SP.036. Let a, b, c be positive real numbers such that

$$3(a+b)(b+c)(c+a) \geq \frac{8}{\sqrt[3]{a^3+b^3+c^3}}$$

Prove that $a+b+c \geq \sqrt[3]{9}$.

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Anas Adlany - El Jadida- Morroco

Solution 2 by Soumitra Mandal – Kolkata – India

Solution 1 by Anas Adlany - El Jadida- Morroco

We have known that

$$(a+b+c)^3 \coloneqq \sum a^3 + 3\prod(a+b)$$

$$\geq \sum a^3 + \frac{8}{\sqrt[8]{a^3+b^3+c^3}}$$

$$\geq 9\sqrt[9]{\left(\sum a^3\right)\left(\frac{1}{\sqrt[8]{a^3+b^3+c^3}}\right)^8}$$

Thus,

$$a+b+c \geq \sqrt[3]{9}$$

Hence proved.

Solution 2 by Soumitra Mandal – Kolkata – India

$$3(a+b)(b+c)(c+a) \geq \frac{8}{\sqrt[3]{a^3+b^3+c^3}}$$

$$\Rightarrow \sum_{cyc} a^3 + 3\prod_{cyc}(a+b) \geq \frac{8}{\sqrt[8]{a^3+b^3+c^3}} + (a^3+b^3+c^3)$$

$$\geq (8+1)\sqrt[9]{\left\{\frac{1}{\sqrt[8]{a^3+b^3+c^3}}\right\}^8 (a^3+b^3+c^3)} = 9$$

$$\Rightarrow (a+b+c)^3 \geq 9 \Rightarrow a+b+c \geq \sqrt[3]{9} \textit{ (proved)}$$

Equality at $a = b = c = \frac{1}{\sqrt[3]{3}}$