

$$7x^2 + 22y^2 + 25z^2 + 4xy + 10xz + 40yz \geq 10(xy + yz + zx)$$

$$7x^2 + 22y^2 + 25z^2 - 6xy + 30yz \geq 0$$

$$22y^2 - 2y(3x - 15z) + 7x^2 + 25z^2 \geq 0$$

***Discriminant to variable y:*** $\Delta = (3x - 15z)^2 - 22(7x^2 + 25z^2)$

$$= -(5x + 9z)^2 - 120x^2 - 244z^2 < 0 \Rightarrow$$

$$22y^2 - 2y(3x - 15z) + 7x^2 + 25z^2 > 0$$

**UP.269. If** $t \geq 0;\ a_n > 0; n \in \mathbb{N}; n \geq 1;\ \lim_{n\to\infty} \frac{a_n}{n^t} = \pi;\ \lim_{n\to\infty} \left(\frac{a_{n+1}}{a_n}\right)^n = e$

**then find:**

$$\Omega = \lim_{n\to\infty} \left( \frac{(n+1)^{t+1}}{a_{n+1}} - \frac{n^t}{a_n} \right)$$

***Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru-Romania***

*Solution by proposers*

$$\lim_{n\to\infty}\left(\left(\frac{n+1}{n}\right)^{t+1}\cdot\frac{a_{n+1}}{a_n}\right)^n=\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{t+1}\cdot\left(\frac{a_{n+1}}{a_n}\right)^n=\frac{e^{t+1}}{e}$$

$$RHS=\frac{(n+1)^{t+1}}{a_{n+1}}-\frac{n^{t+1}}{a_n}=\frac{n^{t+1}}{a_n}\left(\left(\frac{n+1}{n}\right)^{t+1}\cdot\frac{a_n}{a_{n+1}}-1\right)=\frac{n^{t+1}(u_n-1)}{a_n}$$

$$=\frac{n^{t+1}}{a_n}\cdot\frac{u_n-1}{logu_n}\cdot logu_n=\frac{n^t}{a_n}\cdot\frac{u_n-1}{logu_n}\cdot logu_n^n;n\geq 2$$

*where:* $u_n=\left(\frac{n+1}{n}\right)^{t+1}\cdot\frac{a_n}{a_{n+1}}=\frac{(n+1)^t}{a_{n+1}}\cdot\frac{a_n}{n^t}\cdot\frac{n+1}{e}$

$$\lim_{n\to\infty}u_n=\frac{1}{a}\cdot a\cdot 1=1;\ \lim_{n\to\infty}\frac{u_n-1}{logu_n}=1;\ \lim_{n\to\infty}u_n^n=\frac{e^{t+1}}{e}$$

$$\Omega=\lim_{n\to\infty}\left(\frac{(n+1)^{t+1}}{a_{n+1}}-\frac{n^t}{a_n}\right)=\frac{1}{\pi}\cdot 1\cdot log\frac{e^{t+1}}{e}=\frac{t+1-loge}{\pi}=\frac{t}{\pi}$$

**UP.270. If $a,b,c,x,y>0$ then:**

$$a\sqrt{\frac{a(x+y)}{bx+cy}}+b\sqrt{\frac{b(x+y)}{cx+ay}}+c\sqrt{\frac{c(x+y)}{ax+by}}\geq a+b+c$$

***Proposed by D.M.Bătineţu-Giurgiu-Romania***

*Solution 1 by George Florin Şerban -Romania*

$$\sqrt{x}=\sqrt{x\cdot 1}\overset{Gm-Hm}{\geq}\frac{2x\cdot 1}{x+1}\Rightarrow\sqrt{\frac{a(x+y)}{bx+cy}}\geq\frac{\frac{2a(x+y)}{bx+cy}}{\frac{a(x+y)}{bx+cy}+1}=\frac{2a(x+y)}{(a+b)x+(a+c)y}$$

$$a\sqrt{\frac{a(x+y)}{bx+cy}}\geq\frac{2a^2(x+y)}{(a+b)x+(a+c)y}$$

$$\sqrt{\frac{b(x+y)}{cx+ay}}\geq\frac{\frac{2b(x+y)}{cx+ay}}{\frac{b(x+y)}{cx+ay}+1}=\frac{2b(x+y)}{(b+c)x+(a+b)y}$$

$$b\sqrt{\frac{b(x+y)}{cx+ay}}\ge\frac{2b^2(x+y)}{(b+c)x+(a+b)y}$$

$$\sqrt{\frac{c(x+y)}{ax+by}}\ge\frac{\frac{2c(x+y)}{ax+by}}{\frac{c(x+y)}{ax+by}+1}=\frac{2c(x+y)}{(a+c)x+(b+c)y}$$

$$c\sqrt{\frac{c(x+y)}{ax+by}}\ge\frac{2c^2(x+y)}{(a+c)x+(b+c)y}\Rightarrow\sum_{cyc}a\sqrt{\frac{a(x+y)}{bx+cy}}\ge\sum_{cyc}\frac{2a^2(x+y)}{(a+b)x+(a+c)y}=$$

$$=2(x+y)\sum_{cyc}\frac{a^2}{(a+b)x+(a+c)y}\overset{Bergstrom}{\geq}(2x+2y)\frac{(a+b+c)^2}{\sum_{cyc}((a+b)x+(a+c)y)}$$

$$\ge\frac{(2x+2y)(a+b+c)^2}{(a+b+c)(2x+2y)}=a+b+c$$

***Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand***

$$a\sqrt{\frac{a(x+y)}{bx+cy}}+b\sqrt{\frac{b(x+y)}{cx+ay}}+c\sqrt{\frac{c(x+y)}{ax+by}}\ge$$

$$\ge\sqrt{x+y}\cdot\left(\frac{a^2}{\sqrt{abx+acy}}+\frac{b^2}{\sqrt{bcx+aby}}+\frac{c^2}{\sqrt{acx+bcy}}\right)$$

$$\ge\frac{(a+b+c)(a+b+c)\sqrt{x+y}}{\sqrt{abx+acy}+\sqrt{bcx+aby}+\sqrt{acx+bcy}}\ge a+b+c$$

$$Iff\ \sqrt{x+y}(a+b+c)\ge\sqrt{abx+acy}+\sqrt{bcx+aby}+\sqrt{acx+bcy}$$

$$\sqrt{x+y}(a+b+c)\ge\sqrt{3\big(ab(x+y)+bc(x+y)+ca(x+y)\big)}$$

$$a+b+c\ge\sqrt{3(ab+bc+ca)}$$

$$(a+b+c)^2\ge3(ab+bc+ca)$$

***It’s nice to be important but more important it’s to be nice.***

***At this paper works a TEAM.***

***This is RMM TEAM.***

***To be continued!***

***Daniel Sitaru***