SOCIETATEA DE ŞTIINȚE MATEMATICE DIN ROMÂNIA
ROMANIAN MATHEMATICAL SOCIETY

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SP.048. Prove that the following inequality holds for all non-negative real numbers a, b, c

$$(a^4+b^4+c^4)(ab^3+bc^3+ca^3)\geq(a^3b+b^3c+c^3a)(a^2b^2+b^2c^2+c^2a^2)$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Solution 2 by Soumava Chakraborty-Kolkata-India

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar para todos los reales no negativos: a, b, c la siguiente desigualdad:

$$(a^4+b^4+c^4)(ab^3+bc^3+c^3a)\geq(a^3b+b^3c+a^3c)(a^2b^2+b^2c^2+c^2a^2)$$

Siendo: $a, b, c\geq 0$. Por la desigualdad de Cauchy:

$$(a^4+b^4+c^4)(a^2b^2+b^2c^2+c^2a^2)\geq(a^3b+b^3c+a^3c)^2 \text{ ... (A)}$$

$$(ab^3+bc^3+c^3a)(a^3b+b^3c+a^3)\geq(a^2b^2+b^2c^2+c^2a^2)^2 \text{ ... (B)}$$

Multiplicando, se obtiene: (A) × (B):

$$(a^4+b^4+c^4)(ab^3+bc^3+c^3a)\geq(a^3b+b^3c+a^3c)(a^2b^2+b^2c^2+c^2a^2)$$

(LQQD)

Solution 2 by Soumava Chakraborty-Kolkata-India

Case 1

Exactly 1 of $a, b, c = 0$

WLOG, let $c=0. LHS\geq RHS \Leftrightarrow (a^4+b^4)ab^3\geq a^5b^3$

$\Leftrightarrow b^4\geq 0 \to$ true $\Rightarrow$ inequality is valid in this case

Case 2

Exactly 2 of $a, b, c = 0$. Then $LHS = RHS = 0$

And $\because 0 \geq 0, \therefore$ ***inequality is valid in this case.***

Case 3

$$a = b = c = 0. \therefore LHS = RHS = 0 \Rightarrow LHS \geq RHS$$

Case 4

$$a, b, c > 0$$

$$LHS - RHS \geq 0 \Leftrightarrow$$

$$\Leftrightarrow ab^7 + bc^7 + ca^7 + a^4bc^3 + b^4ca^3 + c^4ab^3 \geq$$

$$\geq a^5bc^2 + b^5ca^2 + c^5ab^2 + a^3b^3c^2 + b^3c^2a^2 + c^3a^3b^2 \quad \textit{(1)}$$

$$a^4bc^3 + a^4bc^3 + ab^7 \overset{A-G}{\geq} 3a^3b^3c^2 \quad | \quad ab^7 + b^3a^3c^2 \overset{A-G}{\geq} 2b^5a^2c$$

$$b^4ca^3 + b^4ca^3 + bc^7 \overset{A-G}{\geq} 3b^3c^3a^2 \quad | \quad bc^7 + c^3b^3a^2 \overset{A-G}{\geq} 2c^5b^2a$$

$$c^4ab^3 + c^4ab^3 + ca^7 \overset{A-G}{\geq} 3c^3a^3b^2 \quad | \quad ca^7 + c^3a^3b^2 \overset{A-G}{\geq} 2a^5c^2b$$

Adding the above 6 inequalities, we get:

$$2(ab^7 + bc^7 + ca^7 + a^4bc^3 + b^4ca^3 + c^4ab^3) \geq$$

$$2(a^5bc^2 + b^5ca^2 + c^5ab^2 + a^3b^3c^2 + b^3c^3a^2 + c^3a^3b^2)$$

$$\Leftrightarrow ab^7 + bc^7 + ca^7 + a^4bc^3 + b^4ca^3 + c^4ab^3 \geq$$

$$\geq a^5bc^2 + b^5ca^2 + c^5ab^2 + a^3b^3c^2 + b^3c^3a^2 + c^3a^3b^2$$

$\Rightarrow$ ***(1) is true (Proved)***