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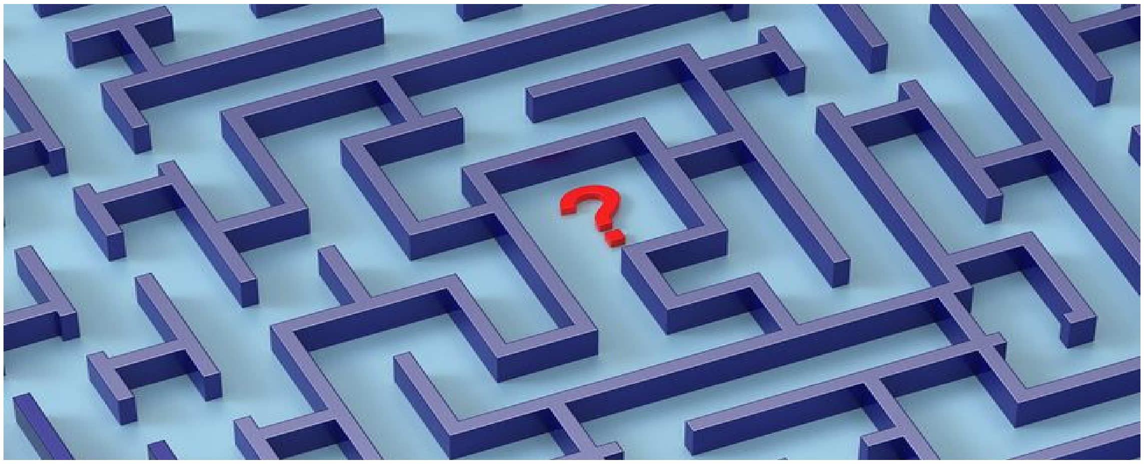

If $x, y, z > 0$ then:

$$\sum_{cyc}\frac{x}{y}+4\sum_{cyc}\frac{y}{x+y}+4\sum_{cyc}\frac{xy}{(x+y)^2}\geq 12$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Tran Hong-Dong Thap-Vietnam, Solution 2 by Serban George Florin-Romania, Solution 3 by Soumava Chakraborty-Kolkata-India, Solution 4 by Sanong Huayrerai-Nakon Pathom-Thailand

Solution 1 by Tran Hong-Dong Thap-Vietnam

Must show that: $\frac{x}{y}+4\cdot\frac{y}{x+y}+4\cdot\frac{xy}{(x+y)^2}\geq 4$ *(etc)*

$$\Leftrightarrow x(x+y)^2+4y^2(x+y)+4xy^2\geq 4y(x+y)^2$$

$$\Leftrightarrow x^3-2x^2y+xy^2\geq 0$$

$$\Leftrightarrow x(x^2-2xy+y^2)\geq 0\Leftrightarrow x(x-y)^2\geq 0\quad \textit{(true)}$$

$$\Rightarrow \sum\frac{x}{y}+4\sum\frac{y}{x+y}+4\sum\frac{xy}{(x+y)^2}\geq 4+4+4=12$$

Proved.

Solution 2 by Serban George Florin-Romania

$$\frac{x}{y}+\frac{4y}{x+y}+\frac{4xy}{(x+y)^2}\geq 4,\ (\forall)x,y>0$$

$$\frac{x}{y}=t\Rightarrow t+\frac{4}{t+1}+\frac{4t}{(t+1)^2}-4\geq 0$$

$$\frac{t(t+1)^2+4(t+1)+4t-4(t+1)^2}{(t+1)^2}\geq 0,\ (\forall)t>0$$

$$\Rightarrow\frac{t^3+2t^2+t+4t+4+4t-4t^2-8t-4}{(t+1)^2}\geq 0$$

$$\frac{t^3-2t^2+t}{(t+1)^2}\geq 0,\ \frac{t(t-1)^2}{(t+1)^2}\geq 0,\ \textit{true}\ (\forall)t>0$$

$$\Rightarrow\sum_{cyc}\left(\frac{x}{y}+\frac{4y}{x+y}+\frac{4xy}{(x+y)^2}\right)=\sum_{cyc}\frac{x}{y}+4\sum_{cyc}\frac{y}{x+y}+4\sum_{cyc}\frac{xy}{(x+y)^2}\geq 4+4+4=12\ \textit{true}$$

Solution 3 by Soumava Chakraborty-Kolkata-India

Given inequality $\Leftrightarrow$

$$\sum\frac{x}{y}+4\sum\left\{\frac{y}{x+y}+\frac{xy}{(x+y)^2}\right\}-12\geq 0$$

$$\Leftrightarrow\sum\frac{x}{y}+4\sum\left\{\frac{y}{x+y}\left(1+\frac{x}{x+y}\right)\right\}-12\geq 0$$

$$\Leftrightarrow\sum\frac{x}{y}+4\sum\left\{\frac{y(2x+y)}{(x+y)^2}-1\right\}\geq 0\Leftrightarrow\sum\frac{x}{y}+4\sum\left[\frac{2xy+y^2-(x+y)^2}{(x+y)^2}\right]\geq 0$$

$$\Leftrightarrow\sum\frac{x}{y}+4\sum\left\{\frac{-x^2}{(x+y)^2}\right\}\geq 0\Leftrightarrow\sum\frac{x}{y}\geq\sum\frac{4x^2}{(x+y)^2}$$

$$\Leftrightarrow\sum\left\{\frac{x}{y}-\frac{4x^2}{(x+y)^2}\right\}\geq 0\Leftrightarrow\sum\left[x\left\{\frac{1}{y}-\frac{4x}{(x+y)^2}\right\}\right]\geq 0$$

$$\Leftrightarrow\sum\left[x\left\{\frac{(x+y)^2-4xy}{y(x+y)^2}\right\}\right]\geq 0\Leftrightarrow\sum\left\{\frac{x(x-y)^2}{y(x+y)^2}\right\}\geq 0\rightarrow\textit{true (proved)}$$

Solution 4 by Sanong Huayrerai-Nakon Pathom-Thailand

For $x,y,z>0$ *give* $a=x+y,b=y+z,c=z+x$

Hence $x=\frac{a+c-b}{2},y=\frac{a+b-c}{2},z=\frac{b+c-a}{2}$ *and we have as follows*

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x}+4\left(\frac{x}{x+z}+\frac{y}{y+x}+\frac{z}{z+x}\right)+4\left(\frac{xy}{(x+y)^2}+\frac{yz}{(y+z)^2}+\frac{zx}{(z+x)^2}\right)$$

$$=\frac{a+c-b}{a+b-c}+\frac{a+b-c}{b+c-a}+\frac{b+c-a}{a+c-b}+2\left(\frac{a+c-b}{c}+\frac{a+b-c}{a}+\frac{b+c-a}{b}\right)+$$

$$+\frac{(a+c-b)}{a^2}+\frac{(a+b-c)(b+c-a)}{b^2}+\frac{(b+c-a)(a+c-b)}{c^2}$$

$$= \left[\frac{a+c-b}{a+b-c} + \frac{(a+c-b)(a+b-c)}{a^2}\right] + \left[\frac{a+b-d}{b+c-a} \cdot \frac{(a+b-c)(b+c-a)}{b^2}\right] +$$

$$+ \left[\frac{b+c-a}{a+c-b} + \frac{(b+c-a)(a+c-b)}{c^2}\right] + 2\left(\frac{a+c-b}{c} + \frac{a+b-c}{a} + \frac{b+c-a}{b}\right)$$

$$\geq 2\left[\frac{b+c-b}{a} + \frac{a+b-c}{a}\right] + 2\left[\frac{a+b-d}{b} + \frac{b+c-a}{b}\right] + 2\left[\frac{b+c-a}{c} + \frac{a+c-b}{c}\right]$$

$$= 2\left[\frac{a+c-b+a+b-c}{a}\right] + 2\left[\frac{a+b-c+b+c-a}{b}\right] + 2\left[\frac{b+c-a+a+c-b}{c}\right]$$

$$= 2\left[\frac{2a}{a} + \frac{2b}{b} + \frac{2c}{c}\right] = 2[2+2+2] = 2 \times 6 = 12 \ \textit{ok}$$

Therefore, it is true.