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Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{n^6}\sum_{1\le i<j<k\le n}\frac{i\cdot j\cdot k}{(n^2+i^2)(n^2+j^2)(n^2+k^2)}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Remus Florin Stanca-Romania, Solution 2 by Tran Hong-Dong Thap-Vietnam

Solution 1 by Remus Florin Stanca-Romania

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{n^6}\sum_{1\le i<j<k\le n}\frac{n^3}{n^6}\cdot\frac{\frac{i}{n}}{1+\left(\frac{i}{n}\right)^2}\cdot\frac{\frac{j}{n}}{1+\left(\frac{j}{n}\right)^2}\cdot\frac{\frac{k}{n}}{1+\left(\frac{k}{n}\right)^2}\right) =$$

$$= \lim_{n\to\infty}\left(\frac{1}{n^9}\cdot\sum_{1\le i<j<k\le n}\frac{\frac{i}{n}}{1+\left(\frac{i}{n}\right)^2}\cdot\frac{\frac{j}{n}}{1+\left(\frac{j}{n}\right)^2}\cdot\frac{\frac{k}{n}}{1+\left(\frac{k}{n}\right)^2}\right)$$

$$\frac{1}{n^9}\cdot\sum_{1\le i<j<k\le n}\frac{\frac{i}{n}}{1+\left(\frac{i}{n}\right)^2}\cdot\frac{\frac{j}{n}}{1+\left(\frac{j}{n}\right)^2}\cdot\frac{\frac{k}{n}}{1+\left(\frac{k}{n}\right)^2} \le$$

$$\le \frac{1}{n^9}\cdot\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left(\sum_{k=1}^{n}\frac{\frac{i}{n}}{1+\left(\frac{i}{n}\right)^2}\cdot\frac{\frac{j}{n}}{1+\left(\frac{j}{n}\right)^2}\cdot\frac{\frac{k}{n}}{1+\left(\frac{k}{n}\right)^2}\right)\right)\right)$$

$$=\frac{1}{n^9}\cdot\left(\sum_{i=1}^{n}\frac{\frac{i}{n}}{1+\left(\frac{i}{n}\right)^2}\right)^3=\frac{1}{n^6}\left(\frac{1}{n}\cdot\sum_{i=1}^{n}\frac{\frac{i}{n}}{1+\left(\frac{i}{n}\right)^2}\right)^3$$

$$\lim_{n\to\infty}\frac{1}{n}\cdot\sum_{i=1}^{n}\frac{\frac{i}{n}}{1+\left(\frac{i}{n}\right)^2}=\int_0^1\frac{x}{x^2+1}dx=\frac{1}{2}\cdot\int_0^1\frac{2x}{x^2+1}dx=\frac{1}{2}\ln(x^2+1)\Big|_0^1=\ln(\sqrt{2})\Rightarrow$$

$$\Rightarrow \lim_{n\to\infty}\frac{1}{n^6}\cdot\left(\frac{1}{n}\sum_{i=1}^{n}\frac{\frac{i}{n}}{1+\left(\frac{i}{n}\right)^2}\right)^3=\frac{\ln(\sqrt{2})}{\infty}=0,\ \textit{but}$$

$$\frac{1}{n^9}\sum_{1\le i<j<k\le n}\frac{\frac{i}{n}}{1+\left(\frac{i}{n}\right)^2}\cdot\frac{\frac{j}{n}}{1+\left(\frac{j}{n}\right)^2}\cdot\frac{\frac{k}{n}}{1+\left(\frac{k}{n}\right)^2}\le\frac{1}{n^6}\left(\frac{1}{n}\sum_{i=1}^{n}\frac{\frac{i}{n}}{1+\left(\frac{i}{n}\right)^2}\right)^3\Rightarrow$$

$$\Rightarrow \lim_{n\to\infty}\frac{1}{n^9}\sum_{1\le i<j<k\le n}\frac{\frac{i}{n}}{1+\left(\frac{i}{n}\right)^2}\cdot\frac{\frac{j}{n}}{1+\left(\frac{j}{n}\right)^2}\cdot\frac{\frac{k}{n}}{1+\left(\frac{k}{n}\right)^2}0\Rightarrow\Omega=0$$

Solution 2 by Tran Hong-Dong Thap-Vietnam

For $i,j,k\ge 1$ we have:

$$(n^2+i^2)(n^2+j^2)(n^2+k^2)\overset{AM-GM}{\ge}2ni\cdot 2nj\cdot 2nk=8n^3ijk$$

$$\Rightarrow\frac{ijk}{(n^2+i^2)(n^2+j^2)(n^2+k^2)}\le\frac{1}{8n^3}$$

$$\Rightarrow 0<\Omega_n=\frac{1}{n^6}\sum_{1\le i<j<k\le n}\frac{ijk}{(n^2+i^2)(n^2+j^2)(n^2+k^2)}\le$$

$$\le\frac{1}{n^6}\sum_{1\le i<j\le k\le n}\frac{ijk}{(n^2+i^2)(n^2+j^2)(n^2+k^2)}\le\frac{1}{n^6}\cdot\frac{1}{8n^3}\cdot n^3=\frac{1}{8n^6}\to 0\ (n\to\infty)$$

$$\Rightarrow\Omega=\lim_{n\to\infty}\Omega_n=0$$