JP.032. Prove the following inequality holds for all non-negative real numbers a, b:

$$\frac{1}{4a+1}+\frac{1}{4b+1}+\frac{6}{2a+2b+1}\ge\frac{4}{3a+b+1}+\frac{4}{3b+a+1}$$

Proposed by Nguyen Viet Hung- Hanoi – Vietnam

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Solution 2 by Soumitra Mandal – Kolkata – India

Solution 3 by Henry Ricardo - New York – USA

Solution 4 by Imad Zak – Saida – Lebanon

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

***Probar la siguiente desigualdad para todos los números reales non negativos:** a, b:*

$$\frac{1}{4a+1}+\frac{1}{4b+1}+\frac{6}{2a+2b+1}\ge\frac{4}{3a+b+1}+\frac{4}{3b+a+1}$$

Sea: $x=4a+1\ge1,\ y=4b+1\ge1,\ x+y=2(2a+2b+1)$

Además:

$$12a+4b+4=3(4a+1)+(4b+1)=3x+y,$$

$$12b+4a+4=3(4b+1)+(4a+1)=3y+x$$

$$\Rightarrow:\frac{1}{x}+\frac{1}{y}+\frac{12}{x+y}\ge\frac{16}{3x+y}+\frac{16}{3y+x}\to\frac{x+y}{xy}+\frac{12}{x+y}\ge\frac{16(3y+x)+16(3x+y)}{(3x+y)(y+3x)}$$

$$\Rightarrow\frac{(x+y)^2+12xy}{xy(x+y)}\ge\frac{64(x+y)}{(3x^2+3y^2+10xy)}\to$$

$$\to[3(x^2+y^2)+10xy][(x+y)^2+12xy]\ge64(x+y)^2xy$$

$$\Rightarrow3(x^2+y^2)(x+y)^2+36xy(x^2+y^2)+10xy(x+y)^2+120x^2y^2\ge64(x+y)^2xy$$

$$\Rightarrow 3(x^2+y^2)(x+y)^2+36xy(x^2+y^2)+10xy(x+y)^2+120x^2y^2 \geq$$

$$\geq 64(x+y)^2xy$$

$$\Rightarrow 3(x^2+y^2)(x+y)^2+36xy(x^2+y^2)+120x^2y^2 \geq 54(x+y)^2xy$$

$$\Rightarrow (x^2+y^2)(x+y)^2+12xy(x^2+y^2)+40x^2y^2 \geq 18(x+y)^2xy$$

$$\Rightarrow (x^2+y^2)^2+14xy(x^2+y^2)+40x^2y^2 \geq 18xy(x^2+y^2)+36x^2y^2$$

$$\Rightarrow (x^2+y^2)^2-4xy(x^2+^2)+4x^2y^2 = \left((x^2+y^2)-(2xy)\right)^2 = (x-y)^4 \geq 0$$

La igualdad se alcanza cuando: $x = y = 4a+1 = 4b+1 \rightarrow a = b$

Solution 2 by Soumitra Mandal – Kolkata – India

$$\frac{1}{4a+1}+\frac{1}{4b+1}+\frac{6}{2a+2b+1} \geq \frac{4}{3a+b+1}+\frac{4}{a+3b+1}$$

$$\Leftrightarrow \int_0^1 x^{4a}\,dx+\int_0^1 x^{4b}\,dx+6\int_0^1 x^{2(a+b)}\,dx \geq 4\int_0^1 x^{3a+b}\,dx+4\int_0^1 x^{a+3b}\,dx$$

$$\Leftrightarrow A^4+B^4+6A^2B^2 \geq 4AB(A^2+B^2)$$

$$\Leftrightarrow (A^2+B^2)^2-4AB(A^2+B^2)+4A^2B^2 \geq 0 \Leftrightarrow (A-B)^4 \geq 0,$$

which is true

$$\frac{1}{4a+1}+\frac{1}{4b+1}+\frac{6}{2a+2b+1} \geq \frac{4}{3a+b+1}+\frac{4}{a+3b+1}$$

(proved)

Solution 3 by Henry Ricardo - New York – USA

Noting that $\frac{1}{4a+1} = \int_0^1 t^{4a}\,dt$***, we see that the given inequality is equivalent to***

$$\int_0^1 t^{4a}+t^{4b}+6t^{2a+2b}dt \geq \int_0^1 4t^{3a+b}+4t^{3b+a}\,dt,$$

or $t^{4a}+t^{4b}+6t^{2a+2b}\geq 4t^{3a+b}+4t^{3b+a}$***. If we let*** $t^a=x$ ***and*** $t^b=y$***, the inequality is equivalent to***

$$x^4+y^4+6x^2y^2\geq 4x^3y+4xy^3, \text{ or } (x-y)^4\geq 0,$$

which is true.

Solution 4 by Imad Zak – Saida – Lebanon

Another attempt:

Let $A=\frac{1}{4a+1}+\frac{3}{2a+2b+1}+\frac{4}{3b+a+1}$ ***and*** $B=\frac{1}{4b+1}+\frac{3}{2a+2b+1}-\frac{4}{3b+a+1}$ ***we want to prove*** $A+B\geq 0$***. We find*** $A=\frac{2(a-b)(5a-b+1)}{(4a+1)(3a+b+1)(2a+2b+2)}$ ***and***

$B=\frac{2(a-b)(a-5b-1)}{(4b+1)(3b+a+1)(2a+2b+1)}$ ***and finally***

$$A+B=\frac{2(a-b)}{2a+2b+1}\cdot\left(\frac{5a-b+1}{(4a+1)(3a+b+1)}+\frac{a-5b-1}{(4b+1)(a+3b+1)}\right)=$$

$$=24(a-b)^{\frac{4}{D}} \text{ where}$$

$$D=(4a+1)(4b+1)(3a+b+1)(a+3b+1)(2a+2b+1)$$

Clearly $A+B\geq 0$ ***. Q.E.D. and equality holds when*** $a=b$***.***