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PROBLEMS FOR JUNIORS

JP.256. If $a,b,c,d>0; a+b+c+d=4; 0\leq n\leq 3$ then:

$$\frac{a}{b^3+b^2+b+n}+\frac{b}{c^3+c^2+c+n}+\frac{c}{d^3+d^2+d+n}+$$
$$+\frac{d}{a^3+a^2+a+n}\geq\frac{4}{n+3}$$

Proposed by Marin Chirciu - Romania

JP.257. Let be:

$$\Omega=\left\{y|y=\frac{\sin^2x}{\cos^2x+\tan^2x}+\frac{\cos^2x}{\sin^2x+\cot^2x}; x\in\left(0,\frac{\pi}{2}\right)\right\}$$

Find:

$$\Omega_1=\inf\Omega; \Omega_2=\sup\Omega$$

Proposed by Marin Chirciu - Romania

JP.258. If $a,b,c>0; abc=1; n\in\mathbb{N}$ then:

$$\frac{1}{b^{n+3}+c^{n+3}+a^n}+\frac{1}{c^{n+3}+a^{n+3}+b^n}+\frac{1}{a^{n+3}+b^{n+3}+c^n}\leq 1$$

Proposed by Marin Chirciu - Romania

JP.259. In $\Delta ABC; BB'; CC'$ - symedians. The circumcircle of $\Delta AB'C'$ is tangent to BC. Prove that:

$$\frac{b^2}{a^2+c^2}+\frac{c^2}{a^2+b^2}>\frac{1}{2}$$

Proposed by Marian Ursărescu - Romania

JP.260. In $\Delta ABC, N$ - Nagel's point, BQ, CP - symedians. Prove that:

$$P,N,G \text{ - collinears } \Leftrightarrow \frac{1}{b^2r_b}+\frac{1}{c^2r_c}=\frac{1}{a^2r_a}$$

Proposed by Marian Ursărescu - Romania

JP.261. If $a,b,c>0; a+b+c=1; n\geq 0$ then:

$$\frac{a}{(n+bc)^2}+\frac{b}{(n+ca)^2}+\frac{c}{(n+ab)^2}\geq\frac{81}{(9n+1)^2}$$

Proposed by Marin Chirciu - Romania

JP.262. If $a,b,c>0; a^{3n-1}+b^{3n-1}+c^{3n-1}=1; n\in\mathbb{N}; n\geq 1$ then:

$$\frac{a^2}{b^{3n}}+\frac{b^2}{c^{3n}}+\frac{c^2}{a^{3n}}\geq (a^n+b^n+c^n)^3$$

Proposed by Marin Chirciu - Romania

JP.263. If $m\geq 1; a,b,c,x,y,z>0$ then:

$$\frac{x^{3m}}{(yz)^m(ay^m+bz^m)}+\frac{y^{3m}}{(zx)^m(az^m+bx^m)}+\frac{z^{3m}}{(xy)^m(ax^m+by^m)}\geq\frac{3}{a+b}$$

Proposed by D.M. Bătineţu - Giurgiu, Daniel Sitaru - Romania

JP.264. If $x,y,z\in\left(0,\frac{\pi}{2}\right)$ then in ΔABC the following relationship holds:

$$a^2\tan x+b^2\tan y+c^2\tan z>4F\sqrt{xy+yz+zx}$$

Proposed by D.M. Bătineţu - Giurgiu, Daniel Sitaru - Romania

JP.265. In ΔABC the following relationship holds:

$$\frac{6}{5}-\frac{9r}{10R}\leq\left(\frac{a}{b+c}\right)^2+\left(\frac{b}{c+a}\right)^2+\left(\frac{c}{a+b}\right)^2\leq\frac{3R}{8r}$$

Proposed by Marin Chirciu - Romania

JP.266. If $a,b,c>0$ then:

$$a\sqrt{a^2+2(b+c)^2}+b\sqrt{b^2+2(c+a)^2}+c\sqrt{c^2+2(a+b)^2}\leq(a+b+c)^2$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

JP.267. In ΔABC the following relationship holds:

$$\frac{a^2}{m_b^2+m_c^2}+\frac{b^2}{m_c^2+m_a^2}+\frac{c^2}{m_a^2+m_b^2}\leq 2$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

JP.268. Let $A'B'C'$ be the intouch triangle of ΔABC. Prove that:

$$A'B'+B'C'+C'A'\leq\frac{\sqrt{ab}+\sqrt{bc}+\sqrt{ca}}{2}\leq s$$

Proposed by Marian Ursărescu - Romania

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JP.269. In ΔABC let $A' \in (BC)$ be such that incircles of $\Delta AA'B$; $\Delta AA'C$ have the same radii. Analogous we obtain the points: $B' \in (CA); C' \in (AB)$. Prove that:

$$AA' + BB' + CC' \geq 9r$$

Proposed by Marian Ursărescu - Romania

JP.270. If $m, n > 0$ then in ΔABC the following relationship holds:

$$\frac{r_a^2 + r_b r_c}{nr_b + mr_c} + \frac{r_b^2 + r_c r_a}{nr_c + mr_a} + \frac{r_c^2 + r_a r_b}{nr_a + mr_b} \geq \frac{18r}{m+n}$$

Proposed by D.M. Bătineţu - Giurgiu, Daniel Sitaru - Romania

PROBLEMS FOR SENIORS

SP.256. Let ABC be a triangle and (I, r) its incircle. The circle (I_A, r_A) is externally tangent to the circle (I, r) and internally tangent to the sides AB and AC of the triangle. The circles (I_B, r_B) and (I_C, r_C) are defined similarly. Prove that:

1. $r_A + r_B + r_C \geq r$

2. $\frac{(r-r_A)^2}{rr_A} + \frac{(r-r_B)^2}{rr_B} + \frac{(r-r_C)^2}{rr_C} \geq 4$

Proposed by George Apostolopoulos - Messolonghi - Greece

SP.257. Let BB', CC' be the internal bisectors in ΔABC. If the circumcircle of $\Delta AB'C'$ is tangent to BC then:

$$\frac{2b}{2a+c} + \frac{2c}{2a+b} < 1$$

Proposed by Marian Ursărescu - Romania

SP.258. Let $A'B'C'$ be the circumcevian triangle of symedians in ΔABC. Prove that:

$$\frac{[A'B'C']}{[ABC]} \leq \left(\frac{R}{2r}\right)^2$$

Proposed by Marian Ursărescu - Romania

SP.259. In ΔABC; Γ - Gergonne's point and BN, CM symedians, $M \in (AB); N \in (AC)$. Prove that:

$$B, \Gamma, N \text{ - collinears } \Leftrightarrow \frac{r_b}{b^2} + \frac{r_c}{c^2} = \frac{r_a}{a^2}$$

Proposed by Marian Ursărescu - Romania

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SP.260. If $a_1, a_2, ..., a_k > 0; k \in \mathbb{N}; k \geq 1; n > 0$ fixed then find the minimum of Ω:

$$\Omega = 2(a_1^3 + a_2^3 + ... + a_k^3) - n(a_1a_2 + a_2a_3 + ... + a_ka_1)$$

Proposed by Marin Chirciu - Romania

SP.261. If $x, y, z > 0$ then in ΔABC the following relationship holds:

$$\frac{(x+y)^2}{z^2} + \frac{(y+z)^2}{x^2} + \frac{(z+x)^2}{y^2} + a^4 + b^4 + c^4 \geq 16\sqrt{3}F$$

Proposed by D.M. Bătineţu - Giurgiu, Daniel Sitaru - Romania

SP.262. If $x, y, z > 0; u \geq 0$ then in ΔABC the following relationship holds:

$$\frac{y+z+u}{x}a^2 + \frac{z+x+u}{y}b^2 + \frac{x+y+u}{x}c^2 \geq 8\sqrt{3}F + \frac{12u\sqrt{3}F}{x+y+z}$$

Proposed by D.M. Bătineţu - Giurgiu, Daniel Sitaru - Romania

SP.263. In ΔABC the following relationship holds:

$$12 \leq \left(\frac{a+b}{c}\right)^2 + \left(\frac{b+c}{a}\right)^2 + \left(\frac{c+a}{b}\right)^2 \leq 18\left(\frac{R}{2r}\right)^2 - 6$$

Proposed by George Apostolopoulos - Messolonghi - Greece

SP.264. In ΔABC the following relationship holds:

$$\frac{1}{4R^4} \leq \frac{r_a^2 + r_b^2}{c^2(a^4+b^4)} + \frac{r_b^2 + r_c^2}{a^2(b^4+c^4)} + \frac{r_c^2 + r_a^2}{b^2(c^4+a^4)} \leq \frac{1}{64r^4}$$

Proposed by George Apostolopoulos - Messolonghi - Greece

SP.265. In acute ΔABC the following relationship holds:

$$(\cos A)^A \cdot (\cos B)^B \cdot (\cos C)^C \leq 2^{-\pi}$$

Proposed by Florentin Vişescu - Romania

SP.266. If $A, B \in M_4(\mathbb{C}); AB = \begin{pmatrix} p & p & p & p \\ 0 & -p & -p & -p \\ 0 & 0 & p & p \\ 0 & 0 & 0 & -p \end{pmatrix}; p \in \mathbb{C};$

$p \neq 0; \Omega_1 = BA; \Omega_2 = (BA)^{-1}$ then find:

$$\Omega = \Omega_1^2 + (p^2\Omega_2^{-1})^2$$

Proposed by Marian Ursărescu - Romania

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SP.267. In ΔABC; AA_1, BB_1, CC_1 - internal bisectors and $A_2B_2C_2$ the circumcevian triangle of incenter. Prove that:

$$\left(\frac{r}{R}\right)^2 \le \frac{[A_1B_1C_1]}{[A_2B_2C_2]} \le \frac{1}{4}$$

Proposed by Marian Ursărescu - Romania

SP.268. If $A \in M_2(\mathbb{R})$; $\operatorname{Tr} A = \det A = 1$ then:

$$\det(A^2 + 3A + 3I_2) \ge 5\operatorname{Tr}(A^{-1}) + 3$$

Proposed by Marian Ursărescu - Romania

SP.269. If in ΔABC; $s = \frac{1}{2}$ then:

$$a \cdot e^{\frac{m_a}{a}} + b \cdot e^{\frac{m_b}{b}} + c \cdot e^{\frac{m_c}{c}} \ge e^{m_a+m_b+m_c}$$

Proposed by Daniel Sitaru - Romania

SP.270. If $x_n > 0$; $n \in \mathbb{N}$ is a sequence such that exists $\lim_{n\to\infty} \frac{x_{n+3}\cdot x_{n+1}^3}{x_{n+2}^2\cdot x_n} = \pi$ then find:

$$\Omega = \lim_{n\to\infty} \sqrt[n^3]{x_n}$$

Proposed by D.M. Bătineţu - Giurgiu, Neculai Stanciu - Romania

UNDERGRADUATE PROBLEMS

UP.256. If $m, n \in \mathbb{N}$; $a, b, c > 0$ then:

$$\left(m + (a+b)^{m+1}\right)\left(n + \frac{1}{c^{n+1}}\right) + \left(m + (b+c)^{m+1}\right)\left(n + \frac{1}{a^{n+1}}\right) +$$

$$+\left(m + (c+a)^{m+1}\right)\left(n + \frac{1}{b^{n+1}}\right) \ge 6(m+1)(n+1)$$

Proposed by D.M. Bătineţu - Giurgiu, Daniel Sitaru - Romania

UP.257. If $A \in M_6(\mathbb{Q})$; $p, q \in \mathbb{R}/\mathbb{Q}$; $\det(A^4 + pA^2 + p^2I_6) = $ $= \det(A^2 + qI_6) = 0$ then find:

$$\Omega = \det A$$

Proposed by Marian Ursărescu - Romania

UP.258. If $m \geq 0; x, y > 0$ then in ΔABC the following relationship holds:

$$\frac{a^{3m+4}}{(ax+by)^m} + \frac{b^{3m+4}}{(bx+cy)^m} + \frac{c^{3m+4}}{(cx+ay)^m} \geq \frac{4^{m+2} \cdot F^{m+2}}{(\sqrt{3})^m \cdot (x+y)^m}$$

Proposed by D.M. Bătineţu - Giurgiu, Daniel Sitaru - Romania

UP.259. If $0 < a \leq b; n \in \mathbb{N}; n \geq 1$ then:

$$\left(\int_0^{\sqrt{ab}} x^n e^{x^2} dx\right)\left(\int_0^{\frac{a+b}{2}} x^{n-1} e^{x^2} dx\right) \leq$$
$$\leq \left(\int_0^{\sqrt{ab}} x^{n-1} e^{x^2} dx\right)\left(\int_0^{\frac{a+b}{2}} x^n e^{x^2} dx\right)$$

Proposed by Daniel Sitaru - Romania

UP.260. If $0 < a \leq b < \frac{\pi}{2}$ then:

$$\left(\int_0^{\sqrt{ab}} \left(\frac{\sin t}{1+e^t}\right) dt\right)\left(\int_0^{\frac{a+b}{2}} \left(\frac{\cos t}{1+e^t}\right) dt\right) \leq$$
$$\leq \left(\int_0^{\sqrt{ab}} \left(\frac{\cos t}{1+e^t}\right) dt\right)\left(\int_0^{\frac{a+b}{2}} \left(\frac{\sin t}{1+e^t}\right) dt\right)$$

Proposed by Daniel Sitaru - Romania

UP.261. If $0 < a \leq b < \frac{\pi}{2}$ then:

$$\left(\int_0^{\sqrt{ab}} e^{-x^2} \sin x dx\right)\left(\int_{\frac{a+b}{2}}^{b} e^{-x^2} \cos x dx\right) \leq$$
$$\leq \left(\int_a^{\sqrt{ab}} e^{-x^2} \cos x dx\right)\left(\int_{\frac{a+b}{2}}^{b} e^{-x^2} \sin x dx\right)$$

Proposed by Daniel Sitaru - Romania

UP.262. If $u, v, w > 0; x_n, y_n, z_n > 0; n \in \mathbb{N}; n \geq 1$ sequences such that:

$$\lim_{n\to\infty} \frac{x_{n+1}}{nx_n} = a; \lim_{n\to\infty} \frac{y_{n+1}}{ny_n} = b; \lim_{n\to\infty} \frac{z_{n+1}}{nz_n} = c$$

where a, b, c are sides in a triangle ABC with circumradii R then in ΔABC the following relationship holds:

$$\lim_{n\to\infty} \left(\frac{1}{n^2}\left(vw\sqrt[n]{x_n^2} + wu\sqrt[n]{y_n^2} + uv\sqrt[n]{z_n^2}\right)\right) \leq \frac{(u+v+w)^2 R^2}{e^2}$$

Proposed by D.M. Bătineţu - Giurgiu, Neculai Stanciu - Romania

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UP.263. If $m > 0; f, g : (0, \infty) \to (0, \infty); \lim_{x\to\infty} \frac{f(x)}{x} = a > 0;$

$$\lim_{x\to\infty} g(x) \cdot x^{\frac{1}{m}} = b > 0 \text{ then find:}$$

$$\Omega = \lim_{x\to\infty}\left(\left(\left(\Gamma(x+2)\right)^{\frac{1}{m(x+1)}} - \left(\Gamma(x+1)\right)^{\frac{1}{mx}}\right)\cdot f(x)\cdot g(x)\right)$$

Proposed by D.M. Bătineţu - Giurgiu, Neculai Stanciu - Romania

UP.264. If $r, s \geq 0; a_n, b_n > 0; n \geq 0; n \in \mathbb{N};$

$$\lim_{n\to\infty} \frac{a_{n+1}}{n^r \cdot a_n} = a > 0; \lim_{n\to\infty} \frac{b_{n+1}}{n^{s+1} \cdot b_n} = b > 0 \text{ then find:}$$

$$\Omega = \lim_{n\to\infty}\left(\left(\frac{\sqrt[n]{a_n}}{n^{r+s}} - \frac{\sqrt[n+1]{a_{n+1}}}{(n+1)^{r+s}}\right)\cdot \sqrt[n]{b_n}\right)$$

Proposed by D.M. Bătineţu - Giurgiu, Neculai Stanciu - Romania

UP.265. If $r, s \geq 0; a_n, b_n > 0; n \geq 0; n \in \mathbb{N};$

$$\lim_{n\to\infty} \frac{a_{n+1}}{n^r \cdot a_n} = a > 0; \lim_{n\to\infty} \frac{b_{n+1}}{n^s \cdot b_n} = b > 0 \text{ then find:}$$

$$\Omega = \lim_{n\to\infty}\left(\left(a_{n+1}^{\frac{n^s}{(n+1)\sqrt[n]{b_n}}} - a_n^{\frac{n^{s-1}}{\sqrt[n]{b_n}}}\right)\cdot n^{1-\frac{r\cdot n^s}{\sqrt[n]{b_n}}}\right)$$

Proposed by D.M. Bătineţu - Giurgiu, Neculai Stanciu - Romania

UP.266. If $m \geq 0; x, y, z > 0$ then in ΔABC the following relationship holds:

$$\left(\frac{a^2x}{y+z}\right)^{m+1} + \left(\frac{b^2y}{z+x}\right)^{m+1} + \left(\frac{c^2z}{x+y}\right)^{m+1} \geq 2^{m-1}\cdot(\sqrt{3})^{1-m}\cdot F^{m+1}$$

Proposed by D.M. Bătineţu - Giurgiu, Neculai Stanciu - Romania

UP.267. If $m \in \mathbb{N}; m \geq 1; x, y, z > 0$ then in ΔABC the following relationship holds:

$$m \cdot \sum_{cyc} \frac{x}{y+z} + \sum_{cyc} \frac{x \cdot a^{4(m+1)}}{y+z} \geq 8(m+1)F^2$$

Proposed by D.M. Bătineţu - Giurgiu - Romania

UP.268. In ΔABC the following relationship holds:

$$20a^4 + 5b^4 + 2c^4 \geq 80F^2$$

Proposed by D.M. Bătineţu - Giurgiu - Romania

UP.269. If $t \geq 0; a_n > 0; n \in \mathbb{N}; n \geq 1;$

$$\lim_{n\to\infty} \frac{a_n}{n^t} = \pi; \lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)^n = e \text{ then find:}$$

$$\Omega = \lim_{n\to\infty}\left(\frac{(n+1)^{t+1}}{a_{n+1}} - \frac{n^{t+1}}{a_n}\right)$$

Proposed by D.M. Bătineţu - Giurgiu, Daniel Sitaru - Romania

UP.270. If $a, b, c, x, y > 0$ then:

$$a\sqrt{\frac{a(x+y)}{bx+cy}} + b\sqrt{\frac{b(x+y)}{cx+ay}} + c\sqrt{\frac{c(x+y)}{ax+by}} \geq a+b+c$$

Proposed by D.M. Bătineţu - Giurgiu - Romania

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