SP.048. Prove that the following inequality holds for all non-negative real numbers \(a, b, c\)

\[(a^4 + b^4 + c^4)(ab^3 + bc^3 + ca^3) \geq (a^3b + b^3c + c^3a)(a^2b^2 + b^2c^2 + c^2a^2)\]

*Proposed by Nguyen Viet Hung – Hanoi – Vietnam*

**Solution 1 by Kevin Soto Palacios – Huarmey – Peru**

**Solution 2 by Soumava Chakraborty-Kolkata-India**

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**Probar para todos los reales no negativos: a, b, c la siguiente desigualdad:**

\[(a^4 + b^4 + c^4)(ab^3 + bc^3 + ca^3) \geq (a^3b + b^3c + c^3a)(a^2b^2 + b^2c^2 + c^2a^2)\]

*Siendo: a, b, c \(\geq 0\). Por la desigualdad de Cauchy:*

\[(a^4 + b^4 + c^4)(a^2b^2 + b^2c^2 + c^2a^2) \geq (a^3b + b^3c + c^3a)^2 \quad \cdots (A)\]

\[(ab^3 + bc^3 + ca^3)(a^3b + b^3c + c^3a) \geq (a^2b^2 + b^2c^2 + c^2a^2)^2 \quad \cdots (B)\]

*Multiplicando, se obtiene: \((A) \times (B)\):

\[(a^4 + b^4 + c^4)(ab^3 + bc^3 + ca^3) \geq (a^3b + b^3c + c^3a)(a^2b^2 + b^2c^2 + c^2a^2)\]

*(LQQD)*

**Solution 2 by Soumava Chakraborty-Kolkata-India**

**Case 1**

*Exactly 1 of a, b, c = 0*

*WLOG, let c = 0. LHS \(\geq\) RHS \(\iff\) \((a^4 + b^4)ab^3 \geq a^5b^3\)*

\(\iff b^4 \geq 0 \rightarrow true \rightarrow \text{inequality is valid in this case})*

**Case 2**

*Exactly 2 of a, b, c = 0. Then LHS = RHS = 0*
And ∴ 0 ≥ 0, ∴ inequality is valid in this case.

Case 3

\[ a = b = c = 0. \therefore LHS = RHS = 0 \Rightarrow LHS \geq RHS \]

Case 4

\[ a, b, c > 0 \]

\[ LHS - RHS \geq 0 \iff \]

\[ \Leftarrow ab^7 + bc^7 + ca^7 + a^4bc^3 + b^4ca^3 + c^4ab^3 \geq a^5bc^2 + b^5ca^2 + c^5ab^2 + a^3b^3c^2 + b^3c^2a^2 + c^3a^3b^2 \quad (1) \]

Adding the above 6 inequalities, we get:

\[ 2(ab^7 + bc^7 + ca^7 + a^4bc^3 + b^4ca^3 + c^4ab^3) \geq 2(ab^7 + bc^7 + ca^7 + a^4bc^3 + b^4ca^3 + c^4ab^3) \geq a^5bc^2 + b^5ca^2 + c^5ab^2 + a^3b^3c^2 + b^3c^2a^2 + c^3a^3b^2 \]

\[ \Rightarrow (1) \text{ is true (Proved)} \]